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COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS OF TYPE (A-1) SATISFYING A CONTRACTIVE CONDITION OF INTEGRAL TYPE

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Abstract

In this paper, common fixed point results for two pairs of compatible mappings of type (A-1) satisfying contractive condition of integral type on metric spaces are established.

1. Introduction and Preliminaries

Fixed point theory is one of the most fruitful and applicable topics of nonlinear analysis, which is widely used not only in other mathematical theories, but also in many practical problems of natural sciences and engineering. The Banach contraction mapping principle [1] is indeed the most popular result of metric fixed point theory. This principle has much application in several domains, such as differential equations, functional equations, integral equations, economics, wild life, and several others.

Branciari [2] gave an integral version of the Banach contraction principles and proved fixed point theorem for a single-valued contractive mapping of integral type in metric space. Afterwards many researchers [3-7] extended the result of Branciari and obtained fixed point and common fixed point theorems for various contractive conditions of integral type on different spaces.

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Now, we recollect some known definitions and results from the literature which are helpful in the proof of our main results.

Definition 1.1. A coincidence point of a pair of self-mapping $A, B : X \to X$ is a point $x \in X$ for which $Ax = Bx$.

A common fixed point of a pair of self-mapping $A, B : X \to X$ is a point $x \in X$ for which $Ax = Bx = x$.

Studies of common fixed points of commuting maps were initiated by Jungck [8]. Jungck [9] further made a generalization of commuting maps by introducing the notion of compatible mappings.

The concept of type A-compatible and S-compatible was given by Pathak and Khan [10].

Pathak et al. [11] renamed A-compatible and S-compatible as compatible mappings of type (A-1) and compatible mappings of type (A-2) respectively.

Definition 1.2. Let (X, d) be a metric space and $A, B, P, Q : X \to X$ be four self- maps. The pairs (A, P) and (B, Q) are said to be compatible mappings of type (A-1) if

$$\lim_{n\to\infty} d(AAx_n, PPx_n) = 0 \text{ and } \lim_{n\to\infty} d(BBy_n, QQy_n) = 0$$

whenever there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z$ for some $z \in PX \cap QX$.

Proposition 1.1. *Let* $S, T : (X, d) \to (X, d)$ *be mappings. Let* S *and* T *are compatible of type* (A-1) *and let* $Sx_n, Tx_n \to z$ *as* $n \to \infty$ *for some* $z \in X$. *Then we have the following*:

(a) $\lim_{n\to\infty} TTx_n = Sz$ *if* S *is continuous at* z,

(b) $\lim_{n\to\infty} SSx_n = Tz$ *if* T*is continuous at* z,

(c) $STz = TSz$ *and* $Sz = Tz$ *if* S *and* T *are continuous at* z.

Proposition 1.2. *Let $S, T : (X, d) \to (X, d)$ be mappings. If Sand Tare compatible of type* (A-1) *and $Sz = Tz$ for some $z \in X$, then $SSz = STz = TTz$.*

Definition 1.3. Let Φ be the family of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following:

(1) ϕ is lower semi continuous.

(2) $\phi(t) > 0$ for all $t > 0$ and $\varphi(0) = 0$.

(3) ϕ is discontinuous at $t = 0$.

2. Common Fixed Point Theorems

In this section, we study common fixed point theorems for compatible mappings of type (A-1).

Theorem 2.1. *Let (X, d) be a metric space and A, B, P, Q be four self maps on X satisfying the following*:

(1) *The pairs (A, P) and (B, Q) are compatible mappings of type* (A-1)

(2) $$\int_0^{d(Ax, By)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(By, Qy)[1+d(Ax, Px)]}{[1+d(Px, Qy)]}} \varphi(t)dt + \beta \int_0^{d(Px, Qy)} \varphi(t)dt, \tag{2.1}$$

where $\alpha, \beta > 0$ with $\alpha + \beta < 1$ and $\varphi : [0, +\infty[\to [0, +\infty[$ is a Lebesgue – integrable mapping which is summable on each compact subset of $[0, +\infty[$, nonnegative and such that

(3) $\int_0^{\varepsilon} \varphi(t)dt > 0, \forall \varepsilon > 0.$

If P and Q are continuous, then A, B, P and Q have a unique common fixed point in X.

Proof. Suppose that the pairs (A, P) and (B, Q) are compatible mappings of type (A-1)

Then $\lim_{n\to\infty} d(AAx_n, PPx_n) = 0$ and $\lim_{n\to\infty} d(BBy_n, QQy_n) = 0$

whenever there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z \text{ for some}$$

$$z \in PX \cap QX. \tag{2.2}$$

Since $z \in PX$, there exists a point $s \in X$ such that $Ps = z$ From (2.2), we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z = Ps \tag{2.3}$$

Now we claim that $As = z$. If not, then $d(As, z) > 0$. Putting $x = s$ and $y = y_n$ in (2.1), we get

$$\int_0^{d(As, By_n)} \varphi(t)dt \le \alpha \int_0^{\frac{d(By_n, Qy_n)[1+d(As, Ps)]}{[1+d(Ps, Qy_n)]}} \varphi(t)dt + \beta \int_0^{d(Ps, Qy_n)} \varphi(t)dt \tag{2.4}$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(As, z)} \varphi(t)dt \le \alpha \int_0^{\frac{d(z, z)[1+d(As, Ps)]}{[1+d(z, z)]}} \varphi(t)dt + \beta \int_0^{d(z, z)} \varphi(t)dt$$

$$\int_0^{d(As, z)} \varphi(t)dt = 0 \tag{2.5}$$

Which from (3) implies that $d(As, z) = 0$

Which contradicts the fact that $(As, z) > 0$, therefore

$$Ps = As = z. \tag{2.6}$$

Similarly, since $z \in QX$, so there exists a point $v \in X$ such that $Qv = z$. Thus (2.2) becomes

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Qy_n = z = Qv \tag{2.7}$$

Now we claim that $Bv = z$. If not then $d(Bv, z) > 0$. Then on putting $x = x_n$ and $y = v$ in (2.1), one can get

$$Bv = Qv = z. \tag{2.8}$$

Therefore, from (2.6) and (2.8), one can write

$$As = Ps = Bv = Qv = z. \tag{2.9}$$

Next, we show that z is a common fixed point of A, B, P, and Q. To this aim, since P and Q are continuous then by proposition 1.1

$\lim_{n\to\infty} AAx_n = Pz$ and $\lim_{n\to\infty} BBx_n = Qz$, then using (2.2) we have

$$Az = Pz, \tag{2.10}$$

and

$$Bz = Qz. \tag{2.11}$$

We will show next that $Az = z$. If not then $d(Az, z) > 0$. Putting $x = z$ and $y = v$ in (2.1), we get

$$\int_0^{d(Az, Bv)} \varphi(t)dt \le \alpha \int_0^{\frac{d(Bv, Qv)[1+d(Az, Pz)]}{[1+d(Pz, Qv)]}} \varphi(t)dt + \beta \int_0^{d(Pz, Qv)} \varphi(t)dt$$

$$\int_0^{d(Az, z)} \varphi(t)dt \le \alpha \int_0^{\frac{d(z, z)[1+d(Az, Pz)]}{[1+d(Pz, z)]}} \varphi(t)dt + \beta \int_0^{d(Az, z)} \varphi(t)dt$$

$$(1-\beta)\int_0^{d(Az, z)} \varphi(t)dt \le 0 \tag{2.12}$$

Which from (3) implies that $d(Az, z) \le 0$, a contradiction.

Hence $Az = z$. From (2.10), we can write

$$Az = Pz = z \tag{2.13}$$

Similarly, setting $x = u$, $y = z$ in 2.1 and using (2.9), (2.11), we can have

$$Bz = Qz = z. \tag{2.14}$$

Therefore from (2.13) and (2.14), it follows that

$$Az = Bz = Qz = Pz = z, \tag{2.15}$$

That is, z is a common fixed point of A, B, Q, and P.

Finally, we prove the uniqueness of the common fixed point of A, B, Q, and P. Assume that z_1 and z_2 are two distinct common fixed points of A, B, Q, and P. Then replacing x by z_1 and y by z_2 in (2.1), we have

$$\int_0^{d(Az_1,\, Bz_2)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(Bz_2,\, Qz_2)[1+d(Az_1,\, Pz_1)]}{[1+d(Pz_1,\, Qz_2)]}} \varphi(t)dt + \beta \int_0^{d(Pz_1,\, Qz_2)} \varphi(t)dt$$

$$(1-\beta)\int_0^{d(Az_1,\, Bz_2)} \varphi(t)dt \leq 0 \tag{2.16}$$

Which from (3) implies that $d(Az_1,\, Bz_2) \leq 0$, a contradiction.

Hence $z_1 = z_2$. Therefore A, B, P and Q have a unique common fixed point in X.

From theorem 2.1, we can easily deduce the following corollaries.

Corollary 2.1. *Let* $(X,\, d)$ *be a metric space and* A, P, Q *be three self maps on* X *satisfying the following*:

(1) *The pairs* $(A,\, P)$ *and* $(A,\, Q)$ *are compatible mappings of type* (A-1)

(2) $$\int_0^{d(Ax,\, Ay)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(Ay,\, Qy)[1+d(Ax,\, Px)]}{[1+d(Px,\, Qy)]}} \varphi(t)dt + \beta \int_0^{d(Px,\, Qy)} \varphi(t)dt,$$

where $\alpha, \beta > 0$ *with* $\alpha + \beta < 1$ *and* $\varphi : [0, +\infty[\to [0, +\infty[$ *is a Lebesgue integrable mapping which is summable on each compact subset of* $[0, +\infty[$, *nonnegative and such that*

(3) $\int_0^{\varepsilon} \varphi(t)dt > 0,\ \forall \varepsilon > 0.$

If P *and* Q *are continuous, then* A, P *and* Q *have a unique common fixed point in* X.

Corollary 2.2. *Let* $(X,\, d)$ *be a metric space and* A, Q *be two self maps on* X *satisfying the following*:

(1) *The mappings* A *and* Q *are compatible of type* (A-1);

(2) $$\int_0^{d(Ax,\, Ay)} \varphi(t)dt \leq \alpha \int_0^{\frac{d(Ay,\, Qy)[1+d(Ax,\, Ax)]}{[1+d(Ax,\, Qy)]}} \varphi(t)dt + \beta \int_0^{d(Ax,\, Qy)} \varphi(t)dt,$$

where $\alpha, \beta > 0$ *with* $\alpha + \beta < 1$ *and* $\varphi : [0, +\infty[\to [0, +\infty[$ *is a Lebesgue-integrable mapping which is summable on each compact subset of* $[0, +\infty[$, *nonnegative and such that*

$$(3) \int_0^\varepsilon \varphi(t)dt > 0, \forall \varepsilon > 0.$$

If the mapping Q is continuous, then A and Q have a unique common fixed point in X.

References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leurs applications aux equations integrals, Fundam. Math. 3 (1922), 133-181.

[2] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 29(9) (2002), 531-536.

[3] A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl. 322 (2006), 796-802.

[4] I. Altun and D. Turkoglu, Some fixed point theorems for weakly compatible mappings satisfying an implicit relation, Taiwan. J. Math. 13 (2009), 1291-1304.

[5] Z. Liu, X. Zou, S. M. Kang and J. S. Ume, Common fixed points for a pair of mappings satisfying contractive conditions of integral type, J. Inequal. Appl. (2014), Article ID 394.

[6] PP. Murthy, S. Kumar and K. Tas, Common fixed points of self maps satisfying an integral type contractive condition in fuzzy metric spaces, Math. Commun. 15 (2010), 521-537.

[7] S. Gulyaz, E. Karapınar, V. Rakocevic and P. Salimi, Existence of a solution of integral equations via fixed point theorem, J. Inequal. Appl. 2013, Article ID 529.

[8] G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly 83 (1976), 261-263.

[9] G. Jungck, Compatible maps and common fixed points, Intern at. J. Math., Math. Sci. 9 (1986), 771-779.

[10] H. K. Pathak and M. S. Khan, J. Indian Pure Appl. Math. 28(4) (1997), 477-485.

[11] M. S. Khan, H. K. Pathak and George Reny, Int. Math. Forum 2(11) (2007), 515-524.