**Advances and Applications in Mathematical Sciences**
Volume 20, Issue 5, March 2021, Pages 807-814


# BOX PRODUCT ON INTERVAL-VALUED PYTHAGOREAN FUZZY GRAPH

**S. YAHYA MOHAMED and A. MOHAMED ALI**

PG and Research Department of Mathematics
Government Arts College
Affiliated to Bharathidasan University
Tiruchirappalli, Tamil Nadu, India
E-mail: yahya_md@yahoo.co.in

Department of Mathematics
G.T.N. Arts College (Autonomous)
Dindigul, Tamil Nadu, India
E-mail: mohamedali1128@gmail.com

**Abstract**

An interval-valued Pythagorean fuzzy graph (IVPFG) can be obtained from two given interval-valued Pythagorean fuzzy graphs (IVPFGs) using box-product. In this paper, the box-product of two IVPFGs is introduced. Also we find the degree of a vertex in IVPFGs obtained by box-product of two IVPFGs. Some proposition related to the above concepts are stated and proved.

## 1. Introduction

The concept of Pythagorean fuzzy sets (PFSs) was introduced by Yager [8] is an effective tool for handling the uncertain information more adequately in real-world situations. In PFSs, the sum of squares of the membership grade and non-membership grade is less than or equal to one. For example, if a decision maker gives the degree of membership 0.7 and degree of non-membership degree 0.8 in his evaluation, then this environment cannot be managed by intuitionistic fuzzy set theory [1] because of $0.7 + 0.8 > 1$.

Naz et al. [3] introduced the notion of Pythagorean fuzzy graphs (PFGs),

2010 Mathematics Subject Classification: 05C72, 03E72, 03F55.
Keywords: box-product, degree, strong-IVPFG, complete-IVPFG.
Received January 8, 2020; Accepted May 28, 2020

as an extended version of intuitionistic fuzzy graphs. Peng and Yang [4] initiated the concept of interval-valued Pythagorean fuzzy sets (IVPFSs) as a generalization of Pythagorean fuzzy sets in the spirit of interval-valued fuzzy sets [38]. S. Dogra [2] defined different types of operations on fuzzy graphs. Yahya and Ali [5, 6, 7] introduced the concept of IVPFG and they studied some operations on graphs in IVPF environment. In this paper, the complete IVPFGs are introduced. In this paper, the box-product of two IVPFGs is introduced. Also we find the degree of a vertex in IVPFGs obtained by box-product of two IVPFGs. Some propositions related to the above concepts are studied.

## 2. Preliminaries

In this section, we recall some definitions and basic results of IVPFGs.

**Definition 2.1.** An interval-valued Pythagorean fuzzy graph with underlying set $V$ is defined to be a pair $\mathcal{G} = (P, Q)$ where

- the functions $\widetilde{M}_P : V \to D[0, 1]$ and $\widetilde{N}_P : V \to D[0, 1]$ denote the degree of membership and non-membership of the element $x \in V$, respectively, such that $0 \leq \widetilde{M}_P^2(x) + \widetilde{N}_P^2(x) \leq 1$ for all $x \in V$.

- the functions $\widetilde{M}_Q : E \subseteq V \times V \to D[0, 1]$ and $\widetilde{N}_Q : E \subseteq V \times V \to D[0, 1]$ are defined by $M_{QL}((x, y)) \leq (M_{PL}(x) \wedge M_{PL}(y))$ and $N_{QL}((x, y)) \geq (N_{PL}(x) \vee N_{PL}(y))$, $M_{QU}((x, y)) \leq (M_{PU}(x) \wedge M_{PU}(y))$ and $N_{QU}((x, y)) \geq (N_{PU}(x) \vee N_{PU}(y))$ such that $0 \leq M_{QU}^2((x, y)) + N_{QU}^2((x, y)) \leq 1, \forall (x, y) \in E$.

**Definition 2.2.** An interval-valued Pythagorean fuzzy graph $\mathcal{G} = (P, Q)$ is called strong if $M_{QL}((x, y)) = M_{PL}(x) \wedge M_{PL}(y)$ and $N_{QL}((x, y)) = N_{PL}(x) \vee N_{PL}(y), M_{QU}((x, y)) = M_{PU}(x) \wedge M_{PU}(y)$ and $N_{QU}((x, y)) = N_{PU}(x) \vee N_{PU}(y), \forall (x, y) \in E$.

**Definition 2.3.** An interval-valued Pythagorean fuzzy graph $\mathcal{G} = (P, Q)$ is called complete if

$$M_{QL}((x, y)) = M_{PL}(x) \wedge M_{PL}(y) \text{ and } N_{QL}((x, y)) = N_{PL}(x) \vee N_{PL}(y)$$

$M_{QU}((x, y)) = M_{PU}(x) \wedge M_{PU}(y)$ and $N_{QU}((x, y)) = N_{PU}(x) \vee N_{PU}(y)$, $\forall x, y \in V$.

**Example 2.4.** Consider the graph $\mathcal{G} = (V, E)$ such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let $P$ be an interval valued Pythagorean fuzzy set of $V$ and let $Q$ be an interval-valued fuzzy set of $E \subseteq V \times V$ defined by

$$P = \{\langle x, [0.5, 0.7], [0.1, 0.6]\rangle, \langle y, [0.6, 0.7], [0.1, 0.5]\rangle, \langle z, [0.4, 0.8], [0.2, 0.5]\rangle\},$$

$$Q = \{\langle xy, [0.5, 0.7], [0.1, 0.6]\rangle, \langle yz, [0.4, 0.6], [0.2, 0.5]\rangle, \langle xz, [0.4, 0.6], [0.2, 0.6]\rangle\}.$$

## 3. Box-Product of Interval-Valued Pythagorean Fuzzy Graphs

**Definition 3.1.** Let $P_1$ and $P_2$ be IVPFSs of $V_1$ and $V_2$ and let $Q_1$ and $Q_2$ be IVPFSs of $E_1$ and $E_2$ respectively and we assume that $V_1 \cap V_2 = \varnothing$. The box-product of two IVPFGs $\mathcal{G}_1$ and $\mathcal{G}_2$ of the graphs $G_1^*$ and $G_2^*$ is denoted by $\mathcal{G}_1 \boxtimes \mathcal{G}_2 : (P_1 \boxtimes P_2, Q_1 \boxtimes Q_2)$ with crisp graph $G^* : (V_1 \times V_2, E)$ where $E = \{((x_1, y_1)(x_2, y_2)) : x_1 = x_2, (y_1, y_2) \notin E_2$ or $(x_1, x_2) \in E_1, (y_1, y_2) \notin E_2\}$ and is defined as follows:

(1) $(M_{P_1L} \boxtimes M_{P_2L})(x, y) = M_{P_1L}(x) \wedge M_{P_2L}(y)$

$(M_{P_1U} \boxtimes M_{P_2U})(x, y) = M_{P_1U}(x) \wedge M_{P_2U}(y)$

$(N_{P_1L} \boxtimes N_{P_2L})(x, y) = N_{P_1L}(x) \vee N_{P_2L}(y)$

$(N_{P_1U} \boxtimes N_{P_2U})(x, y) = N_{P_1U}(x) \vee N_{P_2U}(y), \forall (x, y) \in V_1 \times V_2$.

(2) $(M_{Q_1L} \boxtimes M_{Q_2L})((x_1, y_1), (x_2, y_2)) = M_{P_1L}(x_1) \wedge M_{P_2L}(y_1) \wedge M_{P_2L}(y_2)$

$(M_{Q_1U} \boxtimes M_{Q_2U})((x_1, y_1), (x_2, y_2)) = M_{P_1U}(x_1) \wedge M_{P_2U}(y_1) \wedge M_{P_2U}(y_2)$

$(N_{Q_1L} \boxtimes N_{Q_2L})((x_1, y_1), (x_2, y_2)) = N_{P_1L}(x_1) \vee N_{P_2L}(y_1) \vee N_{P_2L}(y_2)$

$(N_{Q_1U} \boxtimes N_{Q_2U})((x_1, y_1), (x_2, y_2)) = N_{P_1U}(x_1) \vee N_{P_2U}(y_1) \vee N_{P_2U}(y_2)$

$\forall x_1 = x_2 \in V_1, (y_1, y_2) \notin E_2$.

(3) $(M_{Q_1L} \boxtimes M_{Q_2L})((x_1, y_1), (x_2, y_2)) = M_{Q_1L}(x_1 x_2) \wedge M_{P_2L}(y_1) \wedge M_{P_2L}(y_2)$

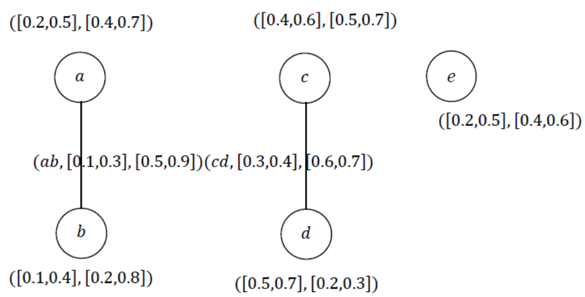
$$(M_{Q_1U} \boxtimes M_{Q_2U})((x_1, y_1), (x_2, y_2)) = M_{Q_1U}(x_1x_2) \wedge M_{Q_2U}(y_1) \wedge M_{P_2U}(y_2)$$

$$(N_{Q_1L} \boxtimes N_{Q_2L})((x_1, y_1), (x_2, y_2)) = N_{Q_1L}(x_1x_2) \vee N_{Q_2L}(y_1) \vee N_{Q_2L}(y_2)$$

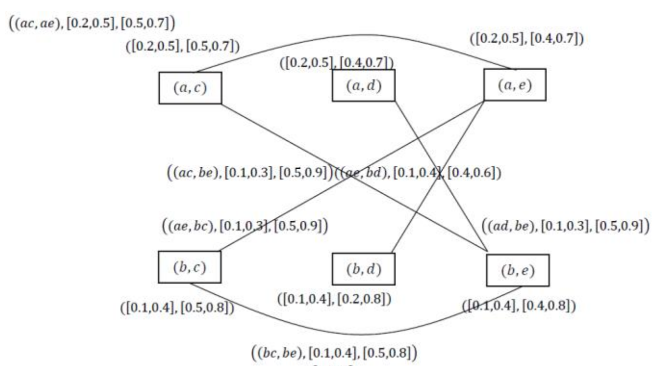
$$(N_{Q_1U} \boxtimes N_{Q_2U})((x_1, y_1), (x_2, y_2)) = N_{Q_1U}(x_1x_2) \vee N_{Q_2U}(y_1) \vee N_{Q_2U}(y_2)$$

$\forall x_1x_2 \in E_1, (y_1, y_2) \notin E_2.$

**Example 3.2.** Consider the two IVPFGs $\mathcal{G}_1 = (P_1, Q_1)$ and $\mathcal{G}_2 = (P_2, Q_2)$ on $V_1 = \{a, b\}$ and $V_2 = \{c, d\}$ respectively




$\mathcal{G}_1$ $\mathcal{G}_2$




$\mathcal{G}_1 \boxtimes \mathcal{G}_2$

**Proposition 3.3.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ are two IVPFGs, then box-product $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ is also an IVPFG.*

**Proof.** Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be IVPFGs.

Let $E = \{(x_1, x_2)(y_1, y_2)/x_1 = y_1, x_2y_2 \notin E_2\} \cup \{(x_1, x_2)(y_1, y_2)/x_1y_1 \in E_1, x_2y_2 \notin E_2\}$. For all $(x, y) \in V_1 \times V_2$,

$(M_{P_1L} \boxtimes M_{P_2L})(x, y) \leq M_{P_1L}(x) \wedge M_{P_2L}(y)$

$(M_{P_1U} \boxtimes M_{P_2U})(x, y) \leq M_{P_1U}(x) \wedge M_{P_2U}(y),$

$(N_{P_1L} \boxtimes N_{P_2L})(x, y) \geq N_{P_1L}(x) \vee N_{P_2L}(y),$

$(N_{P_1U} \boxtimes N_{P_2U})(x, y) \geq N_{P_1U}(x) \vee N_{P_2U}(y).$

If $x_1 = y_1, x_2 y_2 \notin E_2$. Then

$$(M_{Q_1L} \boxtimes M_{Q_2L})((x_1, x_2)(y_1, y_2)) = (M_{P_1L}(x_1) \wedge M_{P_2L}(x_2) \wedge M_{P_2L}(y_2))$$

$$= (M_{P_1L}(x_1) \wedge M_{P_1L}(y_1) \wedge M_{P_2L}(x_2) \wedge M_{P_2L}(y_2))$$

$$= (M_{P_1L}(x_1) \wedge M_{P_1L}(x_2) \wedge M_{P_2L}(y_1) \wedge M_{P_2L}(y_2))$$

$$= (M_{P_1L}(x_1) \wedge M_{P_1L}(x_2)) \wedge (M_{P_2L}(y_1) \wedge M_{P_2L}(y_2))$$

$$= (M_{Q_1L} \boxtimes M_{Q_2L})(x_1, x_2) \wedge (M_{Q_1L} \boxtimes M_{Q_2L})(y_1, y_2)$$

$$(M_{Q_1U} \boxtimes M_{Q_2U})((x_1, x_2)(y_1, y_2)) = (M_{P_1U}(x_1) \wedge M_{P_2U}(x_2) \wedge M_{P_2U}(y_2))$$

$$= (M_{P_1U}(x_1) \wedge M_{P_1U}(y_1) \wedge M_{P_2U}(x_1) \wedge M_{P_2U}(y_2))$$

$$= (M_{P_1U}(x_1) \wedge M_{P_1U}(x_2) \wedge M_{P_2U}(y_1) \wedge M_{P_2U}(y_2))$$

$$= (M_{P_1U}(x_1) \wedge M_{P_1U}(x_2)) \wedge (M_{P_2U}(y_1) \wedge M_{P_2U}(y_2))$$

$$= (M_{Q_1U} \boxtimes M_{Q_2U})(x_1, x_2) \wedge (M_{Q_1U} \boxtimes M_{Q_2U})(y_1, y_2).$$

Similarly, $(N_{Q_1L} \boxtimes N_{Q_2L})((x_1, x_2)(y_1, y_2)) = (N_{Q_1L} \boxtimes N_{Q_2L})(x_1, x_2) \vee (N_{Q_1L} \boxtimes N_{Q_2L})(y_1, y_2)$

$(N_{Q_1U} \boxtimes N_{Q_2U})((x_1, x_2)(y_1, y_2)) = (N_{Q_1U} \boxtimes N_{Q_2U})(x_1, x_2) \vee (N_{Q_1U} \boxtimes N_{Q_2U})(y_1, y_2)$. If $x_1 y_1 \in E_1, x_2 y_2 \notin E_2$. Then

$$(M_{Q_1L} \boxtimes M_{Q_2L})((x_1, x_2)(y_1, y_2)) = (M_{P_1L}(x_1 y_1) \wedge M_{P_2L}(x_2) \wedge M_{P_2L}(y_2))$$

$$\leq (M_{P_1L}(x_1) \wedge M_{P_1L}(y_1) \wedge M_{P_2L}(x_2) \wedge M_{P_2L}(y_2))$$

$$= (M_{P_1L}(x_1) \wedge M_{P_1L}(x_2) \wedge M_{P_2L}(y_1) \wedge M_{P_2L}(y_2))$$

$$= (M_{P_1L}(x_1) \wedge M_{P_1L}(x_2)) \wedge (M_{P_2L}(y_1) \wedge M_{P_2L}(y_2))$$

$$= (M_{Q_1L} \boxtimes M_{Q_2L})(x_1, x_2) \wedge (M_{Q_1L} \boxtimes M_{Q_2L})(y_1, y_2)$$

$$(M_{Q_1U} \boxtimes M_{Q_2U})((x_1, x_2)(y_1, y_2)) = (M_{P_1U}(x_1y_1) \wedge M_{P_2U}(x_2) \wedge M_{P_2U}(y_2))$$

$$\leq (M_{P_1U}(x_1) \wedge M_{P_1U}(y_1) \wedge M_{P_2U}(x_2) \wedge M_{P_2U}(y_2))$$

$$= (M_{P_1U}(x_1) \wedge M_{P_1U}(x_2) \wedge M_{P_2U}(y_1) \wedge M_{P_2U}(y_2))$$

$$= (M_{P_1U}(x_1) \wedge M_{P_1U}(x_2)) \wedge (M_{P_2U}(y_1) \wedge M_{P_2U}(y_2))$$

$$= (M_{Q_1U} \boxtimes M_{Q_2U})(x_1, x_2) \wedge (M_{Q_1U} \boxtimes M_{Q_2U})(y_1, y_2).$$

Similarly,

$$(N_{Q_1L} \boxtimes N_{Q_2L})((x_1, x_2)(y_1, y_2)) \geq (N_{Q_1L} \boxtimes N_{Q_2L})(x_1, x_2)$$

$$\vee (N_{Q_1L} \boxtimes N_{Q_2L})(y_1, y_2)$$

$$(N_{Q_1U} \boxtimes N_{Q_2U})((x_1, x_2)(y_1, y_2)) = (N_{Q_1U} \boxtimes N_{Q_2U})(x_1, x_2)$$

$$\vee (N_{Q_1U} \boxtimes N_{Q_2U})(y_1, y_2).$$

Hence $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ is an IVPFG.

**Theorem 3.4.** *Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are two strong-IVPFGs*, *then box-product* $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ *is a strong-IVPFG.*

**Proof.** This proof is similar to Theorem 3.3.

**Observation 3.5.**

1. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected graphs then $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ is need not be connected.

2. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are complete graphs then $\mathcal{G}_1 \boxtimes \mathcal{G}_2$ is need not be complete.

## 4. Degree of a Vertex in Box-Product of Interval-Valued Pythagorean Fuzzy Graphs

For any vertex $(x_1, x_2) \in V_1 \times V_2$.

The degree of a vertex in $\mathcal{G}_1 \times_m \mathcal{G}_2$ is defined as

$$d^{G_1 \boxtimes G_2}(x_1, x_2) = ((d_{M_L}^{G_1 \boxtimes G_2}(x_1, x_2), d_{M_U}^{G_1 \boxtimes G_2}(x_1, x_2)),$$

$$(d_{N_L}^{G_1 \boxtimes G_2}(x_1, x_2), d_{N_U}^{G_1 \boxtimes G_2}(x_1, x_2)),$$

where

$$d_{M_L}^{G_1 \boxtimes G_2}(x_1, x_2) = \sum_{(x_1, x_2)(y_1, y_2) \in E_1 \times E_2} M_{Q_L}^{G_1 \boxtimes G_2}((x_1, x_2)(y_1, y_2))$$

$$= \sum_{x_1 = y_1,\, x_2 y_2 \notin E_2} M_{P_1 L}(x_1) \wedge M_{P_2 L}(x_2) \wedge M_{P_2 L}(y_2)$$

$$+ \sum_{x_1 y_1 \in E_1,\, x_2 y_2 \notin E_2} M_{Q_1 L}(x_1 y_1) \wedge M_{P_2 L}(x_2) \wedge M_{P_2 L}(y_2)$$

$$d_{M_U}^{G_1 \boxtimes G_2}(x_1, x_2) = \sum_{(x_1, x_2)(y_1, y_2) \in E_1 \times E_2} M_{Q_U}^{G_1 \boxtimes G_2}((x_1, x_2)(y_1, y_2))$$

$$= \sum_{x_1 = y_1 \in E_1,\, x_2 y_2 \notin E_2} M_{P_1 U}(x_1) \wedge M_{P_2 U}(x_2) \wedge M_{P_2 U}(y_2)$$

$$+ \sum_{x_1 y_1 \in E_1,\, x_2 y_2 \notin E_2} M_{Q_1 U}(x_1 y_1) \wedge M_{P_2 U}(x_2) \wedge M_{P_2 U}(y_2)$$

$$d_{N_L}^{G_1 \boxtimes G_2}(x_1, x_2) = \sum_{(x_1, x_2)(y_1, y_2) \in E_1 \times E_2} N_{Q_L}^{G_1 \boxtimes G_2}((x_1, x_2)(y_1, y_2))$$

$$= \sum_{x_1 = y_1 \in E_1,\, x_2 y_2 \notin E_2} N_{P_1 L}(x_1) \vee N_{P_2 L}(x_2) \vee N_{P_2 L}(y_2)$$

$$+ \sum_{x_1 y_1 \in E_1,\, x_2 y_2 \notin E_2} N_{Q_1 L}(x_1 y_1) \vee N_{P_2 L}(x_2) \vee N_{P_2 L}(y_2)$$

$$d_{N_U}^{G_1 \boxtimes G_2}(x_1, x_2) = \sum_{(x_1, x_2)(y_1, y_2) \in E_1 \times E_2} N_{Q_U}^{G_1 \boxtimes G_2}((x_1, x_2)(y_1, y_2))$$

$$= \sum_{x_1 = y_1,\, x_2 y_2 \notin E_2} N_{P_1 U}(x_1) \vee N_{P_2 U}(x_2) \vee N_{P_2 U}(y_2)$$

$$+ \sum_{x_1 y_1 \in E_1,\, x_2 y_2 \notin E_2} N_{Q_1 U}(x_1 y_1) \vee N_{P_2 U}(x_2) \vee N_{P_2 U}(y_2)$$

## Conclusion

IVPFSs are generalization of PFGs and provide a sufficient space for complex decision making situations. In this paper, box product of two IVPFGs is studied. Also, we proved the box-product of two strong interval-valued Pythagorean fuzzy graphs is strong.

## References


[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87-96.

[2] S. Dogra, Different types of products of fuzzy graphs, Progress in Nonlinear Dynamics and Chaos 3(1) (2015), 41-56.

[3] S. Naz, S. Ashraf and M. Akram, A novel approach to decision making with Pythagorean fuzzy information, Mathematics 6 (2018), 1-28.

[4] X. Peng and Y. Yang, Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators, International Journal of Intelligent Systems 31(5) (2016), 444-487.

[5] S. Yahya Mohamed and A. Mohamed Ali, Interval-valued Pythagorean fuzzy graph, Journal of Computer and Mathematical Sciences 9(10) (2018), 1497-1511.

[6] S. Yahya Mohamed and A. Mohamed Ali, Strong Interval-valued Pythagorean fuzzy graph, Journal of Applied Science and Computations 5(10) (2018), 669-713.

[7] S. Yahya Mohamed and A. Mohamed Ali, Some products on interval-valued Pythagorean fuzzy graphs, Malaya Journal of Matematik 7(3) (2019), 566-571.

[8] R. R. Yager, Pythagorean fuzzy subsets, In: Proceedings of Joint IFSA world congress and NAFIPS annual meeting, Edmonton (2013), 57-61.