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CORDIAL LABELING ON THE VERTEX SWITCHING OF JEWEL GRAPH

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Abstract

The Cordial labeling of a graph G is a function $f : V(G) \to \{0, 1\}$ such that every edge uv in G is assigned the label $|f(u) - f(v)|$ with the property $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1,$ where $v_f(i)$ is the number of vertices with label i for $i = 0, 1$ and $e_f(i)$ is the number of edges with label i for $i = 0, 1.$ The graph which satisfies the condition of cordial labeling is called the cordial graph. In this paper, we prove that the vertex switching of jewel graph is cordial, path union of vertex switching of jewel graph is cordial and cycle of vertex switching of jewel graph is cordial.

Introduction

The field of Graph Labeling in Graph Theory is an important research area dealing with non-negative integers assigned to the vertices or edges or both of a graph under some conditions. Cahit [2] introduced cordial labeling

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in 1987. Numerous graphs are shown to be cordial. Cahit [3] has proved that all trees, fans, wheels W_n, for $n \not\equiv 3(\bmod 4)$, complete graphs K_n, for $n \leq 3$, bipartite graphs $K_{m,n}$, friendship graph $C_3^{(t)}$, for $t \not\equiv 2(\bmod 4)$ are cordial. Helms, closed helms, flowers graphs, sunflower graphs are proved as cordial by Andar et al. [1] and they have shown that the one point union of these graphs are also cordial. Rokad and Patadiya [5] proved that the jewel graph is cordial. An extensive survey of cordial labeling is available in Gallian [4]. In this paper, we prove that vertex switching of jewel graph is cordial, the path union of vertex switching of jewel graphs is cordial and cycle of vertex switching of jewel graphs is cordial.

Main Results

First, we define jewel graph, vertex switching of a graph, path union of graphs and cycle of graphs and then we discuss our results.

Definition 1. The *Jewel graph* J_n is the graph with the vertex set $V(J_n) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and the edge set $E(J_n) = \{ux, uy, xy, xv, yu, uu_i, vu_i : 1 \leq i \leq n\}$.

Definition 2. A *vertex switching* G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.

Definition 3. Let $G_1, G_2, \ldots, G_n (n \geq 2)$ be finite graphs. The new graph obtained by adding an edge between a vertex of G_i and a vertex of G_{i+1}, for $i = 1, 2, \ldots, (n-1)$ is called a *path union* of $G_1, G_2, \ldots, G_n$.

Definition 4. Let $G_1, G_2, \ldots, G_n$ be given connected graphs. Then the *cycle of graphs* $C(G_1, G_2, \ldots, G_n)$ is the graph obtained by adding an edge joining G_i to G_{i+1}, for $i = 1, 2, \ldots, (n-1)$ and an edge joining G_n to G_1. When the n graphs are isomorphic to G then it is denoted as $C(n.G)$.

Theorem 1. *The vertex switching of Jewel graph is cordial.*

Proof of Theorem 1. Let $V(J_n) = \{u_1, u_2, u_3, u_4, v_i : 1 \le i \le n\}$ be the vertex set of the jewel graph j_n and $E(J_n) = \{u_1, u_2, u_1, u_4, u_2, u_3, u_4, u_1, u_3, u_2v_i, u_4v_i : 1 \le i \le n\}$ be the edge set as shown in Figure 1.

Let us consider a vertex u_1 as a switching vertex in the Jewel graph J_n. Remove all the edges namely u_1u_2, u_1u_4, u_1u_3 which are incident with u_1 and make u_1 to be adjacent with all the vertices which were not initially adjacent to it. The resultant graph is termed as J'_n, the vertex switching of jewel graph. The vertex set of J'_n, is denoted as $V(J'_n) = \{u_1, u_2, u_3, u_4, v_i : 1 \le i \le n\}$ and $E(J'_n) = \{u_2u_3, u_3u_4, u_1v_i : 1 \le i \le n\}$ is the edge set of J'_n, which is described in Figure 1.

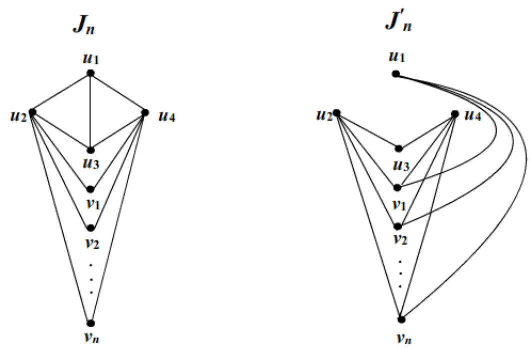

Figure 1. Jewel graph J_n and its vertex switching graph J'_n.

The number of vertices in J'_n is denoted as $p = (n + 4)$ and the number of edges in J'_n is denoted as $q = (3n + 2)$

The vertices are labeled as follows

For $1 \le i \le n$

$$f(v_i) = \begin{cases} 1, & for \quad i \equiv 1 (\text{mod } 2) \\ 0, & for \quad i \equiv 0 (\text{mod } 2) \end{cases}$$

$$f(u_1) = f(u_4) = 1$$

$$f(u_2) = f(u_3) = 0$$

From the above labelings we get

Case (i) When n is even

$$v_f(0) = \frac{p}{2},\ v_f(1) = \frac{p}{2} \text{ and } e_f(0) = \frac{q}{2},\ e_f(1) = \frac{q}{2}$$

$$|v_f(0) - v_f(1)| = \left|\frac{p}{2} - \frac{p}{2}\right| = 0 \text{ and } |e_f(0) - e_f(1)| = \left|\frac{q}{2} - \frac{q}{2}\right| = 0$$

Case (ii) When n is odd

$$v_f(0) = \left\lceil\frac{p}{2}\right\rceil,\ v_f(1) = \left\lfloor\frac{p}{2}\right\rfloor \text{ and } e_f(0) = \left\lceil\frac{q}{2}\right\rceil,\ e_f(1) = \left\lfloor\frac{q}{2}\right\rfloor$$

$$|v_f(0) - v_f(1)| = \left|\left\lceil\frac{p}{2}\right\rceil - \left\lfloor\frac{p}{2}\right\rfloor\right| = 1 \text{ and } |e_f(0) - e_f(1)| = \left|\left\lceil\frac{q}{2}\right\rceil - \left\lfloor\frac{q}{2}\right\rfloor\right| = 1$$

It is clear from the above two cases that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1.$ Hence the vertex switching of jewel graph is proved to be cordial. This is explained below in Figure 2.

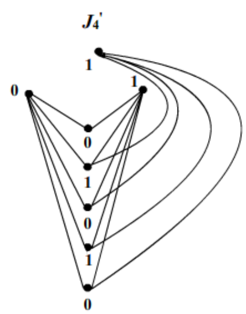

Figure 2. The graph J_4'.

Theorem 2. *Path union of vertex switching of Jewel graph is cordial.*

Proof of Theorem 2. Let us consider m copies of the vertex switching of the Jewel graph J_n' as described in Theorem 1. Denote them as $H_1, H_2, \ldots, H_m$. Let the switching vertex in H_1 be u_1^1. Let $u_2^1, u_3^1, u_4^1, v_1^1, v_2^1, \ldots, v_n^1$ be the remaining vertices of H_1. The switching vertex of H_2 is denoted as u_1^2 and the remaining vertices of H_2 are denoted as $u_2^2, u_3^2, u_4^2, v_1^2, v_2^2, \ldots, v_n^2$. Finally, the switching vertex of the m^{th} copy

H_m is denoted as u_1^m and the remaining vertices of H_m are denoted as $u_2^m, u_3^m, u_4^m, v_1^m, v_2^m, \ldots, v_n^m$. In general, the vertices of H_i are denoted as u_i^j for $(1 \leq i \leq 4), (1 \leq j \leq m)$ and v_i^j for $(1 \leq i \leq n), (1 \leq j \leq m)$.

The switching vertex of each copy is connected by an edge and this forms a path union which is denoted as PJ'_n and shown below in Figure 3.

Let $p = m(n+4)$ denote the number of vertices in graph PJ'_n and $q = 3m(n+1)-1$ denote the number of edges of pJ'_n.

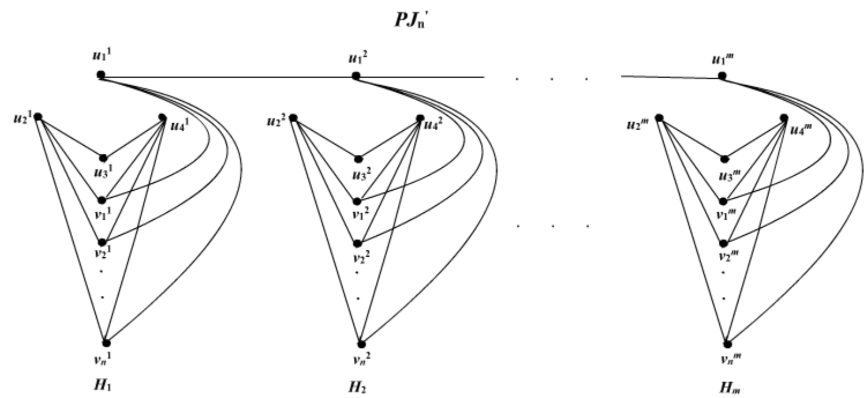

Figure 3. PJ'_n, path union of vertex switching of jewel graph.

The vertices of PJ'_n are labeled as follows

Case (i) When $n \equiv 0 (\bmod 2)$

For $1 \leq j \leq m$ and $1 \leq i \leq n$

$$f(u_1^j) = f(u_4^j) = \begin{cases} 1 & \text{for} \quad j \equiv 1, 2 (\bmod 4) \\ 0 & \text{for} \quad j \equiv 0, 3 (\bmod 4) \end{cases}$$

$$f(u_2^j) = f(u_3^j) = \begin{cases} 0 & \text{for} \quad j \equiv 1, 2 (\bmod 4) \\ 1 & \text{for} \quad j \equiv 0, 3 (\bmod 4) \end{cases}$$

$$f(v_i^j) = \begin{cases} 1 & \text{for} \quad j \equiv 1 (\bmod 2) \\ 0 & \text{for} \quad j \equiv 0 (\bmod 2) \end{cases}$$

From the above labelings we get

Case (a) When m is odd

$$v_f(0) = \frac{p}{2},\ v_f(1) = \frac{p}{2} \text{ and } e_f(0) = \frac{q}{2},\ e_f(1) = \frac{q}{2}$$

$$|\, v_f(0) - v_f(1) \,| = \left| \frac{p}{2} - \frac{p}{2} \right| = 0 \text{ and } |\, e_f(0) - e_f(1) \,| = \left| \frac{q}{2} - \frac{q}{2} \right| = 0$$

It is clear that $|\, v_f(0) - v_f(1) \,| \leq 1$ and $|\, e_f(0) - e_f(1) \,| \leq 1$.

Case (b) When m is even

$$v_f(0) = \frac{p}{2},\ v_f(1) = \frac{p}{2} \text{ and } e_f(0) = \left\lceil \frac{q}{2} \right\rceil,\ e_f(1) = \left\lfloor \frac{q}{2} \right\rfloor$$

$$|\, v_f(0) - v_f(1) \,| = \left| \frac{p}{2} - \frac{p}{2} \right| = 0 \text{ and } \left| e_f(0) - e_f(1) \right| = \left| \left\lceil \frac{q}{2} \right\rceil - \left\lfloor \frac{q}{2} \right\rfloor \right| = 1$$

It is clear that $|\, v_f(0) - v_f(1) \,| \leq 1$ and $|\, e_f(0) - e_f(1) \,| \leq 1$.

Case (ii) When $n \equiv 1 (\bmod 2)$

For $1 \leq j \leq m$ and $1 \leq j \leq n$

$$f(u_1^j) = f(u_4^j) = \begin{cases} 1 & for \quad j \equiv 1 (\bmod 2) \\ 0 & for \quad j \equiv 0 (\bmod 2) \end{cases}$$

$$f(u_2^j) = f(u_3^j) = \begin{cases} 0 & for \quad j \equiv 1 (\bmod 2) \\ 1 & for \quad j \equiv 0 (\bmod 2) \end{cases}$$

For $j \equiv 0 (\bmod 2)$

$$f(v_i^j) = \begin{cases} 1 & for \quad i \equiv 1 (\bmod 2) \\ 0 & for \quad i \equiv 0 (\bmod 2) \end{cases}$$

For $j \equiv 0 (\bmod 2)$

$$f(v_i^j) = \begin{cases} 0 & for \quad i \equiv 1 (\bmod 2) \\ 1 & for \quad i \equiv 0 (\bmod 2) \end{cases}$$

Case (a) When m is odd

$v_f(1) = \left\lceil \frac{p}{2} \right\rceil, v_f(0) = \left\lfloor \frac{p}{2} \right\rfloor$ and $e_f(0) = \left\lceil \frac{q}{2} \right\rceil, e_f(1) = \left\lfloor \frac{q}{2} \right\rfloor$

$| v_f(0) - v_f(1) | = \left| \left\lfloor \frac{p}{2} \right\rfloor - \left\lceil \frac{p}{2} \right\rceil \right| = 1$ and $| e_f(0) - e_f(1) | = \left| \left\lfloor \frac{q}{2} \right\rfloor - \left\lceil \frac{q}{2} \right\rceil \right| = 1$

It is clear that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.

Case (b) When m is even

$v_f(0) = \frac{p}{2}, v_f(1) = \frac{p}{2}$ and $e_f(0) = \left\lceil \frac{q}{2} \right\rceil, e_f(1) = \left\lfloor \frac{q}{2} \right\rfloor$

$| v_f(0) - v_f(1) | = \left| \frac{p}{2} - \frac{p}{2} \right| = 0$ and $| e_f(0) - e_f(1) | = \left| \left\lceil \frac{q}{2} \right\rceil - \left\lfloor \frac{q}{2} \right\rfloor \right| = 1$

It is clear that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.

Therefore, the graph PJ'_n is cordial for the above two cases which is illustrated below in Figure 4.

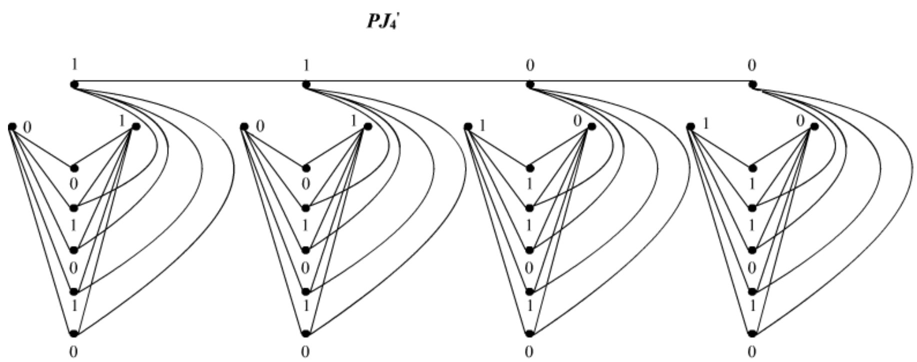

Figure 4. The graph PJ'_4.

Theorem 3. *Let G be the cycle C_m and J be the vertex switching of jewel graph. Then cycle of graphs $C(m \circ J)$ is cordial.*

Proof of Theorem 3. Let J be a vertex switching of jewel graph whose vertices are denoted as u_i, for $(1 \leq i \leq 4)$ and v_i, for $(1 \leq i \leq n)$ as described in Figure 1. Let $J_1, J_2, \ldots, J_m$ be m copies of the vertex switching of jewel graphs. Let $C(m \circ J)$ be the cycle of graphs that has been obtained by considering the cycle C_m whose vertices $k_1, k_2, \ldots, k_m$ considered in anticlockwise direction and replacing each vertex of C_m by the graphs $J_1, J_2, \ldots, J_m$ as shown in Figure 5. In other words, each vertex K_j, for $1 \leq j \leq m$ of the cycle C_m is identified with the switching vertex u_1 of J. The vertices in the first copy J_1 are denoted as $u_2^1, u_3^1, u_4^1, v_1^1, v_2^1, \ldots, v_n^1$. The vertices in the second copy J_2 are described as $u_2^2, u_3^2, u_4^2, v_1^2, v_2^2, \ldots, v_n^2$. Finally, $u_2^m, u_3^m, u_4^m, v_1^m, v_2^m, \ldots, v_n^m$ are vertices in the last copy (that is m^{th} copy) J_m. Thus the vertices in the j^{th} copy are represented as u_i^j for $(1 \leq i \leq 4), (1 \leq j \leq m)$ and u_i^j for $(1 \leq i \leq n), (1 \leq j \leq m)$.

Let $p = m(n+4)$ denote the number of vertices in $C(m \circ J)$ and $p = 3m(n+1)$ denote the number of edges in $C(m \circ J)$.

The vertices of $C(m \circ J)$ are labeled as follows

Case (i) When $n \equiv 0 (\operatorname{mod} 2)$

For $1 \leq j \leq m$ and $1 \leq i \leq n$

$$f(u_1^j) = f(u_4^j) = \begin{cases} 1 & \text{for} \quad j \equiv 1, 2 (\operatorname{mod} 4) \\ 0 & \text{for} \quad j \equiv 0, 3 (\operatorname{mod} 4) \end{cases}$$

$$f(u_2^j) = f(u_3^j) = \begin{cases} 0 & \text{for} \quad j \equiv 1, 2 (\operatorname{mod} 4) \\ 1 & \text{for} \quad j \equiv 0, 3 (\operatorname{mod} 4) \end{cases}$$

$$f(v_i^j) = \begin{cases} 0 & \text{for} \quad i \equiv 1 (\operatorname{mod} 2) \\ 1 & \text{for} \quad i \equiv 0 (\operatorname{mod} 2) \end{cases}$$

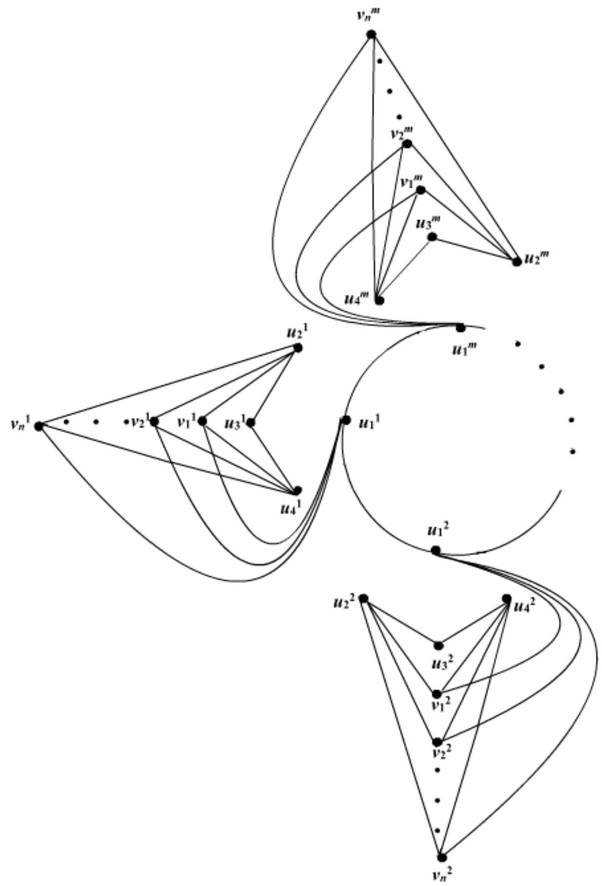

Figure 5. Cycle of vertex switching of jewel graph.

From the above labelings we get

Case (a) $m \equiv 0(\mathrm{mod}\, 4)$

$$v_f(0) = \frac{p}{2},\ v_f(1) = \frac{p}{2} \text{ and } e_f(0) = \frac{q}{2},\ e_f(1) = \frac{q}{2}$$

$$|\, v_f(0) - v_f(1)\,| = \left| \frac{p}{2} - \frac{p}{2} \right| = 0 \text{ and } |\, e_f(0) - e_f(1)\,| = \left| \frac{q}{2} - \frac{q}{2} \right| = 0$$

It is clear that $\left| v_f(0) - v_f(1) \right| \leq 1$ and $\left| e_f(0) - e_f(1) \right| \leq 1.$

Case (b) $m \equiv 1(\mathrm{mod}\, 4)$

$v_f(0) = \frac{p}{2}$, $v_f(1) = \frac{p}{2}$ and $e_f(0) = \left\lceil \frac{q}{2} \right\rceil$, $e_f(1) = \left\lfloor \frac{q}{2} \right\rfloor$

$| v_f(0) - v_f(1) | = \left| \frac{p}{2} - \frac{p}{2} \right| = 0$ and $| e_f(0) - e_f(1) | = \left| \left\lceil \frac{q}{2} \right\rceil - \left\lfloor \frac{q}{2} \right\rfloor \right| = 1$

It is clear that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.

Case (c) $m \equiv 3 \pmod 4$

$v_f(0) = \frac{p}{2}$, $v_f(1) = \frac{p}{2}$ and $e_f(0) = \left\lfloor \frac{q}{2} \right\rfloor$, $e_f(1) = \left\lceil \frac{q}{2} \right\rceil$

$| v_f(0) - v_f(1) | = \left| \frac{p}{2} - \frac{p}{2} \right| = 0$ and $| e_f(0) - e_f(1) | = \left| \left\lfloor \frac{q}{2} \right\rfloor - \left\lceil \frac{q}{2} \right\rceil \right| = 1$

It is clear that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.

Case (ii) When $n \equiv 1 \pmod 2$

For $1 \leq j \leq m$ and $1 \leq i \leq n$

$$f(u_1^j) = f(u_4^j) = \begin{cases} 1 & for \quad j \equiv 1 (\text{mod } 2) \\ 0 & for \quad j \equiv 0 (\text{mod } 2) \end{cases}$$

$$f(u_2^j) = f(u_3^j) = \begin{cases} 1 & for \quad j \equiv 1 (\text{mod } 2) \\ 0 & for \quad j \equiv 0 (\text{mod } 2) \end{cases}$$

For $j \equiv 1 \pmod 2$

$$f(v_i^j) = \begin{cases} 1 & for \quad i \equiv 1 (\text{mod } 2) \\ 0 & for \quad i \equiv 0 (\text{mod } 2) \end{cases}$$

For $j \equiv 0 \pmod 2$

$$f(v_i^j) = \begin{cases} 0 & for \quad i \equiv 1 (\text{mod } 2) \\ 1 & for \quad i \equiv 0 (\text{mod } 2) \end{cases}$$

From the above labelings we get

$v_f(0) = \frac{p}{2}$, $v_f(1) = \frac{p}{2}$ and $e_f(0) = \frac{q}{2}$, $e_f(1) = \frac{q}{2}$

$$|v_f(0) - v_f(1)| = \left|\frac{p}{2} - \frac{p}{2}\right| = 0 \text{ and } |e_f(0) - e_f(1)| = \left|\frac{q}{2} - \frac{q}{2}\right| = 0$$

It is clear that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Thus the cycle of vertex switching of jewel graphs $C(m \circ J)$ is proved to be cordial.

Conclusion

The vertex switching of jewel graph, its path union and cycle of vertex switching of jewel graphs are proved as cordial.

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