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GENERALIZATIONS OF THE CLASSICAL GALTON- WATSON BRANCHING PROCESSES IN VARYING RANDOM ENVIRONMENTS

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Abstract

In this paper, we study a generalization of the classical Galton-Watson branching process with immigration having offspring distribution in all generation. In each generation for which the population size is finite with probability p and e migration; probability q there is not any migration. In this critical case is investigated with an extension when the initial law is attracted to a stable (p) law, $p \leq 1$ or $> 1.$ The asymptotic form of the probability of non-extinction is studied and conditional limit theorems for the population size are obtained, depending on the range of an additional parameter of criticality. Finally we conclude that this paper will be very useful in the society specifically emphasize they can easily access the immigration process.

1. Introduction

A branching process in varying environment, also called time-inhomogeneous branching process is the generalization of the classical Galton-Watson process when the offspring distributions may vary according to the generations. Firstly, Athreya and Karlin (1971) derived several properties of such a process under general settings for the environmental process. Jagers (1974) showed that many of the limiting characteristics of the Galton-Watson process are retained by the varying environment process. Agrestic (1975) and Church (1971) hasstudied the limit behavior of these processes. In this study, we consider the Galton-Watson branching process with immigration having offspring distribution in the varying environments defined as follows: Section 2 is devoted to the definition of a Galton Watson

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Branching Processes with offspring distribution and Probability of Extinction. A condition for the extinction of a Galton Watson branching process is given in lemma 1, while the main condition for survival is given in Theorem has derived the main results in section 3. In section 4discussed classical Galton-Watson Branching process with immigration having off spring distribution and section 5 investigate the asymptotic behavior of the probability of non-extinction up to time n of critical multi-type Galton Watson Branching Processes and introduced some standard notations and definitions.

2. Basic Type of Galton Watson Branching Processes

Definition 1. Let the random variables $X_0, X_1, X_2, \ldots$ denote the size of (or the number of objects in) the $0^{th}, 1^{th}, 2^{nd}, \ldots$ generations respectively. Let the probability that object (irrespective of the generation to which it belongs) generators K similar objects be denoted by P_k, where $P_k \geq 0, k = 0, 1, 2, \ldots \sum p_k = 1$.

The sequence $\{X_n, n = 0, 1, 2, \ldots,\}$ constitute a Galton Watson branching process (or simply a G. W. branching process) with offspring distribution $\{p_K\}$ Our interest lies mainly in the probability distribution of X_n and the probability that $X_n \to 0$ for some n, (i.e.) the probability of ultimate extinction of the family.

Note 1. We shall assume that $X_0 = 1$ (i.e.) the process starts with a single ancestor.

Note 2. The sequence $\{X_n\}$ forms a Markov chain with transition probabilities

$$p_{ij} = P_r\{X_{n+1} = j/X_n = i\}, j = 0, 1, 2, \ldots$$

Note 3. The generating functions prove very useful in the study of branching process.

Definition 2. Probability of Extinction By extinction of the process it is meant that the random sequence $\{X_n\}$ consists of zeros for all except a finite number of values of n. In other words, extinction occurs when $P_r\{X_n = 0\}$, for some value of n, clearly, if $X_n = 0$ for $n = m$, then $X_n = 0$ for $n \geq m$, also $P_r\{X_{n+1} = 0/n = 0\} = 1$.

Definition 3. Conditional limit laws consider a critical (i.e. with $m = 1$) G. W. process. The probability of extinction is 1. Then $P_r\{X_n \to 0\} = 1$ and we also have $Var(X_n) = n\sigma^2 \to \infty$. The distribution of $X_n > 0$ is of considerable interest.

3. Some Main Results of G. W. Branching Processes

Lemma 1. *For a G. W. process with* $m = 1$ *and* $\sigma^2 < \infty$. *Then we have*

$$\lim_{n\to\infty} \frac{1}{n}\left\{\frac{1}{1 - P_n(x)} - \frac{1}{1 - s}\right\} \to \frac{\sigma^2}{2}, \ldots \tag{1}$$

uniformly in $0 \leq s \leq 1$.

Proof. Let $0 \leq s \leq 1$ and $P''(1) < \infty$. Using Taytor's expansion of $P(s)$ in the neighborhood of 1, we get

$$P(s) = s + \frac{\sigma^2}{2}(1 - s)^2 + r(s)(1 - s)^2, \ldots \tag{2}$$

where $r(s) \to 0$ as $s \to 1$. Thus

$$\frac{1}{1 - P(s)} - \frac{1}{1 - s} = \frac{P(s) - s}{(1 - t)(P(s) - s)}$$

$$= \frac{1 - s}{1 - P(s)} - \left\{\frac{\sigma^2}{2} + r(s)\right\}$$

$$= \frac{P(s) - s}{(1 - P(s))(1 - s)}$$

$$= \frac{P(s) - s}{(1 - s)(P(s) - s)}$$

$$= \frac{1 - S}{1 - P(s)}$$

$$= \frac{\sigma^2 + R(s)}{2}$$

$$= \{\sigma^2/2 + r(s)\}\{1 - (1-s)\{\sigma^2/2 + r(s)\}\}^{-1}$$

$$= \sigma^2/2 + R(s) \dots \quad (3)$$

where $R(s) \to 0$ as $s \to 1$ and R is bounded

$$P_2(s) = P(P(s)) = P(s) + \sigma^2/2(1 - P(s))^2 + r(P(s))(1 - P(s))^2.$$

So that

$$\frac{1}{1 - P_2(s)} - \frac{1}{1 - P(s)} = \sigma^2/2 + R(P(s)) \quad (4)$$

and

$$\frac{1}{2}\left\{\frac{1}{1 - P_2(s)} - \frac{1}{1 - P(s)}\right\} = \sigma^2/2 + 1/2r\{R(s) + R(P(s))\}.$$

Iterating one gets $\frac{1}{n}\left\{\frac{1}{1 - P_2(s)} - \frac{1}{1 - P(s)}\right\} = \sigma^2/2 + 1/n\sum_{k=0}^{n} R(P_k(s))$. Since $P_n(0) \le P_n(s) \le 1$ and $P_n(0) \to 1$ from the left the convergences of $P_n(0) \to 1$ is uniform. Hence proved the Lemma.

Theorem 1. *If* $m = 1, \sigma^2 < \infty$. *Then*

(a) $\lim_{n\to\infty} \Pr\{X_n > 0\} = 2/\sigma^2$

(b) $\lim_{n\to\infty} E\{X_n/n | X_n > 0 > 0\} = \sigma^2/2$

(c) $\lim_{n\to\infty} \Pr\{X_n/n > u/X_n > 0\} = \exp(-2u/2), u \ge 0.$

Proof.

(a) $n \Pr\{X_n > 0\} = n\{1 - P_n(0)\}\left[\frac{1}{n}\left\{\frac{1}{1 - P_n(0)}\right\} + \frac{1}{n}\right]^{-1}.$

Thus from the (taking $s = 0$) we get,

$$\lim_{n\to\infty} n \Pr\{X_n > 0\} = \lim_{n\to 0}[\sigma^2/2 + 1/n]^{-1} = 2/\sigma^2$$

(b) We have

$$1 = E\{X_n / X_n > 0\}, \Pr\{X_n > 0\} + 0, \Pr\{X_n = 0\}.$$

So that $E\{X_n/X_n > 0\} = \frac{1}{\Pr\{X_n > 0\}} = n\sigma^2/2$ { form (a)}. Thus $E\{X_n/X_n > 0\} = \sigma^2/2.$

(c) Let $u > 0$ and

$$dF(u) = \Pr\left\{u < \frac{X_n}{n} < u + du/X_n > 0\right\}. \tag{5}$$

Then taking Laplace Transform, we get

$$\int_0^\infty \exp(-au) dF(u) = E\{\exp(-X_n/n/X_n > 0\}. \tag{6}$$

Now $E\left\{\exp\left(\frac{-X_n}{n}\right)\right\} = E\left\{\exp\left(\frac{-X_n}{n}\right)/X_n > 0\right\} > 0, \Pr\{X_n > 0\} + 1, \Pr\{X_n > 0\}$

and since $P_n(S) = E\{S^{\times n}\}$ is the $p.g.f$ of X_n we get $P_n\left(\exp\left(\frac{-a}{n}\right)\right) = E\left\{\exp\left(\frac{-aX_n}{n}\right) | X_n > 0\right\}\{1 - P_n(0)\} + P_n(0).$

Thus

$$E\left\{\exp\frac{(-aX_n)}{n}/X_n > 0\right\} = \frac{P_n(\exp(-a/n)) - P_n(0)}{1 - P_n(0)}$$

$$= 1 - \frac{1 - P_n\left(\exp\left(-\frac{a}{n}\right)\right)}{1 - P_n(0)}. \tag{7}$$

Now as $n \to \infty$, $n\left\{1 - P_n\left(\exp\left(-\frac{a}{n}\right)\right)\right\} \to \frac{2}{\sigma^2}$ (from (a)) and from the basic Lemma (become of uniform convergence), we get

$$\frac{1}{n\left\{1-P_n\left(\exp\left(-\frac{a}{n}\right)\right)\right\}}=\frac{1}{n}\left\{\frac{1}{1-P_n\left(\exp\left(-\frac{a}{n}\right)\right)}-\frac{1}{1-\exp\left(-\frac{a}{n}\right)}\right\}$$

$$+\frac{1/n}{1-\exp\left(-\frac{a}{n}\right)}\to\frac{\sigma^2}{2}+\frac{1}{\alpha}.$$

Thus from (6) and (7) we get as $n\to\infty$, $\int_0^\infty \exp(-\alpha u)dF(a)\to\frac{\frac{\sigma^2}{2}}{\frac{\sigma^2}{2}+\frac{1}{\alpha}}=\frac{1}{1+\alpha\sigma^2/2}$. Since Laplace Transform of $\frac{2}{\sigma^2}\exp\left(\frac{-2u}{\sigma^2}\right)$ is $\frac{1}{1++\alpha\sigma^2/2}$.

We have $\lim_{n\to\infty}\Pr\left\{u<\frac{x_n}{n}<u+du/X_n>0\right\}=\frac{2}{\sigma^2}\exp\left(\frac{-2u}{\sigma^2}\right)$ which established limit law.

Subcritical Process

Theorem 2 (Yagloms Theorem)**.** *For a Galton-Wastson process with*

$$\lim_{n\to\infty}\Pr\{X_n=j/X_n>0\}=b_j,\ j=1,2, \tag{8}$$

exists *and* $\{b_j\}$ *gives a probability distribution whose p.g.f.* $B(s)=\sum_{j=1}^{\infty}b_js^j$ *satisfies the equation*

$$B(P(s))=mP(s)+1-m \tag{9}$$

or,

$$1-B(P(s))=m(1-B(s)). \tag{9a}$$

Further

$$\sum_{j=1}^{\infty}jb_j=1/\varphi(0),$$

where $\varphi(0)=\lim\Pr\{X_n>0\}/m^n$.

Proof. Using Taylor's expansion around $s = 1$ we get

$$P(s) = 1 - m(1 - s)r(s),\ 0 \leq s \leq 1 \tag{10}$$

or

$$\frac{1 - P(s)}{1 - s} = m - r(s). \tag{10a}$$

Consider the function $r(s)$ is $0 \leq s \leq 1$. We have $r(0) = m - P(0) \geq 0$ and $\lim\limits_{s \to 1-0} r(s) = 0.$

Further as $P(s)$ is a convert function $P'(s) \leq \frac{1 - P(s)}{1 - s}$.

So that $r'(s) = (1 - s)^{-1}\left\{\frac{1 - P(s)}{1 - s} - P(s)\right\} \leq 0$. Thus $r(s)$ is monotone decreasing is bounded above by m and $r(s) \to 0$ as $s \to 1$. Replacing s by $P_{k-1}(s)$ in (8a) we get

$$\frac{1 - P_k(s)}{1 - P_{k-1}(s)} = m\{1 - r(P_{k-1}(s)/m)\}. \tag{11}$$

Putting $k = 1, 2, 3, \ldots$ and taking products of both sides we get

$$\frac{1 - P_n(s)}{1 - s} = m^n \prod_{k=0}^{n}\left\{1 - \frac{r(P_k(s))}{m}\right\}. \tag{12}$$

Since $0 \leq r/m \leq 1$ the sequence $\left\{\frac{1 - P_n(s)}{m^n(1 - s)}\right\}$ is monotone decreasing is n, and we have

$$\lim_{n \to \infty} \frac{1 - P_n(s)}{m^n(1 - s)} - \emptyset(s) \geq 0.$$

Putting $s = 0$, we get $\varphi(s) = \lim\limits_{n \to \infty} \frac{1 - P_n(0)}{m^n} = \lim\limits_{n \to \infty} \frac{\Pr\{x_n > 0\}}{m^n}$. Let $b_{jn} \Pr\{X_n = i/X_n > 0\}$ and $B_n(s) = \sum_{j=1}^{\infty} b_{jn} s^j$ be the $p.d.f$ of $\{b_{jn}\}$. Then

$$B_n(s) = \frac{P_n(s) - P_n(0)}{1 - P_n(0)} = 1 - \frac{1 - P_n(s)}{1 - P_n(0)}$$

$$= 1 - (1 - s)\prod_{k=0}^{n} \frac{1 \dfrac{r(P_k(s))}{m}}{1 \dfrac{r(P_k(s))}{m}} \tag{13}$$

(from equation (12)).

Since $P_k(S) > P_k(0)$ and $r(s)$ is monotone decreasing $r(P_k(s)) \leq r(P_k(s))r(P)(0)$, and so each factor of the product on the $r.h.s$ expression is larger then 1. Thus $B_n(s)$ is monotone decreasing and tends to a limit $B(s)$ as $n \to \infty$.

$$B(s) = \sum_{j=1}^{\infty} b_j s^j,$$

Where $b_j = \lim_{n\to\infty} b_{jn} = \lim_{n\to\infty} \Pr\{X_n = i / X_n > 0\}$. Clearly

$$B(0) = 0,\ B(P_k(0)) = \lim_{n\to\infty} B_n(P_n(0))$$

$$= \lim_{n\to\infty}\left\{1 - \frac{1 - P_n(P_k(0))}{1 - P_n(0)}\right\}$$

$$= 1 - \lim_{n\to\infty}\left\{\frac{1 - P_n(P_k(0))}{1 - P_n(0)}\right\}$$

$$= \lim_{n\to\infty}\left\{1 - \frac{1 - P_n(s)}{1 - s}\right\},$$

(since $m < 1,\ P_n(0)$ as $n \to \infty$)

$$= 1 - m^k.$$

It follows that $\lim B(P_k(0)) \to 0$ as $< 1,\ P_n(0)$ as $k \to \infty$.

Hence taking limits of both side of s we get

$$b(P(s)) = 1 - (1 - B(s))m = mB(s) + 1 - m.$$

4. Generalization of the Classical Galton-Watson Process

4.1. Branching process with immigration. For classical Galton-Watson branching processes it is assumed that individuals reproduce independently of each other according to some given offspring distribution. In the setting of this study the off spring distribution varies in a random fashion, independently from one generation to the other. A mathematical formulation of the model is as follows. The value of $E(X_1) = m,$ we have as $\lim_{n\to\infty} \Pr\{X_n \to k = 0\}$ and $\lim_{n\to\infty} \Pr\{X_n = k\} = 0$ for any finite positive integer k for a $G.W.$ process. $\Pr\{X_n \to 0\} = q$ and $\Pr\{X_n \to k = 0\}$ for finite k and $\Pr\{X_n \to \infty\} = 1 - q.$ Let q being the probability of extinction. Further $q = 1$ for critical and subcritical process. Thus left to themselves $G.W.$ populations either die out or grow without limits. Immigration from outside into a critical or subcritical process could have establishing effect on the population size. Apart from this aspect immigration by itself from the point of view of the theory and application. Galton-Watson process with immigration often arise in application in such areas as traffic theory statistical mechanics genetics neurophysiology etc.

Consider a G.W. Process with offspring distribution $\{P_k\}$ (having $p.g.f.$ $P(s)$ and mean $p(1) = m$). Suppose that at time n, i.e., at the time birth of n^{th} generation there is an immigration of Y_r objects into the population and that Y_n, $n = 0, 1, 2, \ldots$ and i.id. random variables with $p.g.f$

$$h(s) = \sum_{j=0}^{\infty} \Pr\{X_{(n)} = j\} s^j$$

(i.e.) with probability h_j, j immigrants enter the n^{th} generation and contribute to the next generation in same way as others already presented. The number of immigrants into successive generations is independent and all objects reproduce independently of each other and of the immigration process. The distribution $\{h_j\}$ will be called immigrant distribution. Let $a = h'(1)$ be the mean of this distribution. Let $X_{(n)}$ be the number of objects at the n^{th} generation and let

$$P_{(n)}(s) = \sum_{j=0}^{\infty} \Pr\{X_{(n)} = j\} s^j$$

be its *p.g.f.* The sequence $\{X_{(n)}, n = 0, 1, 2, \ldots\}$ defines a G.W. immigrant's process. The sequence is a Markov chain whose one-step transition probabilities are given by $P_{ij} = Coeff.\ of\ s^j$ in $h(s)[P(s)]^i$, $i, j \in N$.

Clearly

$$P_{(n)}(s) = h(s) P_{(n-1)}(P(s)). \tag{14}$$

If $\lim_{n\to\infty} P_{(n)}(s) = F(s)$ exists, then one gets

$$F(s) = h(s) F(P(s)) \tag{15}$$

(i.e.) the limit when it exists, satisfies the above functional equation. Corresponding to Yaglom's result for subcritical G.W. Process, Heathcote (1965) obtained the following analogue for G.W. immigrant's process.

Consider a Galton-Watson process with immigration having offspring distribution $\{P_j\}$ with *p.g.f.* $P(s)$ and immigrant distribution $\{h_j\}$ with *p.g.f.* $h(s)$.

If $m < 1$, $a = h'(1) < \infty$, then $\Pr\{X_{(n)} = j\} = d_j$, exists for $j \in N$ and the *p.g.f.*

$$F(s) = \sum_j d_j s^j$$

satisfies the functional equation (13) $F(s) = h(s)F(P(s))$. Further

$$F(1) = 1 \text{ iff } \sum_{j=1}^{\infty} h_j \log(d_j) < \infty.$$

It may be noted that when $F(s) = \dfrac{1 - B(s)}{1 - s}$ and $h(s) = \dfrac{1 - P(s)}{m(1 - t)}$. Then the functional equation

$$B(P(s)) = m(B(s)) + 1 - m$$

reduces to the functional equation (15). Thus, ordinary subcritical G.W. Process may always be regarded as forming a subclass of G.W. immigration process. The result for the critical process, due to Seneta (1970) may be stated as follows.

For a Galton-Watson Process with immigration having $m = 1$, $\sigma^2 < \infty$, $a < \infty$ the random variable $X_{(n)}/n$ converges in distribution as $n \to \infty$ to a random variable having gamma density.

$$\frac{u^{\alpha-1} e^{-u/\beta}}{\mu^{(\alpha)} \beta^{\alpha}},\ u \geq 0 \text{ where } \alpha = \frac{2a}{\sigma^2},\ \beta = \frac{\sigma^2}{2}.$$

If a $h'(1) = \frac{\sigma^2}{2}$, i.e. $\alpha = 1$ the gamma density reduces to an exponential density such that

$$\lim_{n\to\infty} \Pr\left\{\frac{X_n}{n} > u\right\} \to \exp\left\{\frac{-2u}{\sigma^2}\right\}.$$

Again Theorem 2 (c) we get

$$\lim_{n\to\infty} \{X_n > u \mid X_n > 0\} \to \exp\left(\frac{-2u}{\sigma^2}\right),\ u \geq 0.$$

From the G.W. Process $\{X_n\}$ without immigration. Thus in the critical case the effect of the conditioning of non-extinction $\{X_n > 0\}$ is the same as immigration into the process at the rate of $a = \frac{\sigma^2}{2}$.

4.2. Processes in varying and Random Environments. The Galton-Watson process in a varying environment is the generalization of the Galton-Watson process that allows the offspring distribution $\{P_k\}$ remains the same in all the generations and that objects reproduce independently of others. Consider G.W. Process under varying environment the assumption that the distribution $\{P_k\}$ remains the same in all the generations was then replaced by the assumption that off spring distribution for the n^{th} generation is of the form $\{P_{nk}\}$; several interesting results on process in varying environments were obtained among others by Fearn (1979) and Jagers (1974). Smith and Wilkinson (1969) introduced yet another aspect. They postulated that the offspring distribution for each generation is randomly chosen from a class of all reproduction loses and the resulting process are said to be in random environments the topic has been pursued in great details in number of papers by Wilkinson (1969), Athreya and Karlin (1970) Kaplan (1972) and others.

5. Multiple Galton-Watson Processes

A natural extension of Galton-Watson process is concerned with the case where the population consists of finite number of type of objects. Such process are known as multiple Galton-Watson process. Mode (1971) deals exclusively and extensively with such process and discusses several models for applications in various areas.

Suppose that population of individuals (or objects) originates with a single ancestor and that there are k finite types of individuals. Let $P^{(i)}(r_1, r_2, \ldots, r_k)$ be the probability of type i produces r_j offspring's of type j, $j = 1, 2, 3, \ldots k$. We introduce the vector notation.

$$r = (r_1, r_2, \ldots, r_k)$$

$$s = (s_1, s_2, \ldots, s_k)$$

$$e_i = (\delta_{1l}, \delta_{2l}, \ldots, \delta_{kl}).$$

Let $f^{(1)}(s) \sum P^{(1)}(s) S^{r_1, r_2, \ldots, r_k}$ be the $p.g.f.$ of $\{P^{(1)}(r)\}$. Let $X_n = \{X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}\}$ represent the population size of k types is the n^{th} generation. Let m_{ij} be the expected numbers of off spring of type j produced by an object of type i. Then

$$m_{ij} = E\{X_1^{(f)} / X_0 = e_i\} = \frac{\partial f^{(j)}}{\partial s_j} \quad j = 1, 2, \ldots, k$$

The matrix $M = (m_{ij})$ is the matrix of moments. Let $f_n^{(i)}(s)$ denote the $p.d.f.$ of the number of objects in n^{th} generation starting from one object of type i.

A result analogous is

$$f_n^{(i)}(s) = f^{(i)}, \{f_{n-1}^{(1)}(s), f_{n-1}^{(2)}(s), \ldots, f_{n-1}^{(k)}(s)\}, n = 0, 1, 2, \ldots$$

$$f_0^{(i)}(s) = s_i, i = 0, 1, 2, \ldots, k.$$

The population becomes extinct when $X_n = 0$ for some n. Let

$$q_n^{(i)} = \Pr\{X_n = 0/X_0 = e_j\} \text{ and } q^{(i)} = \lim_{n\to\infty} q_n^{(i)};$$

(i.e.) $q^{(i)}$ denote the probability of extinction given the process started with one ancestor of type i, let $q = (q^{(1)}, q^{(2)}, \ldots, q^{(k)})$. We confine ourselves to case where the matrix is moments of certain type.

Let M be a $K \times K$ matrix with non-negative elements such that for some positive integers n, all the elements of M^n as strictly positive. Then there exists a positive eigenvalue ρ of M, which is greater than the absolute value of any other eigenvalue of M; ρ is also called the spectral radius of M.

When M is the matrix (m_{ij}) the Eigen value ρ is the multitype G.W. case plays the role of the mean m of offspring distribution is the simple G.W. case. A multitype G.W. process is said to be subcritical critical or supercricital depending on $p \leq 1$ or > 1.

Let the matrix of moments M be positively regular and let ρ be its spectral radius.

(a) If $p \leq 1$ then $q_i = 1, i = 1, 2, \ldots, k$

(b) If $p > 1$ then $q_i < 1, i = 1, 2, \ldots, k$

(c) In either case $q_i = (q^{(1)}, q^{(2)}, \ldots, q^{(k)})$ is the smallest positive solution of the vector equation $q = f(q)$ (i.e.) $q^{(i)}$ is the smallest positive root of $q^{(i)} = f^{(i)}(q)$ for $i = 1, 2, 3, \ldots, k$.

6. Conclusion

We used a generalization of the Classical Galton-Watson branching with immigration having offspring distribution in all category generation and also the asymptotic form of the probability of non-extinction and conditional limits theorems depends on population size are obtained. Finally we conclude that this paper will be very useful in the society specifically embassy they can easily access the immigration process.

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References

[1] K. J. Adams, The Role of children in the charging socioeconomic strategic of the Guyanese cabs (1981).

[2] A. Agresti, Bounds on the extinction time distribution of a branching process, Adv. Appl. Prob. 6 (1974), 322-325.

[3] A. Agresti, On the extinction times of random and varying environment branching processes, J. Appl. Probab. 12 (1975), 39-46.

[4] K. B. Athreya and S. Karlin, Branching processes with random environments, II: Limit theorems. Ann. Math. Statist. 42 (1971), 1843-1858.

[5] N. T. J. Baiey, The Mathematical Theory of Infection Diseases and its Application, 2nd edition, Griffln, London, 1975.

[6] B. S. Burks, A statistical method for estimating the distribution of size of completed fraternities in a population represented by random sampling of individuals, Journal of American Association 28 (1933), 384-399.

[7] B. D. Coleman, Time dependent mechanical breakdown in bundles of fibers. I. Constant Total Load, Journal of Applied Physics 28(9) (1975), 1058-1064.

[8] J. D. Church, On infinite composition products of probability generating functions, Z. Wahrsch. Verw. Gebiete 15 (1971), 243-256.

[9] D. H. Fearn, Galton-Watson processes with generation dependence, Proc. 6th Berkeley Symp. Math. Statist. Prob. 4 (1972), 159-172.

[10] Donnellyp, Contribution to the discussion, Symposium on stochastic networks, J. R. Statist. Soc. B 47 (1985), 422-423.

[11] M. J. Daddy, Compartmental models with restricted movement, Bull. Math. Biol. 45 (1983), 401-408.

[12] P. Jagers, Galton-Watson processes in varying environments, J. Appl. Probab. 11(1) (1974), 174-178.

[13] A. Joffe, On the Galton-Watson process with mean less than one, Ann. Math. Statist. 38 (1967), 264-266.

[14] N. Kaplan, A. Sudbury and T. S. Nilson, A branching process with disasters, J. Appl. Prob. 12 (1975), 47-59.

[15] C. J. Mode, Multitype branching processes, Theory and applications, Modern Analytic and Computational Methods in Science and Mathematics, No. 34, American Elsevier Publishing Co., Inc., New York, 1971.