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SOME CONTINUOUS FUNCTIONS ON $\mathcal{I}_{g\delta s}$-CLOSED SETS

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Abstract

M. Khan and T. Noiri introduce $\mathcal{I}_{g^*s}$-continuous functions, strongly $\mathcal{I}_{g^*s}$-continuous functions and weakly $\mathcal{I}_{g^*s}$-continuous functions. In this paper we introduce and analyze some additional properties of $\mathcal{I}_{g\delta s}$-continuous function, Strongly $\mathcal{I}_{g\delta s}$-continuous functions and contra $\mathcal{I}_{g\delta s}$-continuous functions and obtain some results.

1. Introduction

M. Khan and T. Noiri introduce $\mathcal{I}_{g^*s}$-continuous functions, strongly $\mathcal{I}_{g^*s}$-continuous functions and weakly $\mathcal{I}_{g^*s}$-continuous functions. It turns

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out that weak $\mathcal{I}_{g^*s}$-continuity is weaker than weak $\mathcal{I}$-continuity defined by Ackgoz et al. In 1996, Dontchev introduced a new class of functions called contra continuous functions. He defined a function $f : X \to Y$ to be contra continuous if the pre image of every open set of Y is closed in X. A new weaker form of this class of functions, called contra-e-continuous functions, contra e-continuous functions, and contra a-continuous functions were introduced and investigated by Ekici. Wadel Al-Omer et al. introduced the concept of contra e-$\mathcal{I}$-continuous functions in ideal topological spaces. In this paper we introduce and analyze some additional properties of $\mathcal{I}_{g^*s}$-continuous function, strongly $\mathcal{I}_{g^*s}$-continuous functions and contra $\mathcal{I}_{g^*s}$-continuous functions and obtain some results.

2. Preliminaries

Definition 2.1 [22]**.** The θ-closure of A, denoted by $Cl_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap Cl(U) \neq \emptyset$ for every open neighbourhood U of X. If $A = Cl_\theta(A)$, then A is called θ-closed. The complement of a θ-closed set is called a θ-open set. The θ-interior of A is defined by the union of all θ-open sets contained in A and is denoted by $Int_\theta(A)$.

Remark 2.2 [22]**.** The collection of δ-open sets in a topological space (X, τ) forms a topology τ_θ on X.

Definition 2.3 [22]**.** The δ-closure of A, denoted by $Cl_\delta(A)$, is defined to be the set of all $x \in X$ such that $A \cap Int(Cl(U)) \neq \emptyset$ for every open neighbourhood U of X. If $A = Cl_\theta(A)$, then A is called a δ-closed set. The complement of a δ-closed set is called δ-open. The δ-interior of A is defined by the union of all δ-open sets contained in A and is denoted by $Int_\delta(A)$,

Remark 2.4 [22]**.** The collection of δ-open sets in a topological space (X, τ) forms a topology τ_δ on X.

Definition 2.5. A subset A of a topological space (X, τ) is said to be

(1) regular open [20] if $A = Int(Cl(A))$,

(2) preopen [18] if $A \subset Int(Cl(A))$,

(3) semiopen [17] if $A \subset Cl(Int(A))$,

The complement of a regular open (resp. preopen, semiopen) set is called a regular closed (resp. preclosed, semiclosed) set. The set of all regular open (resp. preopen, semiopen, regular closed, preclosed, semiclosed) sets of (X, τ) is denoted by $RO(X)$ (resp. $PO(X)$, $SO(X)$, $BO(X)$, $RC(X)$, $PC(X)$, $SC(X)$) $S - \text{int}(A) = \{ \bigcup U : U \subset A$ and U is semi-open sets$\}$ and $scl(A) = \{\bigcap G : A \subset G$ and G is semi-closed$\}$.

Remark 2.6 [22]**.** A set $A \subset X$ is δ-open if and only if it is the union of regular open sets of X.

An ideal $\mathcal{I}$ on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies the following conditions.

(1) $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$ and

(2) $A \in \mathcal{I}$ and $B \in A \Rightarrow A \cup B \in \mathcal{I}$.

Definition 2.7 [12]**.** Let (X, τ) be a topological space with an ideal $\mathcal{I}$ on X and $(\cdot)^*$ be a set operator from $\wp(X)$ to $\wp(X)$. For a subset $A \subset X$, $A^*(\mathcal{I}, \tau) = \{x \in X : U \cup A \notin \mathcal{I}$ for every $U \in \mu(x)\}$ where $\mu(x) = \{U \in \tau : x \in U\}$ is called the local function of A with respect to $\mathcal{I}$ and τ. We will simply write A^* for $A^*(\mathcal{I}, \tau)$.

Definition 2.8 [12]**.** Let (X, τ) be a space with an ideal $\mathcal{I}$ on X. The set operator cl^* is called a $*$-closure and is defined as $cl^*(A) = A \subset A^*$ for $A \subset X$. We will denote by $\tau^*(\mathcal{I}, \tau)$ the generated by cl^*, that is, $\tau^*(\mathcal{I}) = \{U \subset X : cl^*(X - U) = X - U\}$. $\tau^*(\mathcal{I})$ is called $*$-topology which is finer than τ. The elements of $\tau^*(\mathcal{I}, \tau)$ are called $*$-open and the complement of an $*$-open set is called $*$-closed. The interior of a subset A in $(X, \tau^*(\mathcal{I}, \tau))$ is denoted by $\text{int}^*(A)$.

Definition 2.9 [5]**.** A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is called semi-I-open if $S \cup Cl^*(Int(S))$. The complement of a semi-I-open set is called a semi-I-closed set. The family of all semi-I-open (resp. semi-I-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $S\mathcal{I}O(X)$ (resp. $S\mathcal{I}O(X)$). We set $S\mathcal{I}O(X, x) = \{U : U \in S\mathcal{I}O(X)$ and $x \in U\}$ and $S\mathcal{I}O(X, x) = \{U : U \in S\mathcal{I}O(X)$ and $x \in U\}$.

Definition 2.10 [3]**.** The intersection of all semi-I-closed sets containing A is called the semi-I-closure of A and is denoted by $s\mathcal{I}Cl(A)$. A subset A is semi-I-closed if, and only if $s\mathcal{I}Cl(A) = A$. The union of all semi-I-open subsets of (X, τ) contained in $A \subset X$ is called the semi-I-interior of A and is denoted by $s\mathcal{I}Int(A)$.

Definition 2.11 [3]. An ideal topological space $(X, \tau, \mathcal{I})$ is semi-I-normal if for each disjoint closed sets F_1, F_2 of X, there exist disjoint semi-$\mathcal{I}$-open sets W_1, W_2 such that $F_i \subset W_i$, where $i = 1, 2$.

Definition 2.12 [3]. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called a semi-continuous function if for every open subset V of Y, $f^{-1}(V)$ is semi-open in X.

Definition 2.13. A subset A of X is g-closed if $cl(A) \subset U$ whenever $A \subset U$ and U is open.

Lemma 2.14. *The following statements are true Let A be a subset of a topological space (X, τ). Then $A \in PO(X)$ if and only if $sCl(A) = Int(Cl(A))$* [?]

Definition 2.15. A space X is said to be $T\mathcal{I}_{g\delta s}$-space if every $\mathcal{I}_{g\delta s}$-closed set is closed.

Definition 2.16. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be strongly continuous, if $f^{-1}(V)$ is semi-closed in (X, τ) for every closed set V in (Y, σ).

Definition 2.17. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be perfectly continuous, if $f^{-1}(V)$ is clopen in (X, τ) for every open set V in (Y, σ).

Definition 2.18. A function $f : (X, \tau) \to (Y, \sigma)$ is called completely continuous, if the inverse image of every open set in Y is regular open in (X, τ).

Definition 2.19. A function $f : (X, \tau) \to (Y, \sigma)$ is called totally semi-continuous, if the inverse image of every semi-open set in Y is clopen in (X, τ).

Definition 2.20. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called $\mathcal{I}_{g\delta s}$-continuous, if the inverse image of every closed set in Y is $\mathcal{I}_{g\delta s}$-closed in X.

Definition 2.21. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra-continuous if $f^{-1}(V)$ is closed in (X, τ) for each open set V in (Y, σ).

Definition 2.22. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called $\mathcal{I}_{g\delta s}$-irresolute, if the inverse image of every $\mathcal{I}_{g\delta s}$-closed set in Y is $\mathcal{I}_{g\delta s}$-closed in X.

Definition 2.23. A space X is said to be $\mathcal{I}_{g\delta s}$-T_2-space if for any distinct points x and y ox there exists two disjoint $\mathcal{I}_{g\delta s}$-open sets U and V such that $x \in U$ and $y \in V$ respectively.

3. $\mathcal{I}_{g\delta s}$-Continuous Function

In this division, we analyze some additional properties of $\mathcal{I}_{g\delta s}$-continuous function, Strongly $\mathcal{I}_{g\delta s}$-continuous functions and obtain some results.

Definition 3.1. A subset A of X is called generalized δ semi-closed (briefly $\mathcal{I}_{g\delta s}$-closed) set if $\mathcal{I}_{scl}(A) \subset U$ whenever $A \subset U$ and U is δ-open in (X, τ). The family of all $\mathcal{I}_{g\delta s}$-closed subsets of the space X is denoted by $\mathcal{I}_{g\delta s}$-$C(X)$ and $\mathcal{I}_{g\delta s}$-open subsets of the space X is denoted by $\mathcal{I}_{g\delta s}$-$O(X)$.

Definition 3.2. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be $\mathcal{I}_{g\delta s}$-continuous, if the $f^{-1}(V)$ is $\mathcal{I}_{g\delta s}$-closed in $(X, \tau, \mathcal{I})$ for every closed set V in (Y, σ).

Theorem 3.3. *Let* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *be bijective* $\mathcal{I}_{g\delta s}$*-continuous and* $g : (Y, \sigma) \to (Z, \mu)$ *be bijective continuous function then* $g \circ f : (X, \tau, \mathcal{I}) \to (Z, \mu)$ *is* $\mathcal{I}_{g\delta s}$*-continuous function.*

Proof. Let V be any open subset of Z then $g^{-1}(V)$ be open in Y and as f is $\mathcal{I}_{g\delta s}$-continuous $f^{-1}(g^{-1}(V))$ is $\mathcal{I}_{g\delta s}$-open in X i.e., $(g \circ f)^{-1}(V)$ is $\mathcal{I}_{g\delta s}$-open in X implies $g \circ f$ is $\mathcal{I}_{g\delta s}$-continuous function.

Definition 3.4. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is said to be strongly $\mathcal{I}_{g\delta s}$-continuous, if $f^{-1}(V)$ is semi-closed in (X, τ) for every $\mathcal{I}_{g\delta s}$-closed set V in $(Y, \sigma, \mathcal{I})$.

Theorem 3.5. *A function* $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ *is strongly* $\mathcal{I}_{g\delta s}$*-continuous, if and only if the inverse image of each* $\mathcal{I}_{g\delta s}$*-open set in V is a semi open set in U.*

Proof. Suppose $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is strongly $\mathcal{I}_{g\delta s}$-continuous function and V is $\mathcal{I}_{g\delta s}$-open set in Y. Then $Y - V$ is $\mathcal{I}_{g\delta s}$-closed in Y. By hypothesis $f^{-1}(Y - V) = X - f^{-1}(V)$ is a semi-closed set in X and hence $f^{-1}(V)$ is semi-open set in X. On the other hand, if F is $\mathcal{I}_{g\delta s}$-closed set in Y, then $Y - F$ is an $\mathcal{I}_{g\delta s}$-open set in X.

By hypothesis $f^{-1}(Y - F) = X - f^{-1}(F)$ is semi-open set in X, implies $f^{-1}(F)$ is semi-closed set in X. Therefore f is strongly $\mathcal{I}_{g\delta s}$-continuous.

Theorem 3.6. *The following are equivalent for the function* $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$

(1) *The function f is strongly* $\mathcal{I}_{g\delta s}$*-continuous.*

(2) *For each* $x \in U$ *and each* $\mathcal{I}_{g\delta s}$*-open set V in* $(Y, \sigma, \mathcal{I})$ *with* $f(x) \in V$, *there exist a semi open set W in* (X, τ) *such that* $x \in W$ *and* $f(W) \subset V$.

(3) $f^{-1}(V) \subset S - \text{int}(f^{-1}(V))$ *for each* $\mathcal{I}_{g\delta s}$*-open set V of Y.*

(4) $f^{-1}(F)$ *is semi closed in* (X, τ) *for every* $\mathcal{I}_{g\delta s}$*-closed set F of Y.*

Proof.

(1) $\to$ (2) Suppose (1) holds. Let $x \in U$ and V be a $\mathcal{I}_{g\delta s}$-open set in V containing $f(x)$. Since f is strongly $\mathcal{I}_{g\delta s}$-continuous, implies $f^{-1}(V)$ is a semi open set in (X, τ) such that $x \in f^{-1}(V)$. Put $W = f^{-1}(V)$, then $x \in W$ and $f(W) = f(f^{-1}(V))$. Thus (2) holds.

(2) $\to$ (3) Suppose (2) holds. Let V be any $\mathcal{I}_{g\delta s}$-open set in Y and $x \in f^{-1}(V)$. By (2), there exists a semi open set W in (X, τ) such that $x \in W$ and $f(W) \subset V$. This implies $x \in W = S - \text{int}(W) \subset S - \text{int}(f^{-1}(V))$. That is $x \in S - \text{int}(f^{-1}(V))$. Therefore, $f^{-1}(V) \subset S - \text{int}(f^{-1}(V))$.

(3) $\to$ (4) Suppose (3) holds. Let F be any $\mathcal{I}_{g\delta s}$-closed set of Y. Set $V = Y - F$, then V is $\mathcal{I}_{g\delta s}$-open set in Y. By (3) $f^{-1}(V) \subset S - \text{int}(f^{-1}(V))$. That is $f^{-1}(V - F) \subset S - \text{int}(f^{-1}(V - F))$. This implies $X - f^{-1}(F) \subset X - scl(f^{-1}(F))$. This implies $scl(f^{-1}(F)) \subset f^{-1}(F)$. But $f^{-1}(F) \subset scl(f^{-1}(F))$ is always true. Thus, $f^{-1}(F) = scl(f^{-1}(F))$. Therefore, $f^{-1}(F)$ is semi closed in (X, τ).

(4) $\to$ (1) Suppose (4) holds. Let V be any $\mathcal{I}_{g\delta s}$-open set of Y. Set $F = Y - V$. Then F is $\mathcal{I}_{g\delta s}$-closed set of Y. By (iv), $f^{-1}(F)$ is semi closed in (X, τ). But $f^{-1}(F) = f^{-1}(Y - V) = X = f^{-1}(V)$. This implies $f^{-1}(V)$ is a semi open set in (X, τ). Therefore f is strongly $\mathcal{I}_{g\delta s}$-continuous.

4. Perfectly $\mathcal{I}_{g\delta s}$-Continuous Function

Definition 4.1. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is said to be perfectly $\mathcal{I}_{g\delta s}$-continuous, if $f^{-1}(V)$ is clopen in (X, τ) for every $\mathcal{I}_{g\delta s}$-open set V in $(Y, \sigma, \mathcal{I})$.

Theorem 4.2. *If a function* $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ *is perfectly continuous and Y is* $T\mathcal{I}_{g\delta s}$*-space, then f is perfectly* $\mathcal{I}_{g\delta s}$*-continuous.*

Proof. Let G be a $\mathcal{I}_{g\delta s}$-open set in Y. Since Y is $T\mathcal{I}_{g\delta s}$-space, G is an open set in Y. Since f is perfectly continuous, $f^{-1}(G)$ is clopen in (X, τ). Therefore f is perfectly $\mathcal{I}_{g\delta s}$-continuous function.

Theorem 4.3. *Every perfectly* $\mathcal{I}_{g\delta s}$*-continuous function into finite* T_1*-space is strongly continuous.*

Proof. Obvious because every finite T_1-space is discrete space. Therefore every subset A of X is open and hence $\mathcal{I}_{g\delta s}$-open. Since f is perfectly $\mathcal{I}_{g\delta s}$-continuous function, $f^{-1}(A)$ is clopen for every subset A of X. Therefore f is strongly continuous.

Theorem 4.4. *Let X be a discrete topological space, Y be any topological space and* $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ *be a function. Then the following are equivalent.*

(1) *f is perfectly* $\mathcal{I}_{g\delta s}$*-continuous.*

(2) *f is strongly* $\mathcal{I}_{g\delta s}$*-continuous.*

Proof.

(1) $\to$ (2) Obvious because every clopen set is open.

(2) $\to$ (1) Let V is a $\mathcal{I}_{g\delta s}$-open in Y. By hypothesis, $f^{-1}(V)$ is open in (X, τ). Since X is discrete space, $f^{-1}(V)$ is also closed set in (X, τ). Therefore f is perfectly $\mathcal{I}_{g\delta s}$-continuous.

Theorem 4.5. *Let A be any subset of X. If* $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ *is perfectly* $\mathcal{I}_{g\delta s}$*-continuous, then the restriction function* $f|_A : A \to Y$ *is perfectly* $\mathcal{I}_{g\delta s}$*-continuous.*

Proof. Let V be a $\mathcal{I}_{g\delta s}$-open set of Y. Since f is perfectly $\mathcal{I}_{g\delta s}$-

continuous, $f^{-1}(V)$ is clopen set in (X, τ). Then, $(f\,|_A)^{-1}(V) = A \cap f^{-1}(V)$ is clopen in A and hence $f\,|_A$ is perfectly $\mathcal{I}_{g\delta s}$-continuous.

Theorem 4.6. *Let* $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ *and* $f : (Y, \sigma, \mathcal{I}) \to (Z, \mu, \mathcal{J})$ *be two function.*

(1) *If* f, g *are perfectly* $\mathcal{I}_{g\delta s}$*-continuous function, then* $(g \circ f)$ *is perfectly* $\mathcal{I}_{g\delta s}$*-continuous function.*

(2) *If f is perfectly* $\mathcal{I}_{g\delta s}$*-continuous function and g is* $\mathcal{I}_{g\delta s}$*-irresolute, then* $(g \circ f)$ *is perfectly* $\mathcal{I}_{g\delta s}$*-continuous function.*

(3) *If f is perfectly* $\mathcal{I}_{g\delta s}$*-continuous function and g is strongly* $\mathcal{I}_{g\delta s}$*-continuous, then* $(g \circ f)$ *is perfectly* $\mathcal{I}_{g\delta s}$*-continuous function.*

(4) *If f is perfectly* $\mathcal{I}_{g\delta s}$*-continuous function and g is* $\mathcal{I}_{g\delta s}$*-continuous, then* $(g \circ f)$ *is perfectly continuous function.*

(5) *If f is perfectly* $\mathcal{I}_{g\delta s}$*-continuous function and g is* $\mathcal{I}_{g\delta s}$*-continuous, then* $(g \circ f)$ *is totally semi-continuous function.*

(6) *If f is* $\mathcal{I}_{g\delta s}$*-continuous and g is strongly continuous then* $(g \circ f)$ *is* $\mathcal{I}_{g\delta s}$*-continuous.*

(7) *If f is* $\mathcal{I}_{g\delta s}$*-irres-olute and g is perfectly* $\mathcal{I}_{g\delta s}$*-continuous, then* $(g \circ f)$ *is* $\mathcal{I}_{g\delta s}$*-irresolute function.*

Proof.

(1) Suppose F is a $\mathcal{I}_{g\delta s}$-closed set in Z. Since g is perfectly $\mathcal{I}_{g\delta s}$-continuous function, $g^{-1}(F)$ is clopen in Y. Now f is perfectly $\mathcal{I}_{g\delta s}$-continuous function and every closed set is $\mathcal{I}_{g\delta s}$-closed set, implies $g^{-1}(F)$ is $\mathcal{I}_{g\delta s}$-closed set in Y and $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is clopen in (X, τ). Therefore $(g \circ f)$ is perfectly $\mathcal{I}_{g\delta s}$-continuous.

(2) Suppose F is a $\mathcal{I}_{g\delta s}$-closed set in Z. Since g is $\mathcal{I}_{g\delta s}$-irresolute, $g^{-1}(F)$ is $\mathcal{I}_{g\delta s}$-closed set in Y. Now f is perfectly $\mathcal{I}_{g\delta s}$-continuous function, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is clopen in (X, τ). Therefore $(g \circ f)$ is perfectly $\mathcal{I}_{g\delta s}$-continuous.

(3) Suppose U is a $\mathcal{I}_{g\delta s}$-open set in Z. Since g is strongly $\mathcal{I}_{g\delta s}$-continuous $g^{-1}(F)$ is semi-open and hence $\mathcal{I}_{g\delta s}$-open set in Y. Now f is perfectly $\mathcal{I}_{g\delta s}$-continuous function, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is clopen in (X, τ). Therefore $(g \circ f)$ is perfectly $\mathcal{I}_{g\delta s}$-continuous.

(4) Suppose F is an open set in (X, τ). Since g is $\mathcal{I}_{g\delta s}$-continuous $g^{-1}(F)$ is $\mathcal{I}_{g\delta s}$-open set in Y. Now f is perfectly $\mathcal{I}_{g\delta s}$-continuous function, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is clopen in (X, τ). Therefore $(g \circ f)$ is perfectly continuous.

(5) Suppose F is semi open set in Z. Since g is $\mathcal{I}_{g\delta s}$-continuous $g^{-1}(F)$ is $\mathcal{I}_{g\delta s}$-open set in Y. Now f is perfectly $\mathcal{I}_{g\delta s}$-continuous function, $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is clopen in (X, τ). Therefore $(g \circ f)$ is totally semi-continuous.

(6) Let G be an open set in Z. Since g is strongly continuous, $g^{-1}(G)$ is clopen in Y and hence open in Y. Since f is $\mathcal{I}_{g\delta s}$-continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is $\mathcal{I}_{g\delta s}$-open in (X, τ). Hence $(g \circ f)$ is $\mathcal{I}_{g\delta s}$-continuous.

(7) Let G be a $\mathcal{I}_{g\delta s}$-open set in Z. Since g is perfectly $\mathcal{I}_{g\delta s}$-continuous, $g^{-1}(G)$ is clopen and hence it is $\mathcal{I}_{g\delta s}$-open in Y. Again since f is $\mathcal{I}_{g\delta s}$-irresolute, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is $\mathcal{I}_{g\delta s}$-open in (X, τ). Hence $(g \circ f)$ is $\mathcal{I}_{g\delta s}$-irresolute.

Definition 4.7. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is called completely

$\mathcal{I}_{g\delta s}$-continuous, if the inverse image of every $\mathcal{I}_{g\delta s}$-open set in Y is regular open in (X, τ).

Theorem 4.8. *If a function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is completely continuous and Y is $\mathcal{I}_{g\delta s}$-space, then f is completely $\mathcal{I}_{g\delta s}$-continuous.*

Proof. Let G be a $\mathcal{I}_{g\delta s}$-open set in Y. Since Y is $T\mathcal{I}_{g\delta s}$-space, G is an open in V. Since f is completely continuous, $f^{-1}(G)$ is regular open in (X, τ). Therefore, f is completely $\mathcal{I}_{g\delta s}$-continuous function.

Lemma 4.9. *Let V be pre-open subset of X. Then $V \setminus U$ is regular open in X for each regular open set U of X.*

Theorem 4.10. *Let A be pre-open subset of X. If $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ is completely $\mathcal{I}_{g\delta s}$-continuous, then the restriction function $f|_A : A \to Y$ is perfectly $\mathcal{I}_{g\delta s}$-continuous.*

Proof. Let V be a $\mathcal{I}_{g\delta s}$-open set of Y. Then, $(f|_A)^{-1}(V) = A \cap f^{-1}(V)$. Since $f^{-1}(V)$ is regular open and A is pre-open, by lemma 4.9, $(f|_A)^{-1}(V)$ is regular open in the relative topology of A. Hence $f|_A$ is completely $\mathcal{I}_{g\delta s}$-continuous.

Theorem 4.11. *Let $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ and $g : (Y, \sigma, \mathcal{I}) \to (Z, \mu, \mathcal{J})$ be two function. Then*

(1) *If f is completely continuous and g is completely $\mathcal{I}_{g\delta s}$-continuous then $(g \circ f)$ is completely $\mathcal{I}_{g\delta s}$-continuous.*

(2) *If f is completely $\mathcal{I}_{g\delta s}$-continuous and g is $\mathcal{I}_{g\delta s}$-irresolute, then $(g \circ f)$ is completely $\mathcal{I}_{g\delta s}$-continuous.*

(3) *If f is completely $\mathcal{I}_{g\delta s}$-continuous and g is perfectly $\mathcal{I}_{g\delta s}$-continuous, then $(g \circ f)$ is completely $\mathcal{I}_{g\delta s}$-continuous function.*

Proof. (1) Let G be a $\mathcal{I}_{g\delta s}$-open set in Z. Then $g^{-1}(G)$ is regular open in

Y as g is completely $\mathcal{I}_{g\delta s}$-continuo us. So, $g^{-1}(G)$ is open in Y. Since f is completely continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is regular open in (X, τ). Hence $(g \circ f)$ is completely $\mathcal{I}_{g\delta s}$-continuous.

(2) Let G be a $\mathcal{I}_{g\delta s}$-open set in Z. Since g is $\mathcal{I}_{g\delta s}$-irresolute, $g^{-1}(G)$ is $\mathcal{I}_{g\delta s}$-open in Y. Since f is completely $\mathcal{I}_{g\delta s}$-continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is regular open in (X, τ). Hence $(g \circ f)$ is completely $\mathcal{I}_{g\delta s}$-continuous.

(3) Let G be a $\mathcal{I}_{g\delta s}$-open set in Z. As g is perfectly $\mathcal{I}_{g\delta s}$-continuous, $g^{-1}(G)$ is clopen and hence $\mathcal{I}_{g\delta s}$-open in Y. Again since f is completely $\mathcal{I}_{g\delta s}$-continuous, $f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$ is regular open in (X, τ). Hence $(g \circ f)$ is completely $\mathcal{I}_{g\delta s}$-continuous.

5. Contra $\mathcal{I}_{g\delta s}$-continuity

Definition 5.1. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be contra $\mathcal{I}_{g\delta s}$-continuous if $f^{-1}(V)$ is $\mathcal{I}_{g\delta s}$-closed in $(X, \tau, \mathcal{I})$ for each open set V in (Y, σ).

Definition 5.2. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be contra $\mathcal{I}_{rg}$-continuous if $f^{-1}(V)$ is $\mathcal{I}_{rg}$-closed in $(X, \tau, \mathcal{I})$ for each open set V in (Y, σ).

Proposition 5.3. *Every contra g-continuous function is contra $\mathcal{I}_{g\delta s}$-continuous.*

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a contra g-continuous function and let V be any open set in Y. Then, $f^{-1}(V)$ is g-closed in X. Since every g-closed set is $\mathcal{I}_{g\delta s}$-closed, $f^{-1}(V)$ is $\mathcal{I}_{g\delta s}$-closed in X. Therefore f is contra $\mathcal{I}_{g\delta s}$-continuous.

However, converse need not true as seen from the following example.

Example 5.4. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{b, c, d\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra $\mathcal{I}_{g\delta s}$-continuous but not contra g-continuous.

Remark 5.5. The following example shows that $\mathcal{I}_{g\delta s}$-continuity and contra $\mathcal{I}_{g\delta s}$-continuity are independent.

Example 5.6. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\sigma = \{\emptyset, \{b\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra $\mathcal{I}_{g\delta s}$-continuous but not $\mathcal{I}_{g\delta s}$-continuous. The function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ defined by $f(a) = c$, $f(b) = a$ and $f(c) = b$ is $\mathcal{I}_{g\delta s}$-continuous but not contra $\mathcal{I}_{g\delta s}$-continuous.

Proposition 5.7. *Every contra $\mathcal{I}_{g\delta s}$-continuous function is contra $\mathcal{I}_{rg}$-continuous.*

Proof. The proof follows from the fact that every $\mathcal{I}_{g\delta s}$-closed set is $\mathcal{I}_{rg}$-closed in X.

Example 5.8. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, $\sigma = \{\emptyset, \{b\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra Irgcontinuous but not contra $\mathcal{I}_{g\delta s}$-continuous.

Definition 5.9. A map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called contra $*$-continuous if the inverse image of every open set in (Y, σ) is $*$-closed in $(X, \tau, \mathcal{I})$.

Proposition 5.10. *Every contra $*$-continuous function is contra $\mathcal{I}_{g\delta s}$-continuous.*

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a contra $*$-continuous function and let V be any open set in Y. Then, $f^{-1}(V)$ is $*$-closed in X. Since every $*$-closed set is $\mathcal{I}_{g\delta s}$-closed, $f^{-1}(V)$ is $\mathcal{I}_{g\delta s}$-closed in X.

However, converse need not true as seen from the following example.

Example 5.11. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\sigma = \{\emptyset, \{b\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra $\mathcal{I}_{g\delta s}$-continuous but not contra $*$-continuous.

Theorem 5.12. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function. Then the following are equivalent*

(1) *f is contra $\mathcal{I}_{g\delta s}$-continuous.*

(2) *The inverse image of each closed set in Y is $\mathcal{I}_{g\delta s}$-open in X.*

(3) *For each point x in X and each closed set V in Y with $f(x) \in V$, there is an $\mathcal{I}_{g\delta s}$-open set U in X containing x such that $f(U) \subset V$.*

Proof.

(1) $\Rightarrow$ (2). Let F be closed in Y. Then $Y - F$ is open in Y. By definition of contra $\mathcal{I}_{g\delta s}$-continuous, $f^{-1}(Y - F)$ is $\mathcal{I}_{g\delta s}$-closed in X. But $f^{-1}(Y - F) = X - f^{-1}(F)$. This implies $f^{-1}(F)$ is $\mathcal{I}_{g\delta s}$-open in X.

(2) $\Rightarrow$ (3). Let $x \in X$ and V be any closed set in Y with $f(x) \in V$. By (2), $f^{-1}(V)$ is $\mathcal{I}_{g\delta s}$-open in X. Set $U = f^{-1}(V)$. Then there is an $\mathcal{I}_{g\delta s}$-open set U in X containing x such that $f(U) \subset V$.

(3) $\Rightarrow$ (1). Let $x \in X$ and V be any closed set in Y with $f(x) \in V$. Then $Y - V$ is open in Y with $f(x) \in V$. By (3), there is an $\mathcal{I}_{g\delta s}$-open set U in X containing x such that $f(U) \subset V$. This implies $U = f^{-1}(V)$. Therefore, $X - U = X - f^{-1}(V) = f^{-1}(Y - V)$ which is $\mathcal{I}_{g\delta s}$-closed in X.

Theorem 5.13. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \mu)$. Then the following properties hold*

(1) *If f is contra $\mathcal{I}_{g\delta s}$-continuous and g is continuous then $g \circ f$ is contra $\mathcal{I}_{g\delta s}$-continuous.*

(2) *If f is contra* $\mathcal{I}_{g\delta s}$*-continuous and g is contra continuous then* $g \circ f$ *is* $\mathcal{I}_{g\delta s}$*-continuous.*

(3) *If f is* $\mathcal{I}_{g\delta s}$*-continuous and g is contra continuous then* $g \circ f$ *is contra* $\mathcal{I}_{g\delta s}$*-continuous.*

Proof.

(1) Let V be a closed set in Z. Since g is continuous, $g^{-1}(V)$ is closed in Y. Since f is contra $\mathcal{I}_{g\delta s}$-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\mathcal{I}_{g\delta s}$-open in X. Therefore $g \circ f$ is contra $\mathcal{I}_{g\delta s}$-continuous.

(2) Let V be any closed set in Z. Since g is contra continuous, $g^{-1}(V)$ is open in Y. Since f is contra $\mathcal{I}_{g\delta s}$-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\mathcal{I}_{g\delta s}$-closed in X. Therefore $g \circ f$ is $\mathcal{I}_{g\delta s}$-continuous.

(3) Let V be any closed set in Z. Since g is contra continuous, $g^{-1}(V)$ is open in Y. Since f is $\mathcal{I}_{g\delta s}$-continuous, $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is $\mathcal{I}_{g\delta s}$-open in X. Therefore $g \circ f$ is contra $\mathcal{I}_{g\delta s}$-continuous.

Definition 5.14. A space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}_{g\delta s}$-space if every $\mathcal{I}_{g\delta s}$-open set is $*$-open in $(X, \tau, \mathcal{I})$.

Theorem 5.15. *A function* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is contra* $\mathcal{I}_{g\delta s}$*-continuous and X is an* $\mathcal{I}_{g\delta s}$*-space then f is contra* $*$*-continuous.*

Proof. Let V be a closed set in Y. Since f is contra $\mathcal{I}_{g\delta s}$-continuous, $f^{-1}(V)$ is $\mathcal{I}_{g\delta s}$-open in X. Since X is an $\mathcal{I}_{g\delta s}$-space, $f^{-1}(V)$ is $*$-open in X. Therefore f is contra $*$-continuous.

Theorem 5.16. *If* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ *is a contra* $\mathcal{I}_{g\delta s}$*-continuous, closed injection and Y is Ultra normal, then* $(X, \tau, \mathcal{I})$ *is* $\mathcal{I}_{g\delta s}$*-normal.*

Proof. Let F_1 and F_2 be disjoint closed subsets of X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since Y is Ultra normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2 respectively. Hence $F_i \subset f^{-1}(V_i), f^{-1}(V_i) \in \mathcal{I}_{g\delta s}O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus X is $\mathcal{I}_{g\delta s}$-normal.

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