Advances and Applications in Mathematical Sciences
Volume 19, Issue 10, August 2020, Pages 1029-1045

EINSTEIN'S FIELD EQUATIONS WITHIN CONFORMABLE FRACTIONAL DERIVATIVE

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Abstract

In the present paper we obtained the conformable fractional curvature tensor, conformable fractional Riemann tensor and study some of its properties such as symmetry, anti-symmetry and cyclic property. Also we get conformable fractional Ricci tensor and curvature invariant. Lastly by involving conformable fractional derivative Einstein field equations are obtained. Illustrative example is presented.

1. Introduction

The general relativity is the most successful theory of gravitation [1]. There are so many generalizations had been made in the theory of general relativity. A. Einstein [2] generalized the Relativistic theory of gravitation. J. Munkhammar [3] makes the metric complex. Einstein's [4] in the year 1915

2010 Mathematics Subject Classification: 53B20, 53C21 and 26A33.
Keywords: conformable fractional curvature tensor, conformable fractional Riemann tensor, conformable fractional Ricci tensor, curvature invariant, Einstein field equations.
Received November 26, 2019; Accepted August 2, 2020

first obtained field equations in which the energy momentum tensor added on the right hand side. Einstein's field equation describes the space time geometry resulting from the presence of mass, energy and linear momentum. Faisal A. Y. Abdelmohssin [5] modified the Einstein's field equations by introducing general function of Ricci's scalar. In Recent years Fractional calculus has been applied in relativity and cosmology. R. A. EL-Nabulsi [6, 7] has constructed matter field theory in a fractional theory of space with fractional extra dimension and studied Nottale scale relativity and cantorian fractal space time. Using Einstein's Hilbert action of Fractional order V. K. Schigolev [8] derived the cosmological models of a scalar field with dynamical equations containing fractional derivative. Joakim Munkhammar [9] have generalized Einstein's field equation within Riemann-Liouville fractional generalization of ordinary derivative. A. R. El-Nabulsi [10] obtained non-local fractional Einstein's field equations based on fractional derivative inside the geodesic fractional integral. Alireza K. Golmakhaneh et al. [11] have obtained Einstein's Field equations with in local fractional derivative. Recently Khalial et al. [12] have obtained limit based new definition of fractional derivative called as conformable fractional derivative. These new derivative satisfies the derivative of constant is zero, product rule, Quotient rule of two functions and satisfies chain rule [13] similar to the standard derivative. The definition of conformable fractional derivative is simple and more efficient. So it is applied to generalize fractional relativity and cosmology. W. S. Chung [14] has obtained fractional Euler-Lagrange equation involving conformable fractional derivative. Matheus J. Lazo [15] has formulated a fractional action principle. Recently Pawar et al. [16] have studied Riemannian Geometry within conformable fractional derivative. The paper is organised as follows. In section 2 we review the some necessary definitions, formulae and the results on the conformable fractional derivative. In section 3 we give the definitions of conformable fractional Christoffel index symbols and some of its properties. In section 4 we obtained conformable fractional curvature tensor and prove its properties. In section 5 we get conformable fractional Riemann tensor and studied its properties. Conformable fractional Ricci tensor and curvature invariant are obtained in section 6. Lastly by involving conformable fractional derivative Einstein's field equations are obtained in section 7. Section 8 is devoted to our conclusion.

2. A Review of Conformable Fractional Derivative

In this section we review the conformable fractional derivative and its properties. Also we give the conformable fractional derivatives of some standard functions.

2.1. Conformable Fractional Derivative

Suppose $f : [0, \infty) \to R$, the conformable fractional derivative of f of order α is defined as

$$D_\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \text{ for all } t > 0,\ \alpha \in (0, 1]. \tag{1}$$

If the conformable fractional derivative of f of order α exists then we say that f is α differentiable.

2.2. Properties of Conformable Fractional Derivative

Let $\alpha \in (0, 1]$ and f, g be α differentiable functions at point $t > 0$, then

1. $D_\alpha(af + bg) = aD_\alpha(f) + bD_\alpha(g)$

2. $D_\alpha(\lambda) = 0$ for all constant function $f(t) = \lambda$

3. $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$

4. $D_\alpha\left(\frac{f}{g}\right) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$

5. $D_\alpha(t^p) = pt^{p-\alpha}$

6. If f be α differentiable then $D_\alpha(f(t)) = t^{1-\alpha}\frac{df}{dt}$.

2.3. Conformable Fractional Derivative of Certain Functions

The conformable fractional derivatives of some standard functions [20] are

1. $D_\alpha(e^{at}) = at^{1-\alpha}e^{at}, \forall$ real a

2. $D_\alpha(\sin at) = at^{1-\alpha}\cos at, \forall$ real a

3. $D_\alpha(\cos ax) = -at^{1-\alpha} \sin at, \ \forall$ real a

4. $D_\alpha(\tan at) = -at^{1-\alpha} \sec^2 at, \ \forall$ real a

5. $D_\alpha(\cot at) = -at^{1-\alpha} \text{cosec}^2 at, \ \forall$ real a

6. $D_\alpha(\sec at) = at^{1-\alpha} \sec at \cdot \tan at, \ \forall$ real a

7. $D_\alpha(\text{cosec}\, at) = -at^{1-\alpha} \text{cosec}\, at \cdot \cot at, \ \forall$ real a

8. $D_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$

9. $D_\alpha\left(\sin \frac{1}{\alpha}t^\alpha\right) = \cos \frac{1}{\alpha} t^\alpha$

10. $D_\alpha\left(\cos \frac{1}{\alpha}t^\alpha\right) = -\sin \frac{1}{\alpha} t^\alpha$

3. Conformable Fractional Christoffel Index Symbols

In this section we give the definitions of conformable fractional christoffel index symbols of first and second kind and study its properties.

3.1. Conformable Fractional Christoffel Index Symbol of First Kind

Consider a fractional Riemannian manifold (M^α, g^α) and a chart; we can define the conformable fractional christoffel index symbol of first kind as

$$[ij, k]^\alpha = \frac{1}{2}\left\{\frac{\partial^\alpha(g_{ik})}{\partial x^{j\alpha}} + \frac{\partial^\alpha(g_{ik})}{\partial x^{i\alpha}} - \frac{\partial^\alpha(g_{ij})}{\partial x^{k\alpha}}\right\} \tag{2}$$

$$i, j, k = 0, 1, \ldots, N, \ 0 < \alpha \leq 1.$$

Where $[ij, k]^\alpha$, g_{ij} and α are called fractional index symbol of first kind, fractional fundamental metric tensor and fractal dimension respectively.

3.2. Conformable Fractional Christoffel Index Symbol of Second Kind

The Christoffel index symbol of the second kind is defined on fractional

Riemannian manifold (M^{α}, g^{α}) and a given chart as

$$^{\alpha}\Gamma_{jk}^{i} = g^{is}[jk, s]^{\alpha} = \frac{1}{2} g^{is}\left\{\frac{\partial^{\alpha}(g_{is})}{\partial x^{k\alpha}} + \frac{\partial^{\alpha}(g_{ks})}{\partial x^{j\alpha}} - \frac{\partial^{\alpha}(g_{jk})}{\partial x^{s\alpha}}\right\}$$

$$i, j, k = 0, 1, \ldots N, 0 < \alpha \leq 1. \tag{3}$$

3.3. Properties of Conformable Fractional Christoffel Index Symbols

1. The conformable fractional Christoffel index symbols are symmetric in i and j

$$[ij, k]^{\alpha} = [ji, k]^{\alpha} \text{ and } ^{\alpha}\Gamma_{jk}^{i} =^{\alpha} \Gamma_{jk}^{i} \tag{4}$$

2. $[ij, k]^{\alpha} = g_{ks}{}^{\alpha}\Gamma_{ij}^{s}$ (5)

3. The conformable fractional derivative of the metric tensor can be expressed as sum of conformable fractional Christoffel index symbol of first kind, i.e.

$$\frac{\partial^{\alpha}(g_{ij})}{\partial x^{k\alpha}} = [ik, j]^{\alpha} + [jk, i]^{\alpha} \tag{6}$$

4. $\dfrac{\partial^{\alpha}(g^{ij})}{\partial x^{k\alpha}} = -g^{is}\ {}^{\alpha}\Gamma_{sk}^{j} - g^{js}\ {}^{\alpha}\Gamma_{sk}^{i}$ (7)

5. Contraction of the conformable fractional Christoffel index symbol of the second kind

$$^{\alpha}\Gamma_{ij}^{i} = \frac{\partial^{\alpha}}{\partial x^{j\alpha}}(\ln\sqrt{g}) = \frac{\partial^{\alpha}}{\partial x^{j\alpha}}(\ln\sqrt{-g}). \tag{8}$$

4. Conformable Fractional Curvature Tensor and its Properties

In this section we obtained conformable fractional tensor and study its symmetric and cyclic properties.

4.1. Conformable Fractional Curvature Tensor

Let T_r be an arbitrary vector field, then the conformable fractional covariant derivative of T_r is given by

$$^{\alpha}T_{r;m} = \frac{\partial^{\alpha} T_r}{\partial x^{m\alpha}} -^{\alpha} \Gamma^{p}_{rm} T_p \tag{9}$$

which is contra-variant tensor of second order

$$({}^{\alpha}T_{r,m})_n = \frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left(\frac{\partial^{\alpha} T_r}{\partial x^{m\alpha}} - {}^{\alpha}\Gamma^{p}_{rm} T_p\right) - {}^{\alpha}\Gamma^{q}_{rn}\left(\frac{\partial^{\alpha} T_q}{\partial x^{m\alpha}} -^{\alpha} \Gamma^{p}_{qm} T_p\right)$$

$$- {}^{\alpha}\Gamma^{q}_{mn}\left(\frac{\partial^{\alpha} T_r}{\partial x^{q\alpha}} - {}^{\alpha}\Gamma^{p}_{rq} T_p\right)$$

$$\therefore ({}^{\alpha}T_r)_{mn} = \frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left(\frac{\partial^{\alpha} T_r}{\partial x^{m\alpha}}\right) - \frac{\partial^{\alpha}}{\partial x^{n\alpha}}({}^{\alpha}\Gamma^{p}_{rm})T_p - {}^{\alpha}\Gamma^{q}_{rm}\frac{\partial^{\alpha} T_p}{\partial x^{n\alpha}} - {}^{\alpha}\Gamma^{q}_{rn}\frac{\partial^{\alpha} T_q}{\partial x^{m\alpha}}$$

$$+ {}^{\alpha}\Gamma^{q}_{rn}\,{}^{\alpha}\Gamma^{p}_{qm} T_p - {}^{\alpha}\Gamma^{q}_{mn}\frac{\partial^{\alpha} T_r}{\partial x^{q\alpha}} + {}^{\alpha}\Gamma^{q}_{mn}\,{}^{\alpha}\Gamma^{p}_{rq} T_p. \tag{10}$$

Interchanging m and n in Equation (11), we get

$$\therefore ({}^{\alpha}T_r)_{nm} = \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left(\frac{\partial^{\alpha} T_r}{\partial x^{n\alpha}}\right) - \frac{\partial^{\alpha}}{\partial x^{m\alpha}}({}^{\alpha}\Gamma^{p}_{rn})T_p - {}^{\alpha}\Gamma^{p}_{rn}\frac{\partial^{\alpha} T_p}{\partial x^{m\alpha}} - {}^{\alpha}\Gamma^{q}_{rm}\frac{\partial^{\alpha} T_q}{\partial x^{n\alpha}}$$

$$+ {}^{\alpha}\Gamma^{q}_{rm}\,{}^{\alpha}\Gamma^{p}_{qn} T_p - {}^{\alpha}\Gamma^{q}_{nm}\frac{\partial^{\alpha} T_r}{\partial x^{q\alpha}} + {}^{\alpha}\Gamma^{q}_{nm}\,{}^{\alpha}\Gamma^{p}_{rq}\, T_p. \tag{11}$$

Equation (11)-Equation (12) gives

$$\therefore ({}^{\alpha}T_r)_{mn} - ({}^{\alpha}T_r)_{nm}$$

$$= -\left\{\frac{\partial^{\alpha}}{\partial x^{n\alpha}}({}^{\alpha}\Gamma^{p}_{rm}) - {}^{\alpha}\Gamma^{p}_{rn}\ {}^{\alpha}\Gamma^{p}_{qm} - \frac{\partial^{\alpha}}{\partial x^{m\alpha}}({}^{\alpha}\Gamma^{p}_{rn}) + {}^{\alpha}\Gamma^{q}_{rm}\,{}^{\alpha}\Gamma^{p}_{qn}\right\}T_p$$

$$- {}^{\alpha}\Gamma^{p}_{rm}\frac{\partial^{\alpha} T_p}{\partial x^{n\alpha}} - {}^{\alpha}\Gamma^{p}_{rn}\frac{\partial^{\alpha} T_q}{\partial x^{m\alpha}} + {}^{\alpha}\Gamma^{p}_{rn}\frac{\partial^{\alpha} T_q}{\partial x^{m\alpha}} + {}^{\alpha}\Gamma^{q}_{rm}\frac{\partial^{\alpha} T_q}{\partial x^{n\alpha}}. \tag{12}$$

By changing the dummy suffixes p by q and q by p in the last two terms on RHS of equation (12) we get

$$\therefore ({}^{\alpha}T_r)_{mn} - ({}^{\alpha}T_r)_{nm} = {}^{\alpha}R^{p}_{rmn} T_p \tag{13}$$

Where

$$ {}^{\alpha}R^{p}_{rmn} = -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left({}^{\alpha}\Gamma^{p}_{rm}\right) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left({}^{\alpha}\Gamma^{p}_{rn}\right) + {}^{\alpha}\Gamma^{q}_{rn}\,{}^{\alpha}\Gamma^{q}_{qm} - {}^{\alpha}\Gamma^{q}_{rm}\,{}^{\alpha}\Gamma^{p}_{qn} \tag{14}$$

${}^{\alpha}R^{p}_{rmn}$ is a mixed tensor of order four and it is also called as conformable fractional curvature tensor or conformable fractional Riemann Christoffel curvature tensor of second kind.

4.2. Properties of Conformable Fractional Curvature Tensor

The conformable fractional curvature tensor satisfies following properties

Theorem. *The conformable fractional curvature tensor is anti-symmetric i.e.*

$$ {}^{\alpha}R^{p}_{rmn} = -{}^{\alpha}R^{p}_{rmn}. $$

Proof. Consider

$$ {}^{\alpha}R^{p}_{rmn} + {}^{\alpha}R^{p}_{rnm} = -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left({}^{\alpha}\Gamma^{p}_{rm}\right) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left({}^{\alpha}\Gamma^{p}_{rn}\right) - {}^{\alpha}\Gamma^{p}_{rm}\,{}^{\alpha}\Gamma^{p}_{qn} + {}^{\alpha}\Gamma^{p}_{rn}\,{}^{\alpha}\Gamma^{p}_{qm} $$

$$ -\frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left({}^{\alpha}\Gamma^{p}_{rn}\right) + \frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left({}^{\alpha}\Gamma^{p}_{rm}\right) - {}^{\alpha}\Gamma^{q}_{rn}\,{}^{\alpha}\Gamma^{p}_{qm} + {}^{\alpha}\Gamma^{q}_{rm}\,{}^{\alpha}\Gamma^{p}_{qn} = 0 $$

$\therefore {}^{\alpha}R^{p}_{rmn} = -{}^{\alpha}R^{p}_{rmn}.$

Theorem. ${}^{\alpha}R^{p}_{rmn} = +{}^{\alpha}R^{p}_{mnr} + {}^{\alpha}R^{p}_{nrm} = 0$ *Cyclic property.*

Proof. Consider

$$ {}^{\alpha}R^{p}_{rmn} + {}^{\alpha}R^{p}_{mnr} + {}^{\alpha}R^{p}_{nrm} = -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left({}^{\alpha}\Gamma^{p}_{rm}\right) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left({}^{\alpha}\Gamma^{p}_{rn}\right) $$

$$ - {}^{\alpha}\Gamma^{q}_{rm}\,{}^{\alpha}\Gamma^{q}_{qn} + {}^{\alpha}\Gamma^{q}_{rn}\,{}^{\alpha}\Gamma^{p}_{qm} $$

$$ - \frac{\partial^{\alpha}}{\partial x^{r\alpha}}\left({}^{\alpha}\Gamma^{p}_{mn}\right) + \frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left({}^{\alpha}\Gamma^{p}_{mr}\right) $$

$$ - {}^{\alpha}\Gamma^{q}_{mn}\,{}^{\alpha}\Gamma^{p}_{qr} + {}^{\alpha}\Gamma^{q}_{mr}\,{}^{\alpha}\Gamma^{q}_{qn} $$

$$ - \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left({}^{\alpha}\Gamma^{p}_{nr}\right) + \frac{\partial^{\alpha}}{\partial x^{r\alpha}}\left({}^{\alpha}\Gamma^{p}_{nm}\right) $$

$$- {}^{\alpha}\Gamma^{q}_{nr}\,{}^{\alpha}\Gamma^{p}_{qm} + {}^{\alpha}\Gamma^{q}_{nm}\,{}^{\alpha}\Gamma^{q}_{qr} = 0$$

$$\therefore {}^{\alpha}R^{p}_{rmn} + {}^{\alpha}R^{p}_{mnr} + {}^{\alpha}R^{p}_{nrm} = 0.$$

5. Conformable Fractional Riemann Tensor and its Properties

In this section we obtained a conformable fractional Riemann tensor and study its properties like symmetry, anti-symmetry and cyclic property.

5.1. Conformable fractional Riemann tensor

The conformable fractional Riemann tensor is denoted by ${}^{\alpha}R_{prmn}$ and obtained from ${}^{\alpha}R^{p}_{rmn}$ lowering the suffixes by g_{mn} i.e.

$${}^{\alpha}R_{prmn} = g_{ps} + {}^{\alpha}R^{s}_{rmn} \tag{15}$$

$$= g_{ps}\left\{\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left({}^{\alpha}\Gamma^{s}_{rm}\right) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left({}^{\alpha}\Gamma^{s}_{rn}\right) - {}^{\alpha}\Gamma^{q}_{rm}\,{}^{\alpha}\Gamma^{s}_{qn}\,{}^{\alpha}\Gamma^{q}_{rn}\,{}^{\alpha}\Gamma^{s}_{qm}\right\}$$

$$= -g_{ps}\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left({}^{\alpha}\Gamma^{s}_{rm}\right) - {}^{\alpha}\Gamma^{s}_{rm}\frac{\partial^{\alpha}}{\partial x^{n\alpha}}(g_{ps}) + {}^{\alpha}\Gamma^{q}_{rm}\frac{\partial^{\alpha}}{\partial x^{n\alpha}}(g_{ps})$$

$$+ g_{ps}\frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left({}^{\alpha}\Gamma^{s}_{rn}\right) + {}^{\alpha}\Gamma^{s}_{rn}\frac{\partial^{\alpha}}{\partial x^{m\alpha}}(g_{ps})$$

$$- {}^{\alpha}\Gamma^{q}_{rn}\frac{\partial^{\alpha}}{\partial x^{m\alpha}}(g_{ps}) - g_{ps}\,{}^{\alpha}\Gamma^{q}_{rm}\,{}^{\alpha}\Gamma^{s}_{qn} + g_{ps}\,{}^{\alpha}\Gamma^{q}_{rn}\,{}^{\alpha}\Gamma^{s}_{qm}$$

$$= -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left([rm,\ p]^{\alpha}\right) + {}^{\alpha}\Gamma^{s}_{rm}[pn,\ s]^{\alpha} + {}^{\alpha}\Gamma^{s}_{rm}[sn,\ p]^{\alpha}$$

$$+ \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left([rn,\ p]^{\alpha}\right) - {}^{\alpha}\Gamma^{s}_{rn}[pm,\ s]^{\alpha} - {}^{\alpha}\Gamma^{s}_{rn}[sm,\ p]^{\alpha}$$

$$- {}^{\alpha}\Gamma^{q}_{rm}[qn,\ p]^{\alpha} + {}^{\alpha}\Gamma^{s}_{rn}[qm,\ p]^{\alpha}. \tag{16}$$

Replace q by s in the last two terms on RHS of equation (16), we get

$${}^{\alpha}R_{prmn} = -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left([rm,\ p]^{\alpha}\right) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left([rn,\ p]^{\alpha}\right)$$

$$+{}^{\alpha}\,\Gamma^{s}_{rm}[pn,\ s]^{\alpha} - {}^{\alpha}\Gamma^{s}_{rn}[pm,\ s]^{\alpha}. \tag{17}$$

Equation (17) gives the expression for conformable fractional Riemann tensor.

5.2. Properties of Conformable Fractional Riemann Tensor

The conformable fractional Riemann tensor satisfies following properties

Theorem. *The conformable fractional Riemann tensor is anti-symmetric in first two suffixes*, *i.e.*,

$$^{\alpha}R_{prmn} = -^{\alpha}R_{rpmn}.$$

Proof. Consider

$$\begin{aligned}
^{\alpha}R_{prmn} + {}^{\alpha}R_{rpmn} &= -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}([rm,\ p]^{\alpha}) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}([rn,\ p]^{\alpha}) \\
&+^{\alpha}\Gamma^{s}_{rm}[pn,\ s]^{\alpha} - {}^{\alpha}\Gamma^{s}_{rn}[pm,\ s]^{\alpha} \\
&-\frac{\partial^{\alpha}}{\partial x^{n\alpha}}([pm,\ r]^{\alpha}) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}([pn,\ r]^{\alpha}) + {}^{\alpha}\Gamma^{s}_{pm}[rn,\ s]^{\alpha} \\
&- {}^{\alpha}\Gamma^{s}_{pn}[rm,\ s]^{\alpha} \\
&= -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left\{\frac{1}{2}\left(\frac{\partial^{\alpha}g_{rp}}{\partial x^{n\alpha}} + \frac{\partial^{\alpha}g_{mp}}{\partial x^{r\alpha}} - \frac{\partial^{\alpha}g_{rm}}{\partial x^{p\alpha}}\right)\right\} \\
&+ \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left\{\frac{1}{2}\left(\frac{\partial^{\alpha}g_{rp}}{\partial x^{n\alpha}} + \frac{\partial^{\alpha}g_{np}}{\partial x^{r\alpha}} - \frac{\partial^{\alpha}g_{rn}}{\partial x^{p\alpha}}\right)\right\} \\
&+ g^{sq}[rm,\ q]^{\alpha}[pn,\ s]^{\alpha} - g^{sq}[rn,\ q]^{\alpha}[pm,\ s]^{\alpha} \\
&- \frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left\{\frac{1}{2}\left(\frac{\partial^{\alpha}g_{pr}}{\partial x^{m\alpha}} + \frac{\partial^{\alpha}g_{mr}}{\partial x^{p\alpha}} - \frac{\partial^{\alpha}g_{pm}}{\partial x^{r\alpha}}\right)\right\} \\
&+ \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left\{\frac{1}{2}\left(\frac{\partial^{\alpha}g_{pr}}{\partial x^{n\alpha}} + \frac{\partial^{\alpha}g_{nr}}{\partial x^{p\alpha}} - \frac{\partial^{\alpha}g_{pm}}{\partial x^{r\alpha}}\right)\right\} \\
&+ g^{sq}[pm,\ q]^{\alpha}[rn,\ s]^{\alpha} - g^{sq}[pn,\ q]^{\alpha}[rm,\ s]^{\alpha} \\
&= -\frac{1}{2}\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left(\frac{\partial^{\alpha}g_{rp}}{\partial x^{m\alpha}}\right) - \frac{1}{2}\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\left(\frac{\partial^{\alpha}g_{mp}}{\partial x^{r\alpha}}\right)
\end{aligned}$$

$$+ \frac{1}{2} \frac{\partial^\alpha}{\partial x^{n\alpha}} \left(\frac{\partial^\alpha g_{rm}}{\partial x^{p\alpha}} \right) + \frac{1}{2} \frac{\partial^\alpha}{\partial x^{m\alpha}} \left(\frac{\partial^\alpha g_{rp}}{\partial x^{n\alpha}} \right)$$

$$+ \frac{1}{2} \frac{\partial^\alpha}{\partial x^{m\alpha}} \left(\frac{\partial^\alpha g_{np}}{\partial x^{r\alpha}} \right) - \frac{1}{2} \frac{\partial^\alpha}{\partial x^{m\alpha}} \left(\frac{\partial^\alpha g_{rn}}{\partial x^{p\alpha}} \right)$$

$$+ g^{sq}[rn, q]^\alpha [pn, s]^\alpha - g^{sq}[rn, q]^\alpha [pm, s]^\alpha$$

$$- \frac{1}{2} \frac{\partial^\alpha}{\partial x^{n\alpha}} \left(\frac{\partial^\alpha g_{pr}}{\partial x^{m\alpha}} \right) - \frac{1}{2} \frac{\partial^\alpha}{\partial x^{n\alpha}} \left(\frac{\partial^\alpha g_{mr}}{\partial x^{p\alpha}} \right) + \frac{1}{2} \frac{\partial^\alpha}{\partial x^{n\alpha}} \left(\frac{\partial^\alpha g_{pm}}{\partial x^{r\alpha}} \right)$$

$$+ \frac{1}{2} \frac{\partial^\alpha}{\partial x^{m\alpha}} \left(\frac{\partial^\alpha g_{pr}}{\partial x^{n\alpha}} \right)$$

$$+ \frac{1}{2} \frac{\partial^\alpha}{\partial x^{m\alpha}} \left(\frac{\partial^\alpha g_{nr}}{\partial x^{p\alpha}} \right) - \frac{1}{2} \frac{\partial^\alpha}{\partial x^{m\alpha}} \left(\frac{\partial^\alpha g_{pn}}{\partial x^{r\alpha}} \right) + g^{sq}[pm, q]^\alpha [rn, s]^\alpha$$

$$- g^{sq}[pn, q]^\alpha [rm, s]^\alpha$$

$$= g^{sq}[rm, q]^\alpha [pn, s]^\alpha - g^{sq}[rn, q]^\alpha [pm, s]^\alpha$$

$$+ g^{sq}[pm, q]^\alpha [rn, s]^\alpha$$

$$- g^{sq}[pn, q]^\alpha [rm, q]^\alpha.$$

Replacing q by s and s by q in last two terms we get

$$^\alpha R_{prmn} + {}^\alpha R_{rpmn} = 0$$

$$\therefore {}^\alpha R_{prmn} = - {}^\alpha R_{rpmn}.$$

Theorem. *The conformable fractional Riemann tensor is antisymmetric in last two suffixes i.e.*

$$^\alpha R_{prmn} = - {}^\alpha R_{prnm}.$$

Proof. Consider

$$^\alpha R_{prmn} + {}^\alpha R_{prmn} = - \frac{\partial^\alpha}{\partial x^{n\alpha}} ([rm, p]^\alpha) + \frac{\partial^\alpha}{\partial x^{m\alpha}} ([rn, p]^\alpha)$$

$$+ {}^\alpha\Gamma^s_{rm}[pn, s]^\alpha - {}^\alpha\Gamma^s_{rn}[pm, s]^\alpha$$

$$- \frac{\partial^\alpha}{\partial x^{m\alpha}}([rn, p]^\alpha) + \frac{\partial^\alpha}{\partial x^{n\alpha}}([rm, p]^\alpha)$$

$$+ {}^\alpha\Gamma^s_{rn}[pm, s]^\alpha - {}^\alpha\Gamma^s_{rm}[pn, s]^\alpha = 0$$

$\therefore {}^\alpha R_{prmn} = - {}^\alpha R_{prnm}.$

Theorem. *The conformable fractional Riemann tensor is pair wise symmetric i.e.*

$${}^\alpha R_{prmn} = - {}^\alpha R_{mnpr}.$$

Proof. Consider

$$\begin{aligned}
{}^\alpha R_{prmn} - {}^\alpha R_{rpmn} &= \Bigg(-\frac{\partial^\alpha}{\partial x^{n\alpha}}([rm, p]^\alpha) + \frac{\partial^\alpha}{\partial x^{m\alpha}}([rn, p]^\alpha) \\
&\quad + {}^\alpha\Gamma^s_{rm}[pn, s]^\alpha - {}^\alpha\Gamma^s_{rm}[pm, s]^\alpha \Bigg) \\
&\quad - \Bigg(-\frac{\partial^\alpha}{\partial x^{r\alpha}}([np, m]^\alpha) + \frac{\partial^\alpha}{\partial x^{p\alpha}}([nr, m]^\alpha) \\
&\quad + {}^\alpha\Gamma^s_{np}[mr, s]^\alpha - {}^\alpha\Gamma^s_{nr}[mp, s]^\alpha \Bigg) \\
&= -\frac{\partial^\alpha}{\partial x^{n\alpha}}\left\{\frac{1}{2}\left(\frac{\partial^\alpha g_{rp}}{\partial x^{m\alpha}} + \frac{\partial^\alpha g_{mp}}{\partial x^{r\alpha}} - \frac{\partial^\alpha g_{rm}}{\partial x^{p\alpha}}\right)\right\} \\
&\quad + \frac{\partial^\alpha}{\partial x^{m\alpha}}\left\{\frac{1}{2}\left(\frac{\partial^\alpha g_{rp}}{\partial x^{n\alpha}} + \frac{\partial^\alpha g_{np}}{\partial x^{r\alpha}} - \frac{\partial^\alpha g_{rn}}{\partial x^{p\alpha}}\right)\right\} \\
&\quad + \frac{\partial^\alpha}{\partial x^{r\alpha}}\left\{\frac{1}{2}\left(\frac{\partial^\alpha g_{nm}}{\partial x^{p\alpha}} + \frac{\partial^\alpha g_{pm}}{\partial x^{n\alpha}} - \frac{\partial^\alpha g_{np}}{\partial x^{m\alpha}}\right)\right\} \\
&\quad - \frac{\partial^\alpha}{\partial x^{p\alpha}}\left\{\frac{1}{2}\left(\frac{\partial^\alpha g_{nm}}{\partial x^{r\alpha}} + \frac{\partial^\alpha g_{rm}}{\partial x^{n\alpha}} - \frac{\partial^\alpha g_{nr}}{\partial x^{m\alpha}}\right)\right\} \\
&\quad + g^{sq}[rm, q]^\alpha[pn, s]^\alpha - g^{sq}[np, q]^\alpha[mr, s]^\alpha \\
&= g^{sq}[rm, q]^\alpha[pn, s]^\alpha - g^{sq}[np, q]^\alpha[mr, s]^\alpha.
\end{aligned}$$

Replacing q by s and s by q in last term we get

$$^{\alpha}R_{prmn} - {}^{\alpha}R_{mnpr} = 0$$

$\therefore {}^{\alpha}R_{prmn} = {}^{\alpha}R_{mnpr}.$

Theorem. $^{\alpha}R_{prmn} + {}^{\alpha}R_{pmnr} + {}^{\alpha}R_{pnrm} = 0.$ *Cyclic property.*

Proof. Consider

$$\begin{aligned} {}^{\alpha}R_{prmn} + {}^{\alpha}R_{prmm} + {}^{\alpha}R_{pnrm} &= -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}([rm,\ p]^{\alpha}) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}([rn,\ p]^{\alpha}) \\ &+ {}^{\alpha}\Gamma^{s}_{rm}[pn,\ s]^{\alpha} - {}^{\alpha}\Gamma^{s}_{rn}[pm,\ s]^{\alpha} \\ &- \frac{\partial^{\alpha}}{\partial x^{r\alpha}}([mn,\ p]^{\alpha}) \frac{\partial^{\alpha}}{\partial x^{n\alpha}}([mr,\ p]^{\alpha}) \\ &+ {}^{\alpha}\Gamma^{s}_{mn}[pr,\ s]^{\alpha} - {}^{\alpha}\Gamma^{s}_{mr}[pn,\ s]^{\alpha} \\ &- \frac{\partial^{\alpha}}{\partial x^{m\alpha}}([nr,\ p]^{\alpha}) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}([nm,\ p]^{\alpha}) \\ &+ {}^{\alpha}\Gamma^{s}_{nr}[pm,\ s]^{\alpha} - {}^{\alpha}\Gamma^{s}_{nm}[pr,\ s]^{\alpha} = 0 \\ &\therefore {}^{\alpha}R_{prmn} + {}^{\alpha}R_{pmnr} + {}^{\alpha}R_{pnrm} = 0. \end{aligned}$$

6. Conformable Fractional Ricci Tensor and Curvature Invariant

In this section we obtained conformable fractional Ricci tensor and curvature invariant.

6.1. Conformable Fractional Ricci Tensor. The contraction of $^{\alpha}R^{p}_{rmn}$ which is non zero is called as Ricci tensor it is denoted by $^{\alpha}R_{rm}$

$$\therefore {}^{\alpha}R_{rm} = {}^{\alpha}R^{n}_{rmn}. \tag{18}$$

$$= -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}({}^{\alpha}\Gamma^{n}_{rm}) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}({}^{\alpha}\Gamma^{n}_{rn}) - {}^{\alpha}\Gamma^{q}_{rm}\,{}^{\alpha}\Gamma^{n}_{qn} + {}^{\alpha}\Gamma^{q}_{rn}\,{}^{\alpha}\Gamma^{n}_{qm}$$

$$= -\frac{\partial^{\alpha}}{\partial x^{n\alpha}}({}^{\alpha}\Gamma^{n}_{rm}) + \frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left(\frac{\partial^{\alpha}}{\partial x^{r\alpha}}\ln\sqrt{-g}\right) - {}^{\alpha}\Gamma^{q}_{rm}\frac{\partial^{\alpha}}{\partial x^{q\alpha}}\ln\sqrt{-g} + {}^{\alpha}\Gamma^{q}_{rn}\,{}^{\alpha}\Gamma^{n}_{qm}. \quad (19)$$

Equation (21) gives the expression for conformable fractional Ricci tensor.

Theorem. *The conformable fractional Ricci tensor is symmetric.*

Proof. We have

$$\begin{aligned}\therefore {}^{\alpha}R_{rm} = g^{pn}\,{}^{\alpha}R_{prmn} &= g^{pn}\,{}^{\alpha}R_{mnpr}\\ &= -g^{pn}\,{}^{\alpha}R_{mnrp}\\ &= g^{np}\,{}^{\alpha}R_{nmrp}\\ &= {}^{\alpha}R_{mr}.\end{aligned}$$

6.2. Conformable Fractional curvature invariant. The conformable fractional curvature invariant is denoted R^{α} and it is given by

$$R^{\alpha} = g^{mn}\,{}^{\alpha}R_{mn} \quad (20)$$

$$= -g^{mn}\frac{\partial^{\alpha}}{\partial x^{p\alpha}}({}^{\alpha}\Gamma^{p}_{mn}) + g^{mn}\frac{\partial^{\alpha}}{\partial x^{m\alpha}}\left(\frac{\partial^{\alpha}}{\partial x^{n\alpha}}\ln\sqrt{-g}\right) - g^{mn}\,{}^{\alpha}\Gamma^{q}_{mn}\frac{\partial^{\alpha}}{\partial x^{q\alpha}}\ln\sqrt{-g}$$

$$+ g^{mn}\,{}^{\alpha}\Gamma^{q}_{pn}\,{}^{\alpha}\Gamma^{p}_{qm}. \quad (21)$$

Equation (23) gives the expression for conformable fractional curvature invariant.

7. Conformable Fractional Einstein's Field Equation

First we define the conformable fractional Einstein tensor as following

$${}^{\alpha}G_{mn} = {}^{\alpha}R_{mn} - \frac{1}{2}R^{\alpha}g_{mn}. \quad (22)$$

Where ${}^{\alpha}R_{mn}$, R^{α} and g_{mn} are conformable fractional Ricci tensor, curvature invariant and metric tensor respectively it is also written as in the following forms.

$$^{\alpha}G^{mn} = {}^{\alpha}R^{mn} - \frac{1}{2}R^{\alpha}g^{mn},\ {}^{\alpha}G_n^m = {}^{\alpha}R_n^m - \frac{1}{2}R^{\alpha}\delta_n^m. \tag{23}$$

Now we define the conformable fractional Einstein field equations as

$$^{\alpha}G_{mn} + g_{mn}P = {}^{\alpha}T_{mn}Q, \tag{24}$$

where P and Q are conformable fractional space constants.

Einstein's field equations are used to determine the space time geometry resulting from the presence of mass energy and linear momentum.

Example. Consider the fractional line element $(d^{\alpha}s)^2 = -A^2\{(d^{\alpha}x)^2 + (d^{\alpha}y)^2\} - B^2(d^{\alpha}z)^2 + (d^{\alpha}t)^2$ where A and B are α differential functions of time t.

The elements of conformable fractional metric tensors are

$$g_{11} = -A^2,\ g_{22} = -A^2,\ g_{33} = -B^2,\ g_{44} = 1. \tag{25}$$

The reciprocal conformable fractional metric tensors are

$$g^{11} = -\frac{1}{A^2},\ g^{22} = -\frac{1}{A^2},\ g^{33} = -\frac{1}{B^2}\ g^{44} = 1. \tag{26}$$

The non-zero conformal fractional Christoffel symbols are obtained using equation (3) as

$$^{\alpha}\Gamma_{41}^{1} = {}^{\alpha}\Gamma_{41}^{1}\frac{(x^4)^{1-\alpha}A_4}{A} \quad {}^{\alpha}\Gamma_{11}^{4} = (x^4)^{1-\alpha}AA_4 \quad {}^{\alpha}\Gamma_{22}^{4} = (x^4)^{1-\alpha}AA_4$$

$$^{\alpha}\Gamma_{24}^{2} = {}^{\alpha}\Gamma_{42}^{2}\frac{(x^4)^{1-\alpha}A_4}{A} \quad {}^{\alpha}\Gamma_{33}^{4} = (x^4)^{1-\alpha}BB_4 \quad {}^{\alpha}\Gamma_{43}^{3} = {}^{\alpha}\Gamma_{34}^{3}\frac{(x^4)^{1-\alpha}B_4}{B} \tag{27}$$

$$g = |\ g_{mn}\ | = A^4B^2 \quad \therefore \sqrt{-g} = A^2B. \tag{28}$$

In view of equation (21) the conformable fractional Ricci Tensors are

$$\begin{aligned} ^{\alpha}R_{11} = {} & -(1-\alpha)(x^4)^{1-2\alpha}AA_4(x^4)^{2-2\alpha}AA_{44} \\ & -(x^4)^{2-2\alpha}(A_4)^2 - (x^4)^{2-2}\frac{AA_4B_4}{B} \end{aligned} \tag{29}$$

$$
\begin{aligned}
{}^{\alpha}R_{22} &= -(1-\alpha)(x^4)^{1-2\alpha}AA_4 - (x^4)^{2-2\alpha}AA_{44} \\
&\quad - (x^4)^{2-2\alpha}(A_4)^2 - (x^4)^{2-2\alpha}\frac{AA_4B_4}{B}
\end{aligned} \tag{30}
$$

$$
{}^{\alpha}R_{33} = -(1-\alpha)(x^4)^{1-2\alpha}BB_4 - (x^4)^{2-2\alpha}BB_{44} - 2(x^4)^{2-2\alpha}\frac{BA_4B}{A} \tag{31}
$$

$$
\begin{aligned}
{}^{\alpha}R_{33} &= 2(1-\alpha)(x^4)^{1-2\alpha}A_4 + 2(x^4)^{2-2\alpha}\frac{A_{44}}{A} - +(1-\alpha)(x^4)^{1-2\alpha}\frac{B_4}{B} \\
&\quad + (x^4)^{2-2\alpha}\frac{B_{44}}{B}.
\end{aligned} \tag{32}
$$

Using equation (22) the conformable fractional curvature invariant is

$$
\begin{aligned}
R^{\alpha} &= 4(1-\alpha)(x^4)^{1-2\alpha}\frac{A_4}{A} + 4(x^4)^{2-2\alpha}\frac{A_{44}}{A} + 2(x^4)^{2-2\alpha}\frac{(A_4)^2}{A^2} \\
&\quad + 4(x^4)^{2-2\alpha}\frac{A_4B_4}{AB} + 2(1-\alpha)(x^4)^{1-2\alpha}\frac{B_4}{B} + 2(x^4)^{2-2\alpha}\frac{B_{44}}{B}.
\end{aligned} \tag{33}
$$

The conformable fractional Einstein tensors are obtained using equation (24) as

$$
\begin{aligned}
{}^{\alpha}G_{11} &= (1-\alpha)(x^4)^{1-2\alpha}AA_4 + (x^4)^{2-2\alpha}AA_{44} + (x^4)^{2-2\alpha}\frac{AA_4B_4}{B} \\
&\quad + (1-\alpha)(x^4)^{1-2\alpha}\frac{A^2B_4}{B} + (x^4)^{2-2\alpha}\frac{A^2B_{44}}{B}
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
{}^{\alpha}G_{22} &= (1-\alpha)(x^4)^{1-2\alpha}AA_4 + (x^4)^{2-2\alpha}AA_{44} + (x^4)^{2-2\alpha}\frac{AA_4B_4}{B} \\
&\quad + (1-\alpha)(x^4)^{1-2\alpha}\frac{A^2B_4}{B} + (x^4)^{2-2\alpha}\frac{A^2B_{44}}{B}
\end{aligned} \tag{35}
$$

$$
{}^{\alpha}G_{33} = 2(1-\alpha)(x^4)^{1-2\alpha}\frac{A_4B^2}{A} + 2(x^4)^{2-2\alpha}\frac{B^2A_{44}}{A} + 2(x^4)^{2-2\alpha}\frac{(A_4)^2B^2}{A^2} \tag{36}
$$

$$
{}^{\alpha}G_{44} = -(x^4)^{2-2\alpha}\frac{(A_4)^2}{A^2} - 2(x^4)^{2-2\alpha}\frac{A_4B_4}{AB}. \tag{37}
$$

All the result in this example will lead to standard results by choosing $\alpha = 1$.

8. Conclusions

In this work, we obtained curvature and Riemann tensor involving conformable fractional derivative and studied their properties. Conformable fractional Ricci tensors and curvature invariant are defined on fractal space and obtained it in terms of conformable fractional derivative. Einstein's Field equations are suggested and studied using the line element involving conformable fractional derivative.

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