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MAXIMUM DEGREE ENERGY OF THE NON-COMMUTING GRAPHS ASSOCIATED TO THE DIHEDRAL GROUPS

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Abstract

The maximum degree energy $E_M(\Gamma)$ of a simple graph has been defined by Adiga and Smitha [2] as the summation of the absolute values of the maximum degree eigenvalues (the eigenvalues of the maximum degree matrix) of the graph Γ. In this research, the maximum degree energy of the non-commuting graphs associated to the dihedral groups of order $2n$ has been studied. In this paper, exact formulas of the characteristic polynomial, maximum degree eigenvalues and the maximum degree energy of the non-commuting graphs of D_{2n} have been found. Furthermore, the relation between the maximum degree energy and the energy of this graph has been obtained.

1. Introduction

A graph Γ is an ordered pair (V, E) consisting of a set $V(\Gamma)$ of vertices and a set of edges. The cardinalities $|V(\Gamma)| = n$ and $|E(\Gamma)| = m$ often denoted to the number of vertices and the number of edges in the graph, respectively. In graph theory, the study of energy of graphs is a very

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interesting topic because it joins the theory of graph with more than one branches of science. The energy of a graph was first defined by Gutman [9] as the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph. The study in [9] was motivated by the approximation of the total - electron energy of molecules [10] in chemistry. The definition of the energy of a graph is stated as in the following:

Definition 1.1 [4]**.** Let Γ be a simple graph, A be its adjacency matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the graph Γ. By eigenvalues of the graph Γ would mean the eigenvalues of its adjacency matrix. The energy of the graph Γ, $\varepsilon(\Gamma)$, is defined as the sum of the absolute values of its eigenvalues i.e.

$$\varepsilon(\Gamma) = \sum_{i=1}^{n} | \lambda_i |.$$

Motivated by the study on energy of graph, Adiga and Smitha [2] defined a new matrix of graph called the maximum degree matrix of graph and according to this matrix the definition of maximum degree energy of graph is stated as follows:

Definition 1.2 [2]**.** Let Γ be a simple graph with n vertices $v_1, v_2, \ldots v_n$ and let $d_i = \deg(v_i)$ be the degree of v_i, $i = 1, 2, \ldots, n$. The maximum degree matrix of the graph Γ is defined by $M(\Gamma) = [d_{ij}]$, where

$$d_{ij} = \begin{cases} \max\{d_i, d_j\} & \text{if } v_i, v_j \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of the maximum degree matrix $M(\Gamma)$ is defined by

$$\beta(\Gamma; \zeta) = \zeta^n + c_1\zeta^{n-1} + \ldots + c_0,$$

where I is the unit matrix of order n. The roots $\zeta_1, \zeta_2, \ldots, \zeta_n$ assumed in non-increasing order of $\beta(\Gamma; \zeta) = 0$ are the maximum degree eigenvalues of Γ. The maximum degree energy of a graph Γ is defined as $E_M(\Gamma) = \sum_{i=1}^{n} | \zeta_i |$.

Also it is stated that the maximum degree energies of certain classes of graphs are less than the maximum degree energy of the complete graph K_n.

The following results taken from [2] are used in this paper.

Theorem 1.3 [2]. *If is the complete graph* K_n, *then* $-(n-1)$ *and* $-(n-1)^2$ *are maximum degree eigenvalues of* Γ *with multiplicity* $(n-1)$ *and 1, respectively, and* $E_M(K_n) = (n-1)^2$.

Theorem 1.4 [2]. *If the maximum degree energy of a graph is rational, then it must be an even integer*.

The main aim of this paper is to obtain general formulas for the maximum degree energy of the commuting and non-commuting graphs related to the dihedral groups D_{2n}. The definition of the dihedral groups and some properties of groups in general are provided next.

Definition 1.5 [8]. Let be a finite group. The center, $Z(G)$, of a group is the subset of elements of that commute with every element in G. In symbols, $Z(G) = \{a \in G \mid xa = ax, \forall x \in G\}$.

Definition 1.6 [6]. The dihedral group, denoted by D_{2n}, is a group of symmetries of a regular polygon, which include rotations and reactions, and its order is $2n$ where n is an

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Integer and $n \geq 3$. The dihedral groups can be presented in a form of generators and relations given as, $D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e \rangle$.

Proposition 1.7 [6]. *Let be a dihedral group of order* $2n$ *i.e*, $D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e \rangle$. *Then, the center of is as follows*:

$$Z(D_{2n}) = \begin{cases} \{e, a^{\frac{n}{2}}\}, & \text{if } n \text{ is even;} \\ \{e\}, & \text{if } n \text{ is odd.} \end{cases}$$

Recently, the study of algebraic structures, using the properties of graphs, becomes an exciting research topic which is leading to many fascinating results and questions. The concept of commuting graph has been studied in [5], where as in [1], Abdollahi et al. stated that the non-commuting graph was first considered by Paul Erdos.

Next, the definition of the non-commuting graph related to a finite group is stated.

Definition 1.8 [1]**.** Let G be a finite non-abelian group with the center denoted by $Z(G)$. A non-commuting graph of a group G is a simple undirected graph whose vertices are non-central elements of G i.e. $G - Z(G)$. Two vertices v_1 and v_2 are adjacent whenever $v_1 v_2 \neq v_2 v_1$, and is denoted by Γ_G^{ncom}.

In the next, the non-commuting graph associated to the dihedral groups and its maximum degree energy have been discussed.

2. Maximum Degree Energy of the Non-commuting Graph of D_{2n}

In this section, the non-commuting graph of D_{2n} is derived and the maximum degree characteristic polynomial and the maximum degree eigenvalues of the non-commuting graph of D_{2n}, Γ_G^{ncom}, are obtained. Moreover, the exact formula of the maximum degree energy of D_{2n} has been found.

The following proposition is on the non-commuting graph of D_{2n}.

Proposition 2.1. *Let* D_{2n} *be a dihedral group, where* $D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e \rangle$.

Then the non-commuting graph of D_{2n}, $\Gamma_{D_{2n}}^{ncom}$, *is one of the following forms.*

(i) *If n is odd, then* $\Gamma_{D_{2n}}^{ncom} = \overline{K_{n-1}} \vee K_n$, *which is a connected bi-regular graph with vertex of degree* $2n - 2$ *and* $n - 1$ *vertices of degree n.*

(ii) *If n is even, then* $\Gamma_{D_{2n}}^{ncom}$, *forms a connected bi-regular graph with vertex of degree* $2n - 4$ *and* $n - 1$ *vertices of degree n.*

Proof. Suppose D_{2n} is a dihedral group and $\Gamma_{D_{2n}}^{ncom}$ is its non-commuting graph. D_{2n} has two types of elements, a^i and a^ib with $i = 0, 1, 2, \ldots, n-1$. Then

(i) If is odd, then by Proposition 1.7, $Z(D_{2n}) = \{e\}$. Therefore, $\Gamma_{D_{2n}}^{ncom}$ has number of vertices equals $(2n-1)$, where $(n-1)$ of them having form a^i with $i = 1, 2, \ldots, n-1$ which pairwise commute to each other, so they are not adjacent between themselves (i.e. they form $\overline{K_{n-1}}$).

On the other hand, the remaining n vertex of $\Gamma_{D_{2n}}^{ncom}$ having the form a^ib with $i = 0, 1, 2, \ldots, n-1$. which are pairwise non-commuting (so they are pairwise adjacent and form K_n) and also they do not commute with the vertices a^i (thus each one of them is adjacent to all a^i's). Hence from the definition of the join of two graphs, $\Gamma_{D_{2n}}^{ncom} = \overline{K_{n-1}} \vee K_n$ which is a connected bi-regular graph with $n-1$. vertex of degree n and n vertices of full degree $2n-2$.

(ii) If is even, then again from Proposition 1.7, $Z(D_{2n}) = \{e, a^{n/2}\}$. Thus, $\Gamma_{D_{2n}}^{ncom}$ has number of vertices equals $(2n-2)$ in which $(n-2)$ vertex of type a^i with $i = 1, 2, \ldots, n-1, i \neq \frac{n}{2}$ and n vertices of type a^ib with $i = 0, 1, 2, \ldots n-1$. As mentioned before, the vertices are pairwise commuting hence they are not adjacent between themselves. Now, each vertex of type a^ib with $i = 0, 1, 2, \ldots, n-1$. is non-commuting with all the other vertices of $\Gamma_{D_{2n}}^{ncom}$ except one vertex which is $a^{\frac{n}{2}+i}b$ with $i = 0, 1, 2, \ldots, n-1$. Hence, $\Gamma_{D_{2n}}^{ncom}$ forms a connected bi-regular graph with $n-2$ vertex having degree and vertices of degree $\Gamma_{D_{2n}}^{ncom} - 4$

Next, Lemma 2.2 gives the maximum degree characteristic polynomial and the maximum degree eigenvalues of the non-commuting graph associated to the dihedral groups D_{2n} when n is an odd integer.

Lemma 2.2. *Let* $G = D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e \rangle$ *be a dihedral group of order* $2n$ *with* $n \geq 3$ *is an odd integer, and* $\Gamma_{D_{2n}}^{ncom}$ *be its non-commuting graph. Then the characteristic polynomial of the maximum degree matrix of* $\Gamma_{D_{2n}}^{ncom}$ *is as follows,*

$$\beta(\Gamma_{D_{2n}}^{ncom}; \zeta) = \zeta^{n-2}(\zeta + 2(n-1))^{n-1}(\zeta^2 - 2(n-1)^2\zeta - 4n(n-1)^3),$$

and hence the maximum degree eigenvalues of $\Gamma_{D_{2n}}^{ncom}$ *are* $\zeta = 0$ *with multiplicity* $(n-2)$, $\zeta = -2(n-1)$ *with multiplicity* $(n-1)$ *and* $\zeta = (n-1)^2 \pm (n-1)\sqrt{5n^2 - 6n + 1}$.

Proof. Suppose that $G = D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e \rangle$ is a dihedral group of order $2n$. Since n is odd, then by Proposition 1.7, $Z(D_{2n}) = \{e\}$. Therefore, $\Gamma_{D_{2n}}^{ncom}$ has $2n-1$ vertices. Thus, the maximum degree matrix of $\Gamma_{D_{2n}}^{ncom}$ is a $(2n-1) \times (2n-1)$ matrix and the degree of its characteristic polynomial is $(2n-1)$.

As mentioned in the proof of Proposition 2.1, D_{2n} has two types of elements a^i and $a^i b$ with $i = 0, 1, 2, \ldots, n-1$. The elements a^i are pairwise commuting and $a^i b$ are pairwise non-commuting and also non-commuting with the elements a^i in D_{2n} except the identity element. Thus, by the definition of the non-commuting graph, $\Gamma_{D_{2n}}^{ncom}$ has $n-1$ vertex of type a^i with degree n and n vertex of type $a^i b$ with degree $2n-2$.

Now, since $n \geq 3$, then the maximum degree between any adjacent pair of vertices is $2n-2$. Hence, the maximum degree matrix of $\Gamma_{D_{2n}}^{ncom}$ is

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$$M(\Gamma_{D_{2n}}^{ncom}) = \begin{bmatrix} \Theta_{(n-1)\times(n-1)} & \Lambda_{(n-1)\times n}^{T} \\ \Lambda_{(n-1)\times n}^{T} & \Omega_{n\times n} \end{bmatrix},$$

Where $\Theta_{(n-1)\times(n-1)}$ is the $(n-1)\times(n-1)$ zero matrix, $\Lambda_{(n-1)\times n}^{T}$ is an $(n-1)\times n$ matrix in which all the entries are equal $(2n-2)$ and $\Omega_{n\times n}$ is an $(n\times n)$ matrix in which all the diagonal entries are zeros and all other entries are equal $(2n-2)$. Assisted by Maple software, the maximum degree characteristic polynomial of $\Gamma_{D_{2n}}^{ncom}$ is obtained as follows:

$$\beta(\Gamma_{D_{2n}}^{ncom}; \zeta) = \zeta^{n-2}(\zeta + 2(n-1))^{n-1}(\zeta^2 - 2(n-1)^2\zeta - 4n(n-1)^3),$$

and hence the maximum degree eigenvalues of $\Gamma_{D_{2n}}^{ncom}$ are $\zeta = 0$ with multiplicity $(n-2)$, $\zeta = -2(n-1)$ with multiplicity $(n-1)$ and $\zeta = (n-1)^2 \pm (n-1)\sqrt{5n^2 - 6n + 1}$.

Now, Lemma 2.3 gives the maximum degree characteristic polynomial and the maximum degree eigenvalues of the non-commuting graph associated to the dihedral groups D_{2n} when n is an even integer.

Lemma 2.3. *Let $G = D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e\rangle$ be a dihedral group of order $2n$ with $n \geq 4$ is an even integer, and $\Gamma_{D_{2n}}^{ncom}$ be its non-commuting graph. Then the characteristic polynomial of the maximum degree matrix of $\Gamma_{D_{2n}}^{ncom}$ is as follows,*

$$\beta(\Gamma_{D_{2n}}^{ncom}; \zeta) = \zeta^{\frac{3(n-2)}{2}}(\zeta + 4(n-2))^{\frac{n}{2}-1}(\zeta^2 - 2(n-2)^2\zeta - 4n(n-2)^3),$$

and hence the maximum degree eigenvalues of $\Gamma_{D_{2n}}^{ncom}$ are $\zeta = 0$ with multiplicity $\frac{3(n-2)}{2}$, $\zeta = -4(n-1)$ with multiplicity $\frac{n}{2} - 1$ and

$\zeta = (n-2)^2 \pm (n-2)\sqrt{5n^2 - 12n + 4}.$

Proof. Consider $G = D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e \rangle$ to be a dihedral group of order $2n$. Since n is even, then by Proposition 1.7, $Z(D_{2n}) = \{e, a^{n/2}\}$. Thus, $\Gamma_{D_{2n}}^{ncom}$ has number of vertices equals $(2n-2)$. Therefore, the maximum degree matrix of $\Gamma_{D_{2n}}^{ncom}$ is a $(2n-2)\times(2n-2)$ matrix and the degree of its characteristic polynomial is $(2n-2)$. By the definition of the non-commuting graph, the vertices a^i with $i = 1, 2, \ldots, n-1, i \neq \frac{n}{2}$ are pairwise commuting, then they are not adjacent between themselves, and each vertex of type $a^i b$ with $i = 0, 1, 2, \ldots, n-1$ is non-commuting with all the other vertices of $\Gamma_{D_{2n}}^{ncom}$ (of both types a^i and $a^i b$) except only one vertex which is $a^{\frac{n}{2}+i}b$. Hence, $\Gamma_{D_{2n}}^{ncom}$ has $(n-2)$ vertex of type a^i with degree n, and n vertices of type $a^i b$ degrees $(2n-4)$. Since $n \geq 4$, then the maximum degree between each adjacent pair of vertices is $(2n-4)$. Therefore, the maximum degree matrix of $\Gamma_{D_{2n}}^{ncom}$ in this case takes the form

$$M(\Gamma_{D_{2n}}^{ncom}) = \begin{bmatrix} \Theta_{(n-1)\times(n-1)} & \Lambda_{(n-1)\times n} \\ \Lambda_{(n-1)\times n}^T & \Omega_{n\times n} \end{bmatrix},$$

where $\Theta_{(n-2)\times(n-2)}$ is the $(n-2)\times(n-2)$ zero matrix, $\Lambda_{(n-1)\times n}$ is an $(n-2)\times n$ matrix in which all the entries are equal $(2n-4)$ and $\Omega_{n\times n}$ is an $(n \times n)$ matrix in which all the diagonal entries are zeros and the other entries are equal $(2n-4)$ or zero according to the adjacency of the corresponding pair of vertices. Since the rank of $M(\Gamma_{D_{2n}}^{ncom})$ (recall that is an even integer) equals $\left(\frac{n}{2}+1\right)$, then there exist $\left(\frac{n}{2}+1\right)$, non-zero maximum degree eigenvalues of $\Gamma^{ncom}_{D_{2n}}$. Thus, there are $(2n-2)-\left(\frac{n}{2}+1\right)$ zero maximum degree eigenvalues of $\Gamma^{ncom}_{D_{2n}}$. This implies that, $\zeta = 0$ with multiplicity $\frac{3(n-2)}{2}$, $\zeta = -(n-1)$ with multiplicity $\frac{n}{2}-1$ and $\zeta = (n-2)^2 \pm (n-2)\sqrt{5n^2-12n+4}$. Hence,

$$\beta(\Gamma^{ncom}_{D_{2n}};\zeta) = \zeta^{\frac{3(n-2)}{2}}(\zeta+4(n-2))^{\frac{n}{2}-1}(\zeta^2-2(n-2)^2\zeta-4n(n-2)^3).$$

According to Lemma 2.2 and Lemma 2.3, the exact formula of the maximum degree energy of the non-commuting graph associated to the dihedral groups is obtained as in the following.

Theorem 2.4. *Let* $G = D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e\rangle$ *be a dihedral group of order* $2n$ *with* $n \geq 3$ *is an integer, and* $\Gamma^{ncom}_{D_{2n}}$. *be its non-commuting graph. Then the maximum degree energy of* $\Gamma^{ncom}_{D_{2n}}$ *is*

$$E_M(\Gamma^{ncom}_{D_{2n}}) = \begin{cases} 2(n-1)[n-1+\sqrt{5n^2-6n+1}], & \text{if } n \text{ is odd;} \\ 2(n-2)[n-1+\sqrt{5n^2-12n+4}] & \text{if } n \text{ is even.} \end{cases}$$

Proof. The proof is divided into two cases:

Case 1. Suppose n is odd. Then by Lemma 2.2, the maximum degree eigenvalues of the non-commuting graph $\Gamma^{ncom}_{D_{2n}}$ of the dihedral groups D_{2n}, where $n \geq 3$ is an odd integer are $\zeta = 0$ with multiplicity $(n-2)$, $\zeta = -2(n-1)$ with multiplicity $(n-1)$ and $\zeta = (n-1)^2 \pm (n-1)\sqrt{5n^2-6n+1}$. Therefore, by Definition 1.2, the maximum degree energy of the $\Gamma^{ncom}_{D_{2n}}$ when is an odd integer is

$$\begin{aligned} E_M(\Gamma^{ncom}_{D_{2n}}) &= (n-1)|\ -2(n-1)\,| + |\ (n-1)^2 \pm (n-1)\sqrt{5n^2-6n+1}\ | \\ &= 2(n-1)^2 + 2(n-1)\sqrt{5n^2-6n+1} \end{aligned}$$

$$= 2(n-1)[n-1+\sqrt{5n^2-6n+1}].$$

Case 2. Suppose n is even. Then by Lemma 2.3, the maximum degree eigenvalues of the non-commuting graph $\Gamma_{D_{2n}}^{ncom}$ of the dihedral groups D_{2n}, where $n \geq 4$ is an even integer are $\zeta = 0$ with multiplicity $\frac{3(n-2)}{2}$, $\zeta = -4(n-1)$ with multiplicity $\frac{n}{2} - 1$ and $\zeta = (n-2)^2 \pm (n-2)\sqrt{5n^2 - 12n + 4}$. Thus by Definition 1.2, the maximum degree energy of the $\Gamma_{D_{2n}}^{ncom}$ when n is an even integer is

$$E_M(\Gamma_{D_{2n}}^{ncom}) = \left(\frac{n}{2} - 1\right)| -4(n-2)| + |(n-1)^2 \pm (n-2)\sqrt{5n^2 - 12n + 4}|$$

$$= 2(n-2)^2 + 2(n-2)\sqrt{5n^2 - 12n + 4}$$

$$= 2(n-2)[n-2+\sqrt{5n^2-12n+4}].$$

In the following corollary, the relation between the maximum degree energy and the energy of the non-commuting graph associated to the dihedral groups D_{2n} is obtained.

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Corollary 2.5. *Let* $G = D_{2n} \cong \langle ab \mid a^n = b^2 = baba = e\rangle$ *be a dihedral group of order* $2n$ *with* $n \geq 3$ *is an integer, and* $\Gamma_{D_{2n}}^{ncom}$ *be its non-commuting graph. Then*

(i) *If n is odd, then* $E_M(\Gamma_{D_{2n}}^{ncom}) = 2(n-1)\varepsilon(\Gamma_{D_{2n}}^{ncom})$.

(ii) *If n is even, then* $E_M(\Gamma_{D_{2n}}^{ncom}) = 2(n-2)\varepsilon(\Gamma_{D_{2n}}^{ncom})$.

3. Conclusion

In this paper, the non-commuting graph associated to the dihedral groups is provided. In addition, general forms of the maximum degree matrix, the maximum degree characteristic polynomial and the maximum degree eigenvalues of this graph are found. Furthermore, exact formula of the maximum degree energy of the non - commuting graph of the dihedral groups is formulated. Finally, the relation between the energy and the maximum degree energy of the non-commuting graph is obtained.

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References

[1] A. Abdollahi, S. Akbari and H. Maimani, Non-commuting graph of a group, Journal of Algebra 298(2) (2006), 468-492.

[2] C. Adiga and M. Smitha, On maximum degree energy of a graph, Int. J. Contemp. Math. Sciences 4(8) (2009), 385-396.

[3] R. Balakrishnan, The energy of a graph, Linear Algebra and its Applications 387 (2004), 287-295.

[4] R. B. Bapat, Graphs and matrices, Springer (2010), 27.

[5] D. Bundy, The connectivity of commuting graphs, J. Comb. Theory 113(6) (2006), 995-1007.

[6] D. S. Dummit and R. M. Foote, Abstract algebra, Wiley Hoboken, 2004.

[7] Faisal Ali, Muhammad Salman and Shuliang Huang, On the commuting graph of dihedral group, Communications in Algebra 44(6) (2016), 2389-2401.

[8] J. Gallian, Contemporary abstract algebra, Nelson Education, 2012.

[9] I. Gutman, The energy of a graph, Der. Math. stat. Sekt. Forschungszent Graz., 103 (1978), 1-22.

[10] H. Ramaswamy and C. Veena, On the energy of unitary Cayley graphs, The Electronic Journal of Combinatorics 16(1) (2009), N24.

[11] J. Vahidi and A. A. Talebi, The commuting graphs on groups D_{2n} and Q_n, J. Math. Comput. Sci. 1(2) (2010), 123-127.