Advances and Applications in Mathematical Sciences
Volume 21, Issue 1, November 2021, Pages 405-419

PERFORMANCE ANALYSIS OF M/G/1 QUEUING MODEL WITH TWO TYPES OF SERVICE, FEEDBACK, AND SERVER BREAKDOWN

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Abstract

This paper deals with steady state analysis of $M/G/1$ retrial queuing model with two types of service, feedback, and modified vacation with random server break down. For each customer two types of service are available. Each customer can choose type 1 service with probability p_1 or type 2 service with probability p_2. Many queuing situations have the feature that the customers may be served repeatedly for a certain reason. When the service of a customer is unsatisfied, it may be retried again and again until a successful service is completed. These queuing models arise in stochastic modelling of many real-life situations. Queuing system with server breakdown is very common in communication systems and manufacturing systems. The steady state distribution of the server state and the number of customer in the orbit/system are obtained. While the server working with any type of service, it may break down at any instant and the service channel will fail for a short interval of time and it is repaired immediately. Numerical example is presented to illustrate the influence of the parameters on several performance characteristics.

1. Introduction

In a retrial queue, an arriving customer who finds the server busy has to leave the service area and may retry later. Such queues, model many real world situations including web access, call centers, telecommunication networks and computer systems. Jau-Chuanke et al. [3] first introduced the modified vacation policy in retrial queues. Queuing systems with server breakdowns are very common in communication systems and manufacturing

2020 Mathematics Subject Classification: 60K25, 68M20.
Keywords: Retrial queuing, Modified vacation, Feedback, Server breakdown.
Received May 17, 2021; Accepted June 7, 2021

systems. Single server queuing systems with server breakdowns and vacations have been studied by Peishu Chen et al. [7]. P. Rajadurai et al. [9, 10] dealt with $M/G/1$ retrial queue with two types of service, modified vacations, and server breakdowns.

Queuing systems with vacation time and server breakdowns have been found to be useful in modelling the systems in which the server has additional tasks. Single server queuing systems with server breakdowns and Bernoulli vacation have been studied by many researchers including Li et al. [6], Wang et al. [12] and Peishu Chen et al. [7]. Priyanka Kalita et al. [8] analyzed single server queue with modified vacation policy. A comprehensive study on the vacation models can be found in Takagi [11]. Recent papers on queuing theory are Artalejo, J. R., [2] and Le, Q [6], B. K. Kumar, and D. Arivudainambi [4, 5]. Amina Angelika et al. analyzed feedback multiple vacation queuing system with differentiated vacation, vacation interruptions and impatient customers.

2. The System

New customers arrive from outside the system according to a Poisson process with rate λ.

We assume that there is no waiting space and therefore if an arriving customer finds the server free, the arrival begins his service. The customers from the orbit try to request his service later and the inter-retrial times have an arbitrary distribution $A(x)$ with corresponding Laplace-Stieltjes transform (LST) $\widetilde{A}(\theta)$.

There is a single server which provides two types of service. If an arriving customer finds the server free, then he choose first type of service (FTS) with probability p_1 or choose second type of service (STS) with probability $p_2 (p_1 + p_2 = 1)$. It is assumed that the service $S_i (i = 1, 2)$ of the i^{th} type of service follows a random variable with distribution function $S_i(x)$ and Laplace Stieljes transform $\widetilde{S}_i(\theta)$ and nth factorial moments sin.

As soon as a customer served completely he will decide either to join the orbit for another service with probability p or to leave the system with probability q When the server is busy it fails at an exponential rate β is assumed. When the server fails, it is repaired immediately and the customer just being served before server breakdown waits for the server until repair completion in order to accomplish its remaining service. The repair time is a random variable with probability distribution function $R_i(x)$ with corresponding Laplace Stieltjes transform (LST) $\tilde{R}_i(\theta)$ and n^{th} factorial moments r_{in}.

Whenever the orbit is empty, the server leaves for a vacation of random length V. if no customers appear in the orbit when the server returns from the vacation; he again leaves for another vacation of same length. This pattern continues until he returns from a vacation to find at least one customer recorded in the orbit or he has already taken J vacations. If the orbit is empty by the end of j^{th} vacation, the server remains idle for customers in the system. At a vacation completion epoch the orbit is nonempty, the server waits for the customers, if any in the orbit, or for new customers to arrive. The vacation time V has distribution function $V(x)$ corresponding Laplace Stieltjes transform (LST) $\tilde{V}(\theta)$ Various Stochastic processes involved in the system are assumed to be independent of each other.

3. System Analysis

Let $A^0(t), S_i^0(t), R_i^0(t)$ and $V^0(t)$ be the elapsed retrial time, service time, repair time and vacation time respectively at time t. The state of the system at time t can be described by the Bivariate Markov process $\{C(t),\ N(t); t \geq 0\}$ where $C(t)$ denotes the server state $(0, 1, 2, 3, 4, \ldots, J + 4)$ depending if the service is idle, busy on FTS or STS, repair on FTS or STS, I vacation,… and j^{th} vacation and $N(t)$ denotes the number of customers in the orbit. $C(t)$ The function $\theta(x),\ \mu_i(x),\ \gamma_i(y),\ \Omega(x)$ are the conditional completion rates for repeated attempts, for service, repair and vacation time respectively (for $i = 1, 2)$

$$\theta(x) = \frac{dA(x)}{1 - A(x)},\ \mu_i(x) = \frac{ds_i(x)}{1 - S_i(x)},\ \gamma_i(y) = \frac{dR_i(y)}{1 - R_i(y)},\ \omega(x) = \frac{dV(x)}{1 - V(x)}$$

Let $\{t_n; n = 1, 2, \ldots, \}$ be the sequence of epochs at which either a service period completion occurs or a vacation time ends. The sequence of random vectors $Z_n = \{C(t_n +), N(t_n +)\}$ forms a Markov chain which is embedded in the retrial queuing system.

Theorem 1. *The embedded Markov chain* $\{z_n; n \in N\}$ *is ergodic if and only if* $p - p_1 s_1 \lambda(1 + \beta_1 \gamma_1) - p_2 s_2 \lambda(1 + \beta_2 r_2) < \widetilde{A}(\lambda)$.

For the process $\{N(t), t \geq 0\}$ *we define the probabilities*

$$P_0(t) = P\{C(t) = 1, N(t) = 0\}$$

$$P_n(x, t)dx = P\{C(t) = 0, N(t) = n, x \leq A^0(t) < x + dx\}$$

$$\pi_{1,n}(x, t)dx = P\{C(t) = 1, N(t) = n, x \leq S_1^0(t) < x + dx\}$$

$$\pi_{2,n}(x, t)dx = P\{C(t) = 2, N(t) = n, x \leq S_2^0(t) < x + dx\}$$

$$R_{1,n}(x, y, t)dy = P\{C(t) = 3, N(t) = n, x \leq R_1^0(t) < y + dy / S_1^0(t) = x\}$$

$$R_{2,n}(x, y, t)dy = P\{C(t) = 4, N(t) = n, x \leq R_2^0(t) < y + dy / S_2^0(t) = x\}$$

$$\Omega_{j,n}(x, t)dx = P\{C(t) = j + 4, N(t) = n, x \leq V_J^0(t) < x + dx\}, (1 \leq j \leq J)$$

Under the stability condition, it is assumed that

$$P_0 = \lim_{t \to \infty} P_0(t)$$

$$\pi_{i,n}(x) = \lim_{t \to \infty} \pi_{i,n}(x, t)$$

$$\Omega_{j,n}(x) = \lim_{t \to \infty} \Omega_{j,n}(x, t)$$

$$R_{i,n}(x, y) = \lim_{t \to \infty} R_{i,n}(x, y, t).$$

4. Steady State Equations

The Kolmogrov forward equations which govern the system under steady state condition is obtained as

$$\lambda P_0 = \int_0^\infty \Omega_{j,0}(x)\omega(x)dx \tag{1}$$

$$\frac{dP_n(x)}{dx} + [\lambda + \theta(x)]P_n(x) = 0 \tag{2}$$

$$\frac{d\prod_{i,0}(x)}{dx} + [\lambda + \beta_i + \mu_i(x)]\pi_i(x) = \int_0^\infty \gamma_i(y)\beta_{i,0}(x, y)dy \tag{3}$$

$$\frac{d\prod_{i,n}(x)}{dx} + [\lambda + \beta_i + \mu_i(x)] = \int_0^\infty \gamma_i(y)\beta_{i,n}(x, y)dy \tag{4}$$

$$\frac{d\Omega_{j,0}(x)}{dx} + [\lambda + \omega(x)]\Omega_{j,0}(x) = 0 \tag{5}$$

$$\frac{d\Omega_{j,n}(x)}{dx} + [\lambda + \omega(x)]\Omega_{j,n}(x) = \lambda\Omega_{j,n-1}(x) \tag{6}$$

$$\frac{dB_{i,0}(x, y)}{dx} + [\lambda + \gamma(y)]B_{i,0}(x) = 0 \tag{7}$$

$$\frac{dB_{i,n}(x, y)}{dx} + [\lambda + \gamma(y)]B_{i,n}(x) = \lambda B_{i,n-1}(x). \tag{8}$$

The steady state Boundary conditions at $x = 0$ and $y = 0$ are

$$\begin{aligned} &P_n(0) \\ &= \int_0^\infty \Omega_{1,n}(x)\omega(x)dx + \int_0^\infty \Omega_{2,x}(x)\omega(x)dx + \ldots + \int_0^\infty \Omega_{j,n}(x)\omega(x)dx \\ &+ (1-p)[\int_0^\infty \pi_{1,n-1}(x)\mu_1(x) + \pi_{2,n}(x)dx] \\ &+ p[\int_0^\infty \pi_{1,n-1}(x)\mu_1(x) + \pi_{2,n-1}(x)\mu_2(x)dx] \end{aligned} \tag{9}$$

$$\pi_{i,n}(0) = p_i\left[\int_0^{\infty} P_{n+1}(x)\theta(x)dx + \lambda\int_0^{\infty} P_n(x)dx\right] \text{ where } i = 1, 2 \tag{10}$$

$$\Omega_{j,n}(0) = \int_0^{\infty} \pi_{1,0}(x)\mu_1(x)dx + \int_0^{\infty} \pi_{2,0}(x)\mu_2(x)dx \tag{11}$$

$$\Omega_{j,n}(0) = \int_0^{\infty} \Omega_{j-1}(x)\omega(x)dx \tag{12}$$

$$B_{i,n}(x, 0) = \beta_i\pi_{i,n}(x) \tag{13}$$

and the normalization condition

$$P_0 + \sum_{n=0}^{\infty}\int_0^{\infty} P_n(x)dx + \sum_{n=0}^{\infty}\int_0^{\infty} \pi_1(x)dx + \sum_{n=0}^{\infty} \pi_2(x)dx$$
$$+ \sum_{n=0}^{\infty}\int_0^{\infty} B_1(x, y)dy + \sum_{n=0}^{\infty} B_2(x, y)$$
$$+ \sum_{j=1}^{J}\sum_{n=0}^{\infty}\int_0^{\infty} \Omega_{j,n}(x)dx = 1. \tag{14}$$

Theorem 2. *Under the stability condition,* $p - p_1 s_1\lambda(1 + \beta_1 r_1) - p_2 s_2\lambda(1 + \beta_2 r_2) < \tilde{A}(\lambda)$, *the stationary distributions of the number of customers in the system when the server is free, busy, on vacation and on break down are given by*

$$P(z) = \frac{\begin{array}{l} zP_0[1 - \tilde{A}(\lambda)\{N(z) - 1 + (1 - p + pz)[p_1\tilde{S}_1[\lambda(1-z) + \beta_1 - \beta_1\tilde{R}_1(\lambda(1-z)) + \\ p_2\tilde{S}_2[\lambda(1-z) + \beta_2 - \beta_2\tilde{R}_2(\lambda(1-z))]\} \end{array}}{\begin{array}{l} z - \{(1 - p + pz)[\tilde{A}(\lambda) + z(1 - \tilde{A}(\lambda))][p_1\tilde{S}_1[\lambda(1-z) + \beta_1 - \beta_1\tilde{R}_1(\lambda(1-z)) + \\ p_2\tilde{S}_2[\lambda(1-z) + \beta_2 - \beta_2\tilde{R}_2(\lambda(1-z))]\} \end{array}}$$

$$\pi_1(Z) = \frac{\begin{array}{c} \lambda P_0 p_1\{(N(z) - 1)[\tilde{A}(\lambda) + z(1 - \tilde{A}(\lambda)) + z]\} \\ [1 - \tilde{S}_1(\lambda(1-z) + \beta_1 - \beta_1\tilde{R}_1(\lambda(1-z)))] \end{array}}{\begin{array}{l} z - (1 - p + pz)[\tilde{A}(\lambda) + z(1 - \tilde{A}(\lambda))]p_1\tilde{S}_1[\lambda(1-z) + \beta_1 - \beta_1\tilde{R}_1(\lambda(1-z))] \\ + p_2\tilde{S}_2[\lambda(1-z) + \beta_2 - \beta_2\tilde{R}_2(\lambda(1-z))[\lambda(1-z) + \beta_1 - \beta_1\tilde{R}_1(\lambda(1-z))]] \end{array}}$$

$$\pi_2(Z) = \frac{\lambda P_0 p_2\{(N(z)-1)[\widetilde{A}(\lambda) + z(1-\widetilde{A}(\lambda)) + z]\} \\ [1-\widetilde{S}_2(\lambda(1-z) + \beta_2 - \beta_2\widetilde{R}_2(\lambda(1-z)))]}{z-(1-p+pz)[\widetilde{A}(\lambda) + z(1-\widetilde{A}(\lambda))]p_1\widetilde{S}_1[\lambda(1-z) + \beta_1 - \beta_1\widetilde{R}_1(\lambda(1-z))] \\ + p_2\widetilde{S}_2[\lambda(1-z) + \beta_2 - \beta_2\widetilde{R}_2(\lambda(1-z))[\lambda(1-z) + \beta_1 - \beta_1\widetilde{R}_1(\lambda(1-z))]]}$$

$$B_1(z) = \beta_1\pi_1(z)\frac{\{1-\widetilde{R}_1[\lambda(1-z)]\}}{\lambda(1-z)}$$

$$B_2(z) = \beta_2\pi_2(z)\frac{\{1-\widetilde{R}_2[\lambda(1-z)]\}}{\lambda(1-z)}$$

$$\Omega_j(z) = \frac{P_0[1-\widetilde{V}(\lambda(1-z))]}{(1-z)[\widetilde{V}(\lambda)]^{J-j+1}}$$

$$N(z) = \frac{1-[\widetilde{V}(\lambda)]^J[\widetilde{V}(\lambda(1-z))-1]}{[\widetilde{V}(\lambda)]^J(1-\widetilde{V}(\lambda))}$$

Proof. To solve the above equations, we define the generating functions, for $(i = 1, 2)$

$$P(x, z) = \sum_{n=1}^{\infty} P_n(x)z^n; P(0, z) = \sum_{n=1}^{\infty} P_n(0)z^n$$

$$\prod_i(x, z) = \sum_{n=0}^{\infty} \pi_{i,n}(x)z^n; \prod_i(0, z) = \sum_{n=0}^{\infty} \pi_{i,n}(0)z^n;$$

$$\Omega_j(x, z)\sum_{n=0}^{\infty} \Omega_{j,n}(x)(x)z^n; \Omega_j(0, z)\sum_{n=0}^{\infty} \Omega_{j,n}(0)z^n;$$

$$B_i(x, y, z) = \sum_{n=0}^{\infty} B_{i,n}(x, y)z^n; B_i(x, 0, z) = \sum_{n=0}^{\infty} B_{i,n}(x, 0)z^n$$

Now multiply equation (2) by z^n and summing over $(n = 1, 2, 3, \ldots)$,

$$\frac{dP(x,z)}{dx} + [\lambda + \theta(x)]P(x,z) = 0.$$

Multiply equation (3) by $z, z^2, \ldots$ and sub $n = 1, 2, 3, \ldots$ and summing

$$\frac{d}{dx}\sum_{n=0}^{\infty}\pi_{i,n}(x)z^n + [\lambda + \beta_i + \mu_i(x)]\sum_{n=0}^{\infty}\pi_{i,n}(x)z^n$$

$$= \lambda\sum_{n=1}^{\infty}\pi_{i,n-1}(x)z^n + \sum_{n=0}^{\infty}z^n\int_0^{\infty}\gamma_i(y)\beta_{i,n}(x, y)dy.$$

Multiply equation (4) by $z, z^2, \ldots,$ and sub $n = 1, 2, 3, \ldots,$ and summing

$$\frac{d}{dy}\left(\sum_{n=0}^{\infty}B_{i,n}(x, y)z^n + [\lambda + \gamma_i(y)]\sum_{n=0}^{\infty}B_{i,n}(x)z^n = \lambda\sum_{n=1}^{\infty}B_{i,n}(x)z^n\right).$$

Multiply equation (9) by $z, z^2, \ldots,$ and sub $n = 1, 2, 3, \ldots,$ and summing

$$P(0, z) = \sum_{j=1}^{J}\int_0^{\infty}\Omega_j(x, z)\omega(x)dx - \lambda P_0 - \sum_{j=1}^{J}\Omega_{j,0}(0)$$

$$+ (1 - p + pz)[\int_0^{\infty}\pi_1(x, z)\mu_1(x)dx\int_0^{\infty}\pi_2(x, z)\mu_2(x)dx] \tag{15}$$

Multiply equation (10) by $z, z^2, \ldots,$ and sub $n = 1, 2, 3, \ldots,$ summing to get

$$\pi_i(0, z) = p_i\left[\frac{1}{z}\int_0^{\infty}P(x, z)\theta(x)dx + \left(\int_0^{\infty}P(x, z)dx + P_0\right)\right] \tag{16}$$

Multiply equation (10) by $z, z^2, \ldots,$ and sub $n = 1, 2, 3, \ldots,$ and summing

$$B_i(x, 0, z) = \beta_i\pi_i(x, z) \tag{17}$$

Equation (15) implies

$$\frac{dP(x, z)}{dx} + [\lambda + \theta(x)]P(x, z) = 0 \tag{18}$$

Solving we get,

$$P(x, z) = e^{-\lambda x}[1 - A(x)]P(0, z) \tag{19}$$

Similarly solving (18) and (17) we get

$$B_i(x,\ y,\ z) = B_i(x,\ 0,\ z)[1 - R_i(y)]e^{-\lambda[1-z]y} \tag{20}$$

Equation (20) implies

$$B_i(x,\ 0,\ z) = \beta_i \prod_i (x,\ z)$$

$$B(x,\ y,\ z) = \beta_i \pi_i(x,\ z)[1 - R_i(y)]e^{-\lambda[1-z]y} \tag{21}$$

$$\Omega_j(x,\ z) = \Omega_j(0,\ z)e^{-\lambda(1-z)x}[1 - v(x)] \tag{22}$$

Substituting (21) in (16)

$$\pi_i(x,\ z) = \pi_i(0,\ z) + [1 - S_i(x)]e^{-[\lambda(1-z)+\beta_i\beta_i\widetilde{R}_i[\lambda(1-z)]]x} \tag{23}$$

Therefore partial differential equations are:

$$\left\{\begin{array}{l}
(\mathrm{a})\ P(x, z) = e^{-\lambda x}[1 - A(x)]P(0, z) \\
(\mathrm{b})\ \pi_i(x, z) = \pi_i(0, z) + [1 - S_i(x)]e^{-[\lambda(1-z)+\beta_i-\beta_i\widetilde{R}_i[\lambda(1-z)]]x}\ i = 1, 2 \\
(\mathrm{c})\ \Omega_j(x, z) = \Omega_j(0, z)e^{-\lambda(1-z)x}[1 - v(x)] \\
(\mathrm{d})\ B_i(x, y, z) = \beta_i\pi_i(x\ z)[1 - R_i(y)]e^{\lambda[1-z]y}\ i = 1, 2
\end{array}\right\} \tag{24}$$

Solving the above partial differential equations we get result of the theorem. Since P_0 is the probability that the server is idle when no customer in the orbit and it can determine by using the normalizing condition

$$P_0 + P(1) + \pi_1(1) + \pi_2(1) + \beta_1(1) + \beta_2(1) + \sum_{j=1}^{J} \Omega_j(1) = 1 \tag{25}$$

Thus by setting $z = 1$ in $(A) - (D)$ and applying L-Hospital's rule whenever necessary and we get,

$$P(1) = \frac{P_0[1 - \widetilde{A}(\lambda)\{N'(1) + P + p_1 s_1 \lambda(1 + \beta_1 r_1) + p_2 s_2 \lambda(1 + \beta_2 r_2)\}}{[1 - \{P + [1 - \widetilde{A}(\lambda)] + p_1 s_1 \lambda(1 + \beta_1 r_1) + p_2 s_2 \lambda(1 + \beta_2 r_2)}$$

$$\pi_1(1) = \frac{\lambda P_0 p_1[\widetilde{A}(\lambda) + N'(1)]s_1}{1 - \{p + [1 - \widetilde{A}(\lambda)] + p_1 s_1 \lambda(1 + B_1 r_1) + p_2 s_2 \lambda(1 + B_2 r_2)\}}$$

$$\pi_2(1) = \frac{\lambda P_0 p_2[\widetilde{A}(\lambda) + N'(1)]s_2}{1 - \{p + [1 - \widetilde{A}(\lambda)] + p_1 s_1 \lambda(1 + B_1 r_1) + p_2 s_2 \lambda(1 + B_2 r_2)\}}$$

$$B_1(1) = \frac{\beta_1 \lambda P_0 p_1 [\widetilde{A}(\lambda) + N'(1)] s_1 r_1}{1 - \{p + [1 - \widetilde{A}(\lambda)] + p_1 s_1 \lambda (1 + B_1 r_1) + p_2 s_2 \lambda (1 + B_2 r_2)\}}$$

$$B_2(1) = \frac{\beta_2 \lambda P_0 p_2 [\widetilde{A}(\lambda) + N'(1)] s_2 r_2}{1 - \{p + [1 - \widetilde{A}(\lambda)] + p_1 s_1 \lambda (1 + B_1 r_1) + p_2 s_2 \lambda (1 + B_2 r_2)\}}$$

$$\sum_{j=1}^{J} \Omega_j(1) = P_0 N'(1)$$

$$P_0 = \frac{[\widetilde{A}(\lambda) - p - p_1 s_1 \lambda (1 + \beta_1 r_1) - p_2 s_2 \lambda (1 + \beta_2 r_2)]}{2\widetilde{A}(\lambda)(p_1 s_{11} \lambda \beta_1 r_{21} \lambda \beta_2 r_{21} + b(1-p) + bN'(1) + N'(1) - 2N'(1) + N'(1) - 2N'(1)p}$$

Where $N'(1) = \dfrac{\lambda v_1 (1 - \widetilde{V}(\lambda))^J}{(1 - V(\lambda))(\widetilde{V}(\lambda)^J}$

5. Performance Measures

In this section, system performance measures like, the mean number of customer in the orbit (L_q) the mean number of customer in the system (L_s) can be obtained. Numerical illustrations are presented to investigate the effect of system parameters. The probability generating function of the number of customer in the orbit is

$$\emptyset_q(z) = P_0 + P(z) + \sum_{j=1}^{J} \Omega_j(z) + \pi_1(z) + \pi_2(z) + B_1(z) B_2(z) \tag{26}$$

The mean number of customer in the orbit under steady state condition is obtained by differentiating $\emptyset_q(z)$ with respect to z and evaluating at $z = 1$, to get

$$L_q = \emptyset_q'(1) = P_0 \left\{ \frac{Nr'''(1) Dr''(1) - Dr'''(1) Nr'''(1)}{3[Dr''(1)]^2} \right\},$$

where

$$Dr'' = \widetilde{A}(\lambda) - p - p_1 s_1 \lambda (1 + \beta_1 r_1) - p_2 s_2 \lambda (1 + \beta_2 r_2)$$

$$Dr''' = -2p(1 - \widetilde{A}(\lambda)) - 2p_1 s_{11} \lambda (1 - p - \widetilde{A}(\lambda))(1 + \beta_1 r_{11})$$

$$-2p_2s_{21}\lambda(1-p-\widetilde{A}(\lambda))(1+\beta_2r_{21})$$

$$-\lambda^2(p_1s_{11}\beta_1r_{12}+p_2s_{21}\lambda\beta_2r_{22})-p_1s_{12}\lambda^2(1+\beta_1r_{11})^2$$

$$-p_2s_{22}\lambda^2(1+\beta_2r_{21})^2$$

$$Nr''=2\widetilde{A}(\lambda)\{\lambda(p_1s_{11}\beta_1r_{11}+p_2s_{21}\beta r_{22})+(1-p)(1+2N'(1))\}$$

$$Nr'''=6pN'(1)(\widetilde{A}(\lambda)-1)-6\lambda\widetilde{A}(\lambda)\{p_1s_{11}\beta_1r_{12}+p_2s_{21}\beta_2r_{22}\}+2p\{(1+\beta_1r_{11})+(1+\beta_2r_{12})\}-bN'(1)p\lambda(p_1s_{11}(1+\beta_1r_{11}+p_2s_{21}(1+\beta_2r_{21})$$

The Probability generating function of the number of customer in the system is

$$\phi_s(z)=P_0+P(z)+\sum_{j=1}^{J}\Omega_j(z)+z[\pi_1(z)+\pi_2(z)+B_1(z)+B_2(z)] \tag{27}$$

The mean number of customers in the orbit under steady state condition is obtained by differentiating $\varphi_S(z)$ with respect to z and evaluating at $z=1.$ The mean number of customers in the system is given by

$$L_s=\emptyset_s'(1)=P_0\left\{\frac{Nr'''(1)Dr''(1)-Dr'''(1)Nr''(1)}{3[Dr''(1)]^2}\right\},$$

where

$$Nr''=(2\widetilde{A}(\lambda)+2N'(1))(1-p)$$

$$Dr'''=-2p(1-\widetilde{A}(\lambda))-2\lambda(1+p-\widetilde{A}(\lambda))(p_1s_{11}(1+\beta_1r_{11})+p_2s_{21}(1+\beta_1r_{11}))$$
$$-\lambda^2(p_1s_{11}\beta_1r_{12}+p_2s_{21}\beta_2r_{22})-\lambda^2(p_1s_{12}(1+\beta_1r_{11})^2p_2s_{22})(1+\beta_2r_{21})^2$$

$$Dr''=\widetilde{A}(\lambda)-p-\lambda(p_1s_1(1+\beta_1r_1)+p_2s_2(1+\beta_2r_2))$$

$$Nr''=-3p(N''(1)+N''(1))+6N'(1)\lambda(1-p)(p_1s_{11}(1+\beta_1r_{11})$$

$$+p_2s_{21}(1+\beta_2r_{21}))$$

6. Special Cases

Case 1. Single type service

Let $p_2 = 1$; then our model can be reduced to an $M/G/1$ retrial queuing system with a modified vacations and server breakdowns. The results of this analysis are coincided with the result in Chen et al. [7].

7. Numerical Example

In this section, a numerical example is presented in order to illustrate the effects of various parameters in the system performance measures of our system where all retrial times, service times, vacation times and repair times are exponentially distributed. The table values for different values of λ, β, p is given below, also the graphs drawn for different values of V on P_0, L_q, L_s.

Table 1. The effect of P_0, L_q, L_s. on λ, β, p.

λ	β	P	V	P_0	L_q	L_s
0.4	0.4	0.1	2	0.7286	0.0877	0.1603
			4	0.7487	0.0390	0.1170
			6	0.7526	0.0303	0.1089
			8	0.7564	0.0218	0.1010
			10	0.7602	0.0136	0.0930
	0.5		2	0.7314	0.0910	0.1517
			4	0.7499	0.0481	0.1129
			6	0.7535	0.0404	0.1058
			8	0.7569	0.0329	0.0987
			10	0.7604	0.0257	0.0919
			2	0.7331	0.0934	0.1461
			4	0.7505	0.0545	0.1104

			6	0.7538	0.0474	0.1038
			8	0.7571	0.0407	0.0974
			10	0.7603	0.0341	0.0911
0.4	0.5	0.1	2	0.7314	0.0910	0.1517
			4	0.7499	0.0481	0.1129
			6	0.7535	0.0404	0.1058
			8	0.7569	0.0329	0.0987
			10	0.7604	0.0257	0.0919
0.5			2	0.6876	0.1102	0.1738
			4	0.7066	0.0637	0.1334
			6	0.7102	0.0577	0.1258
			8	0.7138	0.0499	0.1185
			10	0.7173	0.0423	0.1112
0.6			2	0.6485	0.1282	0.1953
			4	0.6677	0.0821	0.1533
			6	0.6714	0.0737	0.1454
			8	0.6751	0.0656	0.1377
			10	0.6787	0.0577	0.1302
	0.5	0.1	4	0.6677	0.0821	0.1533
			6	0.6714	0.0737	0.1454
			8	0.6751	0.0656	0.1377
			10	0.6787	0.0577	0.1302
		0.2	2	0.6566	0.0692	0.1971
			4	0.6620	0.0532	0.1824
			6	0.6673	0.0377	0.1681

			8	0.6726	0.0228	0.1542
			10	0.7086	0.0102	0.1403
		0.3	2	0.6091	0.2331	0.3886
			4	0.6446	0.0874	0.2605
			6	0.6514	0.0618	0.2364
			8	0.6747	0.0739	0.1389
0.6			10	0.6782	0.0665	0.1318

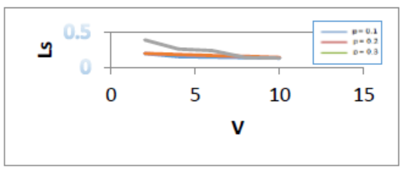

Figure 1. V versus L_s.

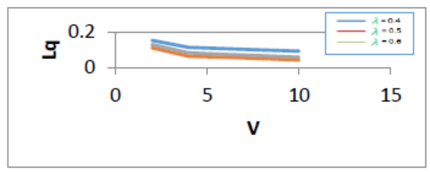

Figure 2. V versus L_q.

Figure 1 shows that as mean system size L_s decreases, number of vacations increases for different values of p. Figure 2 shows as mean orbit size L_q decreases number of increases for different values of λ. Figure 3 shows that the as probability of the system is idle increases, number of vacations increases.

8. Conclusion

In this paper, a retrial queue with two types of service, modified vacations, feedback, and server breakdown is studied. For this model the explicit expression for the probability generating functions of the server state and the number of customers in the system / orbit is derived analytically. The system performance measures are also obtained. A numerical example to study the effect of various parameters on the system characteristics is performed.

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