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SOME RESULTS ON COMMON FIXED POINT IN FUZZY 2-NORMED LINEAR SPACE

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Abstract

The major goal of this paper is to prove the common fixed point theorems for a complete fuzzy 2-normed linear space using the weak commutating condition and the A-contraction type condition, as well as to develop certain inclusion relations between these notions.

1. Introduction

S. Gahler [6] first proposed the concept of linear 2-normed space in 1964. In the setting of linear 2-normed space, a number of mathematicians have worked on different ideas of fixed point theory. In 1965 [13], Zadeh was initiated the concept of fuzzy set. A satisfactory theory of fuzzy norm and α-norm has been established by Bag and Samanta in [1, 2, 3], J. Zhang [14] has defined fuzzy normed linear space in a varies context. Further convergence and completeness in fuzzy 2-normed space in teams of all fuzzy points was

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discussed by Meenakshi [8]. Recently, So many researchers developed in 2-normed spaces and fuzzy 2-normed spaces like [4, 5, 7, 9, 10, 12].

Here we prove a common fixed point theorem for two pairs of weakly commuting mappings using the idea of A-contraction and then extend the theorem for a family of self-mappings in a fuzzy 2-normed linear space. Before proving our main theorem we need to state some preliminary ideas and definitions of weakly commuting mappings in a fuzzy 2-normed linear space.

1.1. Fuzzy 2-normed linear space.

Definition 1.1 [8]. Let X be a vector space over a field K (where K is R or C) and $*$ be a continuous t-norm. A fuzzy set $N \in X^2 \times [0, \infty]$ is called a fuzzy 2-norm on X if it satisfies the following conditions:

(i) $N(x, y, t) = 0 \, \forall \, x, y \in X$

(ii) $N(x, y, t) = 1, \forall \, t > 0$ and at least two among the three points are equal.

(iii) $N(x, y, t) = N(y, x, t)$

(iv) $N(x + y + z, t_1 + t_2 + t_3) \geq N(x, y, t_1) * N(x, y, t_3) \, \forall \, x, y, z \in X$ and $t_1, t_2, t_3 \geq 0$

(v) $\forall \, x, y \in X, N(x, y, t)$ is left continuous and $\lim_{t \to \infty} N(x, y, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy 2-normed linear space (F2-NLS).

Definition 1.2 [8]. A sequence $\{x_n\}$ in a F2-NLS $(X, N, *)$ is converge to $x \in X$ if and only if $\lim_{t \to \infty} N(x, y, t) = 1, \forall \, t > 0$.

Definition 1.3 [8]. A sequence $\{x_n\}$ in a F2-NLS $(X, N, *)$ is said to be fuzzy Cauchy sequence if and only if $\lim_{t \to \infty} N(x_n, x_m, t) = 1, \forall \, t > 0$.

Definition 1.4 [8]. A fuzzy 2-normed linear space in which every fuzzy Cauchy sequence is convergent is called a fuzzy 2-Banach space.

Definition 1.5. Let S and T be two mappings from a fuzzy 2-normed linear space $(X, N(\cdot, \cdot, t))$ into itself. Then a pair of mappings (S, T) is said to be weakly commuting on x, if $N(STx - TSx, u, t) \geq N(Tx - Sx, u, t)$ for all $u \in X$.

Definition 1.6. Let a non-empty set A consisting of all fuzzy functions $\lambda : R_+^3 \to [0, 1]$ satisfying

(i) λ is continuous on the set R_+^3 of all triplets of non-negative real's.

(ii) $\alpha \geq \lambda\beta$ for some $\lambda \in [0, 1)$, whenever $\alpha \geq \lambda(\alpha, \beta, \beta)$ or $\alpha \geq \lambda(b, \alpha, \beta)$ or $\alpha \geq \lambda(\beta, \beta, \alpha)$, for all α, β.

Definition 1.7. A self map T on a fuzzy 2-normed linear space $(X, N, *)$ is said to be A-contraction if it satisfies the condition:

$$N(Tx, Ty, t) \geq \lambda(N(x, y, t), N(x, Tx, t), N(y, Ty, t)). \tag{1.1}$$

for all $x, y \in X$ and some λA.

2. Main Result

Theorem 2.1. *Let I, J, S and T be four self mappings of a complete fuzzy 2-normed linear space $(X, N(\cdot, \cdot, t))$ satisfying*

$$I(X) \subseteq T(X) \text{ and } J(X) \subseteq S(X). \tag{2.1}$$

For $\lambda \in A$ and for all $x, y, u \in X$

$$N(Ix - Jy, u, t) \geq \lambda(N(Sx - Ty, u, t), N(Sx - Ix, u, t), N(Ty - Jy, u, t)). \tag{2.2}$$

If one of I, J, S and T is continuous and if I and J weakly commute with S and T respectively, then I, J, S and T have a unique common fixed point z in X.

Proof. Let x_0 be an arbitrary element of X. We define $Ix_{2n+1} = y_{2n+2}$, $Tx_{2n} = y_n$ and $Jx_{2n} = y_{2n+1}$, $Sx_{2n+1} = y_{2n+1}$, $n = 1, 2, \ldots$. Taking $x = x_{2n+1}$ and $y = x_{2n}$ in (2.2) we have

$$N(Ix_{2n+1} - Jx_{2n}, u, t) \geq \lambda(N(Sx_{2n+1} - Tx_{2n}, u, t), N(Sx_{n+1} - Ix_{2n+1}, u, t),$$

$$N(Tx_{2n} - Jx_{2n}, u, t))$$

or

$$N(y_{2n+2} - y_{2n+1}, u, t) \geq \lambda(N(y_{2n+1} - y_{2n}, u, t), N(y_{n+1} - y_{2n+2}, u, t),$$
$$N(y_{2n} - y_{2n+1}, u, t)).$$

So by axiom (1) of function λ,

$$N(y_{2n+1} - y_{2n+2}, u, t) \geq a \cdot N(y_{2n} - y_{2n+1}, u, t) \text{ where } k \in [0, 1) \tag{2.3}$$

Similarly by putting $x = x_{2n-1}$ and $y = x_{2n}$ in (2.2) we get

$$N(Ix_{2n-1} - Jx_{2n+2}, u, t) \geq \lambda(N(Sx_{2n-1} - Tx_{2n}, u, t), N(Sx_{2n-1} - Ix_{2n-1}, u, t),$$
$$N(Tx_{2n-1} - Jx_{2n}, u, t))$$

or

$$N(y_{2n} - y_{2n+1}, u, t) \geq \lambda(N(y_{2n-1} - y_{2n}, u, t), N(y_{n-1} - y_{2n+2}, u, t),$$
$$N(y_{2n} - y_{2n+1}, u, t)).$$

So by axiom (2) of function λ,

$$N(y_{2n+1} - y_{2n+2}, u, t) \geq a \cdot N(y_{2n} - y_{2n+1}, u, t) \text{ where } k \in [0, 1) \tag{2.4}$$

So by (2.3) and (2.4) we get

$$N(y_{2n+1} - y_{2n+2}, u, t) \geq a \cdot N(y_{2n} - y_{2n+1}, u, t) \geq a^2 \cdot N(y_{2n-1} - y_{2n}, u, t).$$

Proceeding in this way

$$N(y_{2n+1} - y_{2n+2}, u, t) \geq a^{2n+1} \cdot N(y_0 - y_1, u, t)$$

and

$$N(y_{2n} - y_{2n+1}, u, t) \geq a^{2n} \cdot N(y_0 - y_1, u, t)$$

So in general

$$N(y_n - y_{n+1}, u, t) \geq a^n \cdot N(y_0 - y_1, u, t) \tag{2.5}$$

Then using property (4) of fuzzy 2-normed linear space we get

$$N(y_n - y_{n+1},\, u,\, t) \geq N(y_n - y_{n+2},\, y_{n+1},\, t) * N(y_n - y_{n+1},\, u,\, t)$$

$$*N(y_{n+1} - y_{n+2},\, u,\, t) \tag{2.6}$$

$$\geq \bigwedge_{r=0}^{1} (N(y_n - y_{n+2},\, y_{n+1},\, t) * N(y_{n+r} - y_{n+r+1},\, u,\, t)). \tag{2.7}$$

Here we consider two possible cases to show that $N(y_n - y_{n+2},\, y_{n+1},\, t) = 0$.

Case I

$n = \text{even} = 2m$ (say), therefore

$$N(y_n - y_{n+2},\, y_{n+1},\, t) = N(y_{2m} - y_{2m+2},\, y_{2m+1},\, t)$$

$$= N(y_{2m+2} - y_{2m+1},\, y_{2m},\, t)$$

$$\geq N(Ix_{2m+1} - Jx_{2m},\, y_{2m},\, t)$$

$$\geq \lambda(N(Sx_{2m+1} - Tx_{2m},\, y_{2m},\, t)$$

$$N(Sx_{2m+1} - Ix_{2m},\, y_{2m},\, t),\, N(Tx_{2m} - Jx_{2m},\, y_{2m},\, t))$$

$$\geq \lambda(N(y_{2m+1} - y_{2m},\, y_{2m},\, t),\, N(y_{2m+1} - y_{2m},\, y_{2m},\, t),$$

$$N(y_{2m} - y_{2m+1},\, y_{2m},\, t))$$

$$\geq \lambda(0,\, N(y_{2m+1} - y_{2m},\, y_{2m},\, t),\, 0).$$

So by axiom (1) of function λ,

$N(y_n - y_{n+2},\, y_{n+1},\, t) = N(y_{2m} - y_{2m+2},\, y_{2m+1},\, t) \geq a \cdot 0$ where $k \in [0, 1)$ which implies $N(y_n - y_{n+2},\, y_{n+1},\, t) = 0$.

Case II

$n = \text{odd} = 2m + 1$ (say), therefore

$$N(y_n - y_{n+2},\, y_{n+1},\, t) = N(y_{2m+1} - y_{2m+3},\, y_{2m+2},\, t)$$

$$= N(y_{2m+3} - y_{2m+2},\, y_{2m+1},\, t)$$

$$\geq N(Jx_{2m+2} - Ix_{2m+1},\, y_{2m+1},\, t)$$

$$\geq \lambda(N(Sx_{2m+1} - Tx_{2m+2}, y_{2m+1}, t), N(Sx_{2m+1} - Ix_{2m+1}, y_{2m+1}, t),$$

$$N(Tx_{2m+2} - Jx_{2m+2}, y_{2m+1}, t))$$

$$\geq \lambda(N(y_{2m+1} - y_{2m+2}, y_{2m+1}, t), N(y_{2m+1} - y_{2m+2}, y_{2m+1}, t),$$

$$N(y_{2m+2} - y_{2m+3}, y_{2m+1}, t))$$

$$\geq \lambda(0, 0, N(y_{2m+2} - y_{2m+3}, y_{2m+1}, t)).$$

So by axiom (1) of function λ,

$$N(y_n - y_{n+2}, y_{n+1}, t) = N(y_{2m+1} - y_{2m+3}, y_{2m+2}, t) \geq a \cdot 0 \text{ where } k \in [0, 1)$$

So in either cases $N(y_n - y_{n+2}, y_{n+1}, t) = 0$. Therefore from (2.6) we have

$$N(y_n - y_{n+2}, u, t) \geq \overset{1}{\underset{r=0}{\wedge}} N(y_{n+r} - y_{n+r+1}, u, t).$$

Proceeding in the same fashion we have for any $p > 0$,

$$N(y_n - y_{n+p}, u, t) \geq \overset{p-1}{\underset{r=0}{\wedge}} N(y_{n+r} - y_{n+r+1}, u, t).$$

Then by (2.5) we get

$N(y_n - y_{n+p}, u, t) \geq k^n N(y_0 - y_1, u, t) \to 0$ as $n \to \infty$, $p > 0$ and $k^n = \dfrac{a^n}{1-a} \in [0, 1)$.

Hence $\{y_n\}$ is a fuzzy Cauchy sequence. Then by completeness of X, $\{y_n\}$ converges to a point $z \in X$ i.e. $y_n \to z \in X$ as $n \to \infty$.

Since $\{y_n\}$ is a fuzzy Cauchy sequence and taking limit as $n \to \infty$, we get $Ix_{2n} = Tx_{2n+1} \to z$, $Jx_{2n-1} = Sx_{2n} \to z$ and also $Jx_{2n+1} \to z$. Next suppose that S is continuous. Then $\{SIx_{2n}\}$ converges to Sz. Then by property (4) of fuzzy 2-normed linear space, we have

$$N(ISx_{2n} - Sz, u, t) \geq N(ISx_{2n} - Sz, SIx_{2n}, t) * N(ISx_{2n} - SIx_{2n}, u, t)$$

$$*N(SIx_{2n} - Sz, u, t)$$

$$\geq N(SIx_{2n} - Sz, SIx_{2n}, t) * N(Sx_{2n} - Ix_{2n}, u, t) * N(SIx_{2n} - Sz, u, t)$$

since I and S weakly commute.

Letting $n \to \infty$, it follows that $\{SIx_{2n}\}$ converges to Sz. Again by using (2.2)

we have

$$N(ISx_{2n} - Jx_{2n+1}, u, t) \geq N(ISx_{2n} - Sz, SIx_{2n}, t) * N(ISx_{2n} - SIx_{2n}, u, t)$$

$$*N(SIx_{2n} - Sz, u, t)$$

$$\geq \lambda(N(S^2x_{2n} - Tx_{2n+1}, u, t) * N(S^2x_{2n} - ISx_{2n}, u, t)$$

$$*N(Tx_{2n+1} - Jx_{2n+1}, u, t)).$$

Since λ is continuous, taking limit as $n \to \infty$ we get

$$N(Sz - z, u, t) \geq \lambda(N(Sz - z, u, t), N(Sz - Sz, u, t), N(z - z, u, t))$$

implies

$$N(Sz - z, u, t) \geq \lambda(N(Sz - z, u, t), 0, 0)$$

So by axiom (1) of function λ,

$$N(Sz - z, u, t) \geq a \cdot 0 = 0 \text{ which gives } Sz = z. \tag{2.8}$$

Again using the inequality (2.2) we have

$$N(Iz - Jx_{2n+1}, u, t) \geq \lambda(N(Sz - Tx_{2n+1}, u, t), N(Sz - Iz, u, t),$$

$$N(Tx_{2n+1} - Jx_{2n+1}, u, t)).$$

Passing limit as $n \to \infty$ we get

$$N(Iz - z, u, t) \geq \lambda(N(Sz - z, u, t), N(z - Iz, u, t), N(z - z, u, t))$$

implies

$$N(Iz - z, u, t) \geq \lambda(0, N(z - Iz, u, t), 0).$$

Then by axiom (1) of function λ,

$$N(Sz - z, u, t) \geq a \cdot 0 = 0 \text{ which gives } Iz = z. \tag{2.9}$$

Since $I(X) \subseteq T(X)$, there exists a point $\in X$ such that $z = Iz$, so by (2.2) we have

$$N(z - Jz, u, t) = N(z - Jz, u, t)$$

$$\geq \lambda(N(Sz - Tz, u, t), N(Sz - Iz, u, t), N(Tz - Jz, u, t))$$

$$= \lambda(N(z - z, u, t), N(z - z, u, t), N(z - Jz, u, t))$$

$$= \lambda(0, 0, N(z - Jz, u, t))$$

Then by axiom (1) of function λ,

$$N(z - Jz, u, t) \geq a \cdot 0 = 0 \text{ which implies } Jz = z.$$

As J and T weakly commute

$$N(JTz - TJz, u, t) \geq N(Tz - Jz, u, t)$$

which gives $JTz = TJz$ implies

$$Jz = JTz = TJz = Tz \tag{2.10}$$

$$N(z - Tz, u, t) = N(Iz - Jz, u, t)$$

$$\geq \lambda(N(Sz - Tz, u, t), N(Sz - Iz, u, t), N(Tz - Jz, u, t))$$

$$= \lambda(N(z - Tz, u, t), 0, 0).$$

Then by axiom (1) of function λ,

$$N(z - Tz, u, t) \geq a \cdot 0 = 0 \text{ which implies } Tz = z. \tag{2.11}$$

So by (2.8), (2.9), (2.10) and (2.11) we conclude that z is a common fixed point of I, J, S and T.

For uniqueness, Let w be another common fixed point in X such that

$$Iz = Jz = Sz = Tz = z \text{ and } Iw = Jw = Sw = Tw = w.$$

Then by (2.2) we have

$$N(w - z, u, t) = N(Iw - Jz, u, t)$$

$$\geq \lambda(N(Sw - Tz, u, t), N(Sw - Iw, u, t), N(Tz - Jz, u, t))$$

$$= \lambda(N(w - z, u, t), 0, 0).$$

Then by axiom (1) of function λ,

$$N(z - Tz, u, t) \geq a \cdot 0 = 0 \text{ which implies } w = z.$$

So uniqueness of z is proved. The same result holds if any one of I, J and T is continuous. □

Corollary 2.2. *Let S, T, I and J be four self mappings of a complete fuzzy 2-normed linear space $(X, N(\cdot, \cdot, t))$ satisfying*

$$I(X) \subseteq T(X) \text{ and } J(X) \subseteq S(X) \tag{2.12}$$

$$N(Ix - Jy, u, t) \geq c \cdot \max\{N(Ix - Jy, u, t), N(Sx - Ix, u, t), N(Ty - Jy, u, t)\}. \tag{2.13}$$

for all x, y, u in X, where $0 \geq c < 1$. If one of S, T, I and J is continuous and if I and J weakly commute with S and T respectively, then I, J, S and T have a unique common fixed point z in X.

Lemma 2.3. *Let I, J, S and T be four self mappings of a complete fuzzy 2-normed linear space $(X, N(\cdot, \cdot, t))$. If the inequality (2.2) holds for $\lambda \in A$ and for all $x, y, u \in X$.*

Then $(F_S \wedge F_T) \wedge F_I = (F_S \wedge F_T) \wedge F_J$.

Proof. Let $x \in (F_S \wedge F_T) \wedge F_I$. Then by (2.2)

$$N(x - Jx, u, t) = N(Ix - Jx, u, t)$$

$$\geq \lambda(N(Sx - Tx, u, t), N(Sx - Ix, u, t), N(Tx - Jx, u, t))$$

$$= \lambda(0, 0, N(x - Jx, u, t)).$$

Then by axiom (1) of function λ,

$$N(x - Jx, u, t) \geq a \cdot 0 = 0 \text{ implies } x = Jx$$

thus

$$(F_S \wedge F_T) \wedge F_I \subseteq (F_S \wedge F_T) \wedge F_J.$$

Similarly we have

$$(F_S \wedge F_T) \wedge F_J \subseteq (F_S \wedge F_T) \wedge F_I.$$

and so $(F_S \wedge F_T) \wedge F_I \subseteq (F_S \wedge F_T) \wedge F_J$. □

Theorem 2.4. *Let* S, T *and* $\{I_n\}_{n \in N}$ *be mappings from a complete fuzzy 2-normed space* $(X, N(\cdot, \cdot, t))$ *into itself satisfying*

$$I_1(X) \subseteq T(X) \text{ and } \wedge I_2(X) \subseteq S(X) \tag{2.14}$$

For $\lambda \in A$ *and for all* $x, y, u \in X$,

$$N(I_n x - I_{n+1} y, u, t) \geq \lambda(N(Ix - Jy, u, t), N(Sx - I_n x, u, t),$$
$$N(Ty - I_{n+1} y, u, t)). \tag{2.15}$$

holds for all $n \in N$. *If one of* S, T, I_1 *and* I_2 *is continuous and if* I_1 *and* I_2 *weakly commute with S and T respectively, then* S, T *and* $\{I_n\}_n \in N$ *have a unique common fixed point z in X.*

Proof. By Theorem (2.1), S, T, I_1 and I_2 have a unique common fixed point z in X. Now z is a unique common fixed point of S, T, I_1 and also by Lemma (2.3), $(F_S \wedge F_T) \wedge F_{I_1} = (F_S \wedge F_T) \wedge F_{I_2}$, z is a common fixed point of S, T, I_2. Also z is unique common fixed point of S, T, I_2. If not, let w be another common fixed point of S, T, I_2. Then by (2.15)

$$N(z - w, u, t) = N(I_1 z - I_2 w, u, t)$$
$$\geq \lambda(N(Sz - Tw, u, t), N(Sz - I_1 z, u, t), N(Tw - I_2 w, u, t))$$
$$= \lambda(N(z - w, u, t), N(z - z, u, t), N(w - w, u, t))$$
$$= \lambda(N(z - w, u, t), 0, 0)$$

Then by axiom (1) of function λ,

$$N(z - w, u, t) \geq a \cdot 0 = 0 \text{ implies } z = w$$

In the similar manner we can show that z is a unique common fixed point of S, T and I_2. Continuing in this way, we arrive at desired result. □

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