Advances and Applications in Mathematical Sciences
Volume 21, Issue 8, June 2022, Pages 4253-4263

ON $\mathcal{C}$-PERFECTION OF CARTESIAN PRODUCT AND LEXICOGRAPHIC PRODUCT OF GRAPHS

GOKUL S. JAYAKUMAR and V. SANGEETHA

Department of Mathematics
CHRIST (Deemed to be) University
Bangalore-560029, Karnataka, India
E-mail: gokul.jayakumar@res.christuniversity.in
sangeetha.shathish@christuniversity.in

Abstract

Given a graph G, $\mathcal{C}_{\mathcal{G}}$ denotes the set of all induced cycles in G and let $\mathcal{C}$ be any subclass of $\mathcal{C}_{\mathcal{G}}$. For any graph $H \subseteq G$, $\alpha_{\mathcal{C}}(H)$ denotes the maximum number of vertices in H such that no two of them lie in the same induced cycle in $\mathcal{C}$ and $\theta_{\mathcal{C}}(H)$ denotes the minimum number of induced cycles in $\mathcal{C}$ required to cover the vertices in H. A graph G is $\mathcal{C}$-perfect if $\alpha_{\mathcal{C}}H \subseteq \theta_{\mathcal{C}}(H)$ for all $H \in \mathcal{C}_{\mathcal{G}}$. This paper deals with the study on cycle perfection of Cartesian product and Lexicographic product of graphs and explores the properties of those $\mathcal{C}$-perfect graphs. Also, we obtain a characterization for $\mathcal{C}$-perfection of both Cartesian and Lexicographic product of graphs.

1. Introduction

The graphs considered in this paper are finite, simple and undirected unless stated otherwise. Also the terminologies not defined in this paper are followed as in [6], [7] and [14]. The concept of -perfect graphs was introduced by Ravindra in [4], in the year 2011. It is an extension of perfect graphs introduced and studied in [1] by Berge. In [1] Berge has defined two types of perfection:

(i) G is γ-perfect $\Leftrightarrow \chi(H) = \omega(H)\ \forall$ induced subgraph H of G.

2020 Mathematics Subject Classification: 05C17, 05C45, 05C69, 05C70 05C83.
Keywords: $\mathcal{C}$-perfect graph; Graph minors; Product Graphs; Cartesian product; Lexicographic product.
Received January 10, 2022; Accepted March 2, 2022

(ii) G is α-perfect $\Leftrightarrow \alpha(H) = \theta(H)\ \forall$ induced subgraph H of G.

In general, γ-perfect graphs are alternatively referred to as perfect graphs. Further, it is proved in the literature [14] that G is γ-perfect if and only if G is α-perfect (or) equivalently, G is perfect if and only if G is α-perfect. It can be observed that $\alpha(G)$ is the independence number of G and $\theta(H)$ is the clique covering number of G. The concept of perfect graphs was introduced and a characterization of perfect graphs was obtained in [9] by Lovasz in the year 1972.

Cycle perfect graphs or $\mathcal{C}$-perfect graphs are an extension of F-perfect graphs, [4] which was presented by Ravindra. The concept of graph minor was introduced and studied by Lovasz in [11]. "A graph H is a minor of G if H can be molded from G by removing vertices, edges and also by contracting edges." The inception of graph minors arose with Wagner's theorem [13]. Graph product is a binary operation on graphs. It is an operation that takes two graphs G_1 and G_2 and creates a new graph G such that $V(G) = V(G_1) \times (G_2)$ and has properly defined adjacency condition. In this paper we extend the notion of $\mathcal{C}$-perfect graphs to graph products like Cartesian and Lexicographic product. Cartesian products of graphs were brought up in 1912 by Whitehead and Russell and later further studies were done by Imrich and Klavžar in [7]. They were repeatedly rediscovered later by Gert Sabidussi in the year 1960 [12]. The definition of Cartesian product of graphs follows:

A graph $G_1 \square G_2$ is the Cartesian product of the graphs G_1 and G_2 such that

(i) $V(G_{1G_2}) = V(G_1) \times V(G_2)$ and

(ii) Any two vertices (u_1, v_1) and (u_2, v_2) in G_{1G_2} are adjacent $\Leftrightarrow u_1 = u_2$ and $v_1 \leftrightarrow u_2$ in G_2 or $u_1 \leftrightarrow u_2$ in G_1 and $v_1 = v_2$.

Cartesian product is also called box product and this term was coined by Harary in [6] in the year 1969. It has also been identified that the operation of Cartesian product on graphs is commutative. The lexicographic product was presented by Felix Hausdorff in 1914. Lexicographic product of two graphs is defined as follows:

A graph $G \cdot H$ is a Lexicographic product of graphs G and H if

(i) $V(G \cdot H) = V(G) \times V(H)$ and

(ii) any two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \cdot H \Leftrightarrow$ either u_1 is adjacent with u_2 in G or $u_1 = u_2$ and v_1 is adjacent with v_2 in H.

Lexicographic product of graphs is also called graph composition and it is not commutative. Continuing the above, in this paper we studied on the $\mathcal{C}$-perfection of Cartesian product and Lexicographic product of graphs.

1.1. Preliminary Results

The following results in [8] are used in proving some of the main results in this paper.

Definition 1.1 [11]**.** An undirected graph H is a minor of the G if H can be molded from G by removing edges and vertices and contracting edges.

Definition 1.2 [8]**.** The basic parameters leading to the definitions of $\mathcal{C}$-perfect graphs are as follows:

(i) **Induced cycle-independent set or $\mathcal{C}$-independent set** is a collection of vertices in G such that no two of them belong to the same induced cycle and is denoted by $I_{\mathcal{C}}(G)$ (It must be noted that $I_{\mathcal{C}}(G)$ is not unique for a graph).

(ii) **Induced cycle independence number** denoted by $\alpha_{\mathcal{C}}(G)$ is the cardinality of the largest $I_{\mathcal{C}}(G)$.

(iii) **Induced cycle-cover OR $\mathcal{C}$-cover** is a collection of elements in $\mathcal{C}$ of a graph G whose union is G. Let $T_{\mathcal{C}}(G)$ denote any smallest set of induced cycles in G that forms a $\mathcal{C}$-cover. (It must be noted that $T_{\mathcal{C}}(G)$ is not unique for a graph).

(iv) **$\mathcal{C}$-covering number**, denoted by $\theta_{\mathcal{C}}(G)$, is the least number of elements in $\mathcal{C}$ that can cover $V(G)$.

Definition 1.3 [8]. A graph G is $\mathcal{C}$-perfect if $\alpha_{\mathcal{C}}(H) = \theta_{\mathcal{C}}(H)$ for all induced subgraphs H of G, where every vertex in G, belongs to at least one cycle in H.

Lemma 1.4 [8]. *$P_n \square P_2$ is $\mathcal{C}$-perfect for all $n \in N$.*

Lemma 1.5 [8]. *If G is $\mathcal{C}$-perfect, then G is K_4 minor free.*

Corollary 1.6 [8]. *If G is not $\mathcal{C}$-perfect then all graphs with an induced G is not $\mathcal{C}$-perfect.*

2. Results on $\mathcal{C}$-perfection of Cartesian products

This section deals with the study on $\mathcal{C}$-perfection of Cartesian product. In order that the definition of $\mathcal{C}$-perfect graphs is meaningful, a property was formed in [8] that all graphs (including their subgraphs) must possess.

Property $\mathcal{P}$ [8]. Every graph G (inclusive of its subgraphs H) considered in this paper are in such a way that every vertex in G (or H) belongs to at least one cycle in G (or H).

Since, Cartesian product of any two graphs satisfy property all possible graphs can be considered for the study in this paper.

Lemma 2.1. *$P_n \square P_m$ is $\mathcal{C}$-perfect if and only if $n = 2$ or $m = 2$.*

Proof of Lemma 2.1.

From Theorem 1.4 it is obtained that $P_n \square P_2$ is $\mathcal{C}$-perfect.

Conversely, let $P_n \square P_m$ be $\mathcal{C}$-perfect, it is to be proved that $P_n \square P_m$ is a ladder graph. Analyzing the graph $P_3 \square P_3$, it can be clearly seen that it is the wounded wheel graph $W^{1,8}_{\{v_2, v_4, v_6, v_8\}}$, where v is the central vertex. Also, $P_3 \square P_3$ is a K_4-minor. Consequently, from Lemma 1.5 it is obtained that $P_3 \square P_3$ is not $\mathcal{C}$-perfect and therefore by Corollary 1.6 any graph containing $P_3 \square P_3$ is not $\mathcal{C}$-perfect. Hence there exist no $\mathcal{C}$-perfect grid graph $P_n \square P_m$

such that $n \geq 3$ and $m \geq 3$, as it contains an induced $P_3 \square P_3$. Therefore, if any grid graph G is $\mathcal{C}$-perfect, then it is a ladder graph.

Corollary 2.2. *If $G_1 \square G_2$ is $\mathcal{C}$-perfect only if it is $P_3 \square P_3$-free.*

Corollary 2.3. *$G_1 \square G_2$ is $\mathcal{C}$-perfect only if either $G_1 \cong K_2$ or $G_2 \cong K_2$.*

Lemma 2.4. *$G_1 \square G_2$ is $\mathcal{C}$-perfect only if both G_1 and G_2 are acyclic.*

Proof of Lemma 2.4. Let G be the Cartesian product of G_1 and G_2 and let G be $\mathcal{C}$-perfect. We need to prove that both G_1 and G_2 is acyclic. Without loss of generality, let G_1 be cyclic graph. This implies that G_1 contains at least one induced $C_i; i \geq 3$, also G_2 contains a K_2, since it is a non-trivial connected graph. Consequently, G contains an induced $C_i i \square 3$, That is, G contains a prism graph which is a K_4-minor and hence by Lemma 1.5 is not $\mathcal{C}$-perfect which in turn implies that G is not $\mathcal{C}$-perfect. This contradicts the hypothesis, proving our assumption false. Therefore, both G_1 and G_2 is acyclic.

The following Theorem characterizes $\mathcal{C}$-perfect Cartesian product graphs.

Theorem 2.5. *$G_1 \square G_2$ is $\mathcal{C}$-perfect if and only if either one of G_1 or G_2 is a tree and the other is isomorphic to K_2.*

Proof of Theorem 2.5. The forward implication to the characterization results as a direct consequence of lemma 2.3 and corollary 2.4, hence implying that G is $\mathcal{C}$-perfect only if G_1 and G_2 is acyclic and either one is a K_2. Now, it remains to prove the converse. Let $G = G_1 \square K_2$, where G_1 is a tree of order n. It is to be proved that G is $\mathcal{C}$-perfect. Draw $G_1 \square K_2$ in such a way that the tree G_1 and its copy appear as mirror image of each other. Let us label the Cartesian product graph $G_1 \square K_2$ as given in [7]. See Figure 1.

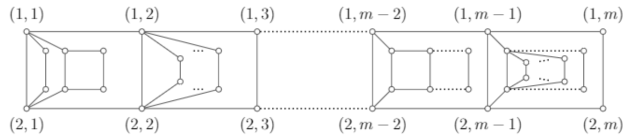

Figure 1. $G_1 \square K_2$.

As per the rules of Cartesian product each vertex $(1, i)$ is adjacent to its corresponding copy $(2, i)$, following this structure it is observed that any two adjacent vertices in G_1 forms an induced C_4 with its corresponding copies in G. This structure leads to a graph that does not contain any subdivision of $K_{2,3}$ as an induced subgraph. Also, since G is formed be parallel connection of two trees, it can be inferred that G is series parallel and therefore it is K_4 minor free. Consequently, from Lemma 1.5 it is obtained that G is $\mathcal{C}$-perfect.

Alternate proof for converse part: We observe that $\mathcal{C}$-perfection of G is analogous to Berge's perfection in G_1. This implies that the $\theta_{\mathcal{C}}(G)$ and $\alpha_{\mathcal{C}}(G)$ values of G are equal to the clique covering number and independence number of G_1 respectively. It is known that trees are perfect graphs, consequently $\alpha(G_1) = \theta(G_1)$. Therefore, $\alpha_{\mathcal{C}}(G_1) = \alpha(G_1) = \theta(G_1) = \theta_{\mathcal{C}}(G_1)$.

Corollary 2.6. *The generalized Cartesian product* $G_1 \square G_2 \square G_3 \square \ldots \square G_n$ *is* $\mathcal{C}$*-perfect if and only if* $n = 2$, G_i *is acyclic for all* $i = 1, 2$ *and* $Gi \sim= K_2$ *for some* $i \in \{1, 2\}$.

Proof of Corollary 2.6. Let $G_1 \square G_2$ be $\mathcal{C}$-perfect. That is $G_1 \square G_2 \cong G_1 \square K_2$, where G_1 is a tree. Clearly, the graph $G_1 \square K_2$ is cyclic. Therefore, by Lemma 2.4 $G_1 \square G_2 \square G_3$ is not $\mathcal{C}$-perfect. This in turn implies that any Cartesian product of graphs with more than two components is not $\mathcal{C}$-perfect. Consequently, $n = 2$. The rest of the proof results as a direct consequence of Theorem 2.5.

3. Results on $\mathcal{C}$-perfection of Lexicographic Product

This section deals with the study $\mathcal{C}$-perfection of Lexicographic product. It is known that Lexicographic product does not satisfy commutative property. That is $G_1 \cdot G_2 \cong G_2 \cdot G_1$ for all G_1 and G_2.

Lemma 3.1. *$P_n \cdot P_m$ is $\mathcal{C}$-perfect if and only if $m = 2$.*

Proof of Lemma 3.1. Let P_n be a path on n vertices and P_2 be the complete graph on two vertices, then $P_n \cdot P_m$ is isomorphic to a ladder graph as shown in Figure 2.

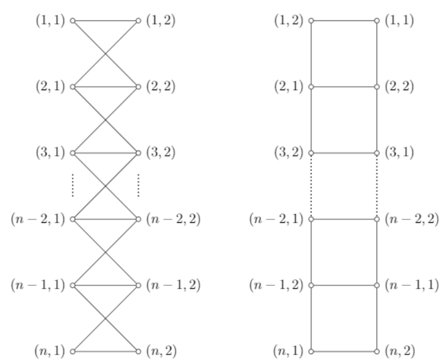

Figure 2. $P_n \cdot P_2 \cong P_n \square P_2$.

From Lemma 1.4 is obtained that $P_n \square P_2$ are $\mathcal{C}$-perfect and hence, $P_n \cdot P_2$ is $\mathcal{C}$-perfect.

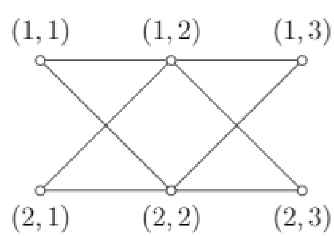

Figure 3. $P_2 \cdot P_3$.

Conversely, let $P_n \cdot P_m$ be $\mathcal{C}$-perfect. Assume that $m \geq 3$. Consider $P_2 \cdot P_3$, as shown in Figure 3. It can be observed that any two vertices from $P_2 \cdot P_3$, belong to a common induced cycle in it and hence $\alpha_{\mathcal{C}}(P_2 \cdot P_3) = 1$. But clearly, $P_2 \cdot P_3$ cannot be covered by a single induced cycle in it, therefore $\theta_{\mathcal{C}}(P_2 \cdot P_3) \geq 2$. Consequently, $P_2 \cdot P_3$ is not $\mathcal{C}$-perfect. Now, any graph $P_n \cdot P_m$; where $m \geq 3$ contains an induced $P_2 \cdot P_3$, hence it is obtained by Corollary 1.6 that $P_n \cdot P_m; m \geq 3$ is not $\mathcal{C}$-perfect. This is a contradiction to the hypothesis. Therefore, it can be concluded that if $P_n \cdot P_m$ is $\mathcal{C}$-perfect then, $m = 2$.

Corollary 3.2. *$G_1 \cdot G_2$ is $\mathcal{C}$-perfect only if it is $P_2 \cdot P_3$-free.*

Lemma 3.3. *A graph $G_1 \cdot G_2$ is $\mathcal{C}$-perfect only if both G_1 and G_2 are acyclic.*

Let $G_1 \cdot G_2$ be a $\mathcal{C}$-perfect graph. Since lexicographic product is not commutative, two cases are to be analysed in order to prove the lemma.

Case 1. Assume that G_1 contains at least one cycle. Then clearly, $C_i \cdot P_2$ is an induced subgraph of $G_1 \cdot G_2$, where C_i is an induced cycle in G_1. But from Lemma 1.5, $C_i \cdot P_2$ is not $\mathcal{C}$-perfect as it is a K_4 minor. Consequently, $G_1 \cdot G_2$ is not $\mathcal{C}$-perfect.

Case 2. Assume that G_2 contains at least one cycle. Then clearly, $P_2 \cdot C_i$ is an induced subgraph of $G_1 \cdot G_2$, where C_j is an induced cycle in G_2. But from Lemma 1.5, $P_2 \cdot C_i$ is not $\mathcal{C}$-perfect as it is a K_4 minor. Consequently, $G_1 \cdot G_2$ is not $\mathcal{C}$-perfect. Both the above cases lead to a contradiction in the hypothesis. Therefore, our assumption is wrong, implying that G_1 and G_2 are acyclic.

The following Theorem characterizes $\mathcal{C}$-perfect Lexicographic product of graphs.

Theorem 3.4. *$G_1 \cdot G_2$ is $\mathcal{C}$-perfect if and only if $G_1 \cong$ Tree and $G_2 \cong K_2$.*

It can be easily observed that $G_1 \cdot K_2$, where G_1 is a Tree, as shown in figure 4, is isomorphic to $G_1 \square K_2$ as shown in figure 1. Consequently from Theorem 2.5 it is obtained that $Tree \cdot K_2$ is $\mathcal{C}$-perfect.

Conversely, let $G_1 \cdot G_2$ be $\mathcal{C}$-perfect. From Lemma 3.3 is obtained that G_1 and G_2 are isomorphic to trees. Also, it is inferred from Corollary 3.2 that $G_1 \cdot G_2$ is $P_2 \cdot P_3$-free. Consequently, order of $G_2 = 2$, since any other $G_1 \cdot G_2$, where order of $G_2 \geq 3$ would contain an induced $P_2 \cdot P_3$. Therefore, $G_2 \cong K_2$.

Corollary 3.5. *The generalized lexicographic product $G_1 \cdot G_2 \cdot G_3 \cdot \ldots \cdot G_n$ is $\mathcal{C}$-perfect if and only if $n = 2$, G_i is a tree for all $i = 1, 2$ and $G_2 \cong K_2$.*

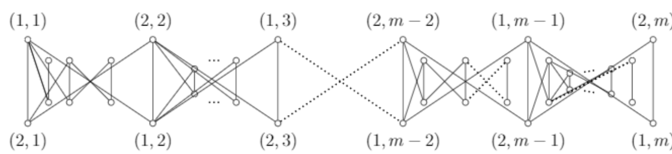

Figure 4 $G_1 \cdot K_2$.

Let $G_1 \cdot G_2$ be $\mathcal{C}$-perfect. That is $G_1 \cdot G_2 \cong G_1 \cdot K_2$, where G_1 is a tree. Clearly, the graph $G_1 \cdot K_2$ is cyclic. Therefore by Lemma 3.3 $G_1 \cdot G_2 \cdot G_3$ is not $\mathcal{C}$-perfect. This in turn implies that any lexicographic product of graphs with more than two components is not $\mathcal{C}$-perfect. Consequently, $n = 2$. The rest of the proof results as a direct consequence of Theorem 3.4.

Corollary 3.6. *If $G_1 \cdot G_2$ is $\mathcal{C}$-perfect then $G_1 \square G_2$ is $\mathcal{C}$-perfect.*

Theorem 3.7. *Let $G_1 \square G_2$ be $\mathcal{C}$-perfect then, $G_1 \cdot G_2$ is $\mathcal{C}$-perfect if and only if $G_2 \cong K_2$.*

Let $G_1 \square G_2$ be $\mathcal{C}$-perfect. Consequently, from Theorem 2.5 it is obtained that both G_1 and G_2 are trees and either one is K_2. Consequently, the result follows directly from Theorem 3.4.

3. Conclusion

Perfect graphs are a new and growing area in graph theory. In this paper we have extended the notion of $\mathcal{C}$-perfect graphs to the well-known graph products namely Cartesian product and Lexicographic product. The main motivation to this extension was to study various structural properties of $\mathcal{C}$-perfect graphs in detail. In this paper we were able to obtain a characterization of $\mathcal{C}$-perfection for both Cartesian product of graphs as well as Lexicographic product of graphs. And we have obtained almost similar characterization for both the graph products. Further we have obtained a relation between Cartesian product and lexicographic product with respect to $\mathcal{C}$-perfection. The same concept can be further extended to various other graph operations, which are currently in progress.

References

[1] Withdrawn: Chapter 16 perfect graphs. In Claude Berge, editor, Graphs and Hypergraphs, volume 6 of North-Holland Mathematical Library, pages Elsevier (1973), 360-385.

[2] Joan Feigenbaum and Alejandro A. Schaeffer, Recognizing composite graphs is equivalent to testing graph isomorphism, SIAM Journal on Computing 15(2) 1986.

[3] T. Gallai, JL. Ramırez-Alfonsın and BA Reed, Perfect graphs, series in discrete mathematics and optimization, 2001.

[4] G. Ravindra, f-perfect graphs, http://www.adma.co.in/news/openproblems/SecondIndiaTaiwanConferenceOpenProblemsSession.pdf

[5] András Gyárfás, Problems from the world surrounding perfect graphs, Number 177, MTA Számítástechnikai és Automatizálási Kutató Intézet, 1985.

[6] Frank Harary, Graph Theory (on Demand Printing Of 02787), CRC Press 3 2018.

[7] Wilfried Imrich and S. Klavzar, Product Graphs, Structure and Recognition, 01 2000.

[8] Gokul Jayakumar, V. Sangeetha and A. Senthil Thilak, Cycle perfect graphs, (Communicated).

[9] László Lovász, A characterization of perfect graphs. Journal of Combinatorial Theory, Series B 13(2) (1972), 95-98.

[10] László Lovász, Normal hypergraphs and the perfect graph conjecture, discrete Mathematics 2(3) (1972), 253-267.

[11] László Lovász, Graph minor theory, Bulletin of the American Mathematical Society 43, 2006.

[12] Gert Sabidussi, Graph multiplication, Mathematische Zeitschrift 72 (1959).

[13] K. Wagner, Über eine eigenschaft der ebenen komplexe, Mathematische Annalen 114 (1937), 570-590.

[14] Douglas Brent West et al., Introduction to Graph Theory, volume 2, Prentice-Hall, Upper Saddle River, 2001.