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SOME FUZZY ASPECTS OF FIXED POINTS IN METRIC SPACES

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Abstract

During the course of present investigation efforts have been made to analyze the role of fuzziness in the context metric-spaces. A nonempty set jointly with a distance function (d) is known as metric-spaces. A fuzzy set can be defined by objects with a scale of membership class. Intention of the manuscript is to look at the advancement of fuzzification in several "metric spaces" and "fixed-point theorems". Further we discuss diverse generalizations of FMS (fuzzy-metric space) such as IFMS (Intuitionistic-fuzzy metric-space), PMS (Probabilistic metric-space), M-FMS (M-fuzzy metric space), fixed-points theorems in several spaces etc. in concise.

1. Introduction and Preliminaries

Fuzzy-set theory was first projected by eminent L. A. Zadeh in year 1965 which was an expansion of the classical set (Zadeh, 1965). With the proposed methodology, Zadeh introduced a mathematical method with which decision-making using fuzzy descriptions of some information becomes possible. The foundation of this term "fuzzy set", is a set that does not have values 1 or zero as in crisp set but it provides membership values based on strength of membership which can vary between zero to one; in other words, elements

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can have a definite degree of association. Hence, appropriate functions are used, namely, "membership functions" to calculate the "membership-degree" of each element in a fuzzy-set. If we consider an input variable x with a field of definition S, the fuzzy-set A' in S can be well defined by,

" $A = \{x, \mu_A(x) \mid x \in S\}$, where $\mu_A(x)$ is the membership function of x in fuzzy set A' and may range from 0 to 1".

Next, L. A. Zadeh showed how elementary set operations, such as "union and intersection" are defined on these "fuzzy-sets". The hypothesis of "fuzzy-set" is one of the major primary and significant tool in "computational-intelligence". Fuzzy-sets can offer answer to an extensive series of problems of "face-recognition", "control-theory", "artificial-intelligence", computer vision, reasoning, planning, and pattern classification.

The pioneer of metric-spaces idea was Maurice Fréchet. In year 1906 Maurice Frechet [13] introduced the important notion of metric spaces which actually is a distance-function (d). The concept of metric space plays a very significant role in functional analysis and topology. Fixed-point assumption in metric spaces has concerned researchers due to its high relevance in various areas such as variational-inequalities, optimization-theory, control-theory and many more. Later on many more generalizations had been made in the field such as using statistical approach Menger established Menger's Space whereas Fuzziness in Metric-space had been studied by many others from different approaches.

The initiative of a PM Space (i.e. Probabilistic Metric Space) was first proposed by Menger. It took a broad views that of a metric-space: a "distribution-function $f(p, q)$" is connected with every pair of elements p and q of a non-empty set S, rather than a positive constant. The concept of probabilistic metric spaces was added to mathematics as a tool which deals with conditions wherein the distance between two points is uncertain or unsure. The uncertainty was assumed to be resulting from randomness.

t-norm ("Triangular norms") is an operation in FMS("fuzzy-metric space") and PMS(probabilistic metric-space) that confirms the exact explanation of the intersection-operator using fuzzy logics. t-norm had been first proposed in PMS(probabilistic metric-space) and later in FMS. t-norm plays very essential role in various problem solving in Mathematics as well as in statistics.

In this review paper, a comprehensive study of the fixed-point theory is proposed and furthermore we present chronological advancement of various generalizations of fuzzy metric spaces such as IFMS, M-FMS and many more.

2. Fixed Point

Fixed point can be understood as the point keeps unaltered when applying any transformation. Historically fixed-point theory was first introduced by the French mathematician Poincare in the study of vector distribution of a function. Fixed point idea acts very powerful role in recent geometry and analysis. Numerous Fixed Point Theorems are derived by many researchers concerning the existence and approximation of fixed points in several analytical spaces. The beauty of Fixed point theory is that it is not only the combination of topology, geometry and analysis but also has countless consequences in diverse fields like as "engineering-problems", "Science-solutions", "Economics-puzzles", "Game-theory", "Biological-Science", "Chemical-theory", "Optimization-theory", "approximation-theory" and many more. Fixed-point theory Showed very significant role in enlargement of various branches of mathematics for many decades.

Definition. "Let X be a non-empty set. A function $T : X \to X$ is called a self map on X. A point $z \in X$ is called a fixed point of a self map $T : X \to X$, if $T(z) = z$."

Examples.

(1) The mapping $g : R \to R$ defined by $g(z) = z^3$ has three fixed-points these are 1, -1 and 0.

(2) Let $g : R \to R$ and $g(t) = ct$, where c is a "non-zero constant", then 0 is the fixed point of g in R.

(3) Function $\mu : R \to R$ defined by $\mu(y) = y + K$, where k is a "non-zero constant" has no fixed point.

The pioneer of "fixed-point theory" is Poincare. In 19^{th} Century the

"Fixed-Point" notion is introduced and later some noteworthy contributions had been made in the field by several researchers such as Kakutani, Brouwer, Schauder, Tarski, Kannan, Banach, and many more.

3. Brouwer's Fixed-Point

Mathematician Brouwer introduced his well known fixed-point theorem in year 1912. Brouwer's fixed-point theorem is one of the most signifiant theorem in functional-analysis. The notion is very powerful means in finding the solutions and existence of several functional equations.

Statement. "If B is a closed unit ball in R^n and if $T : B \to B$ is continuous then T has a fixed point in B."

Note. The "fixed point theorem" defined by Brouwer has drawback that it could not provide any affirmation about the uniqueness of the "fixed-point" however it ensure the existence of fixed-point.

4. Schauder's Fixed-Point

In 1930 Schauder introduced his quite known "fixed-point in an infinite-dimensional Banach-space". The statement is offered below:

Statement. "If $T : B \to B$ is a continuous function on a compact, convex subset B of a Banach space X then f has a fixed point."

Remark. The schauder fixed point theorem is very important and has several applications in economical-hypothesis, game-theory, approximation-theory and several other fields. In his statement Schauder imposed a stronger condition of "compactness on B".

5. Metric Space

In 1906 Maurice Frechet introduced the notion of metric spaces as distance function. The concept of metric space plays a very significant role in functional analysis and topology. Fixed point theory in metric spaces has attracted researchers due to its applications in various areas such as variational inequalities, optimization theory and control theory.

Definition. "Let x be a non empty set and $d : X \times X \to R$ be such a function.

(i) $d(x, y) \geq 0, \forall x, y \in X$ (Non Negativity)

(ii) $d(x, y) = 0$ if and only if $x = y$ (Non degeneracy)

(iii) $d(x, y) = d(y, x)$ (Symmetry)

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ (Triangle inequality)

The pair (X, d) is called a metric space. The function d is called metric or sometimes the distance function".

Examples.

1. In set of real numbers R metric d is defined by $d(a, b) = | a - b |$.

2. Discrete metric d can be described as: $d(x, y) = 0$ if $x = y$ in any other cases $d(x, y) = 1$.

Contraction mapping. "Let (X, d) be a metric space. Then a mapping $f : X \to X$ is called a contraction mapping on X, if there exists $q \in [0, 1)$ such that, $d(fx, fy) \leq q \cdot d(x, y)$, for all $x, y \in X$."

Example. Let $\omega : \left[0, \frac{1}{4}\right] \to \left[0, \frac{1}{4}\right]$ defined by $\omega(z) = z^2$ W. R. T. the usual metric, a contraction mapping.

Solution. Let $z_1, z_2 \in \left[0, \frac{1}{4}\right]$ then $d(\omega(z_1), \omega(z_2)) = | \omega(z1), \omega(z2) |$

$$| z_1^2 - z_2^2 |$$

$$= | z_1 + z_2 || z_1 - z_2 |$$

$$\leq \{| z_1 | + | z_2 |\}| z_1 - z_2 |$$

$$\leq \frac{1}{2} | z_1 - z_2 |$$

$$\leq \frac{1}{2} d(z_1, z_2) | z_1 - z_2 |$$

Hence ω is a "Contraction Mapping".

6. Banach's Fixed Point Theorem

The result proved by Banach is a milestone in metric fixed-point theory.

Statement. "Let (X, d) be a non-empty complete metric space then every contraction mapping in X has a unique fixed point."

7. Fuzzy Set

Definition. "A fuzzy set A on universe (domain) X is defined by a membership function $f_A(x)$ which is a mappings from the universe (domain) X to the range as unit interval $[0, 1]$ i.e.

$f_A(x) : X \to [0, 1]$, where $f_A(x)$ is known as the grade of membership of x in A."

Note. "If $g(x)$ denotes the set of all fuzzy sets on X. Fuzzy-set allows for a membership-degree of an element in a set. The output of the membership function is called the membership grades (degrees). It varies between 0 and 1."

Example. Let $A = \{s_1, s_2, s_3, s_4, s_5\}$ be the set of students in a class and let $\widetilde{A}$ denotes the fuzzy set of "Intelligent" students, where 'Intelligent' is a fuzzy term. $\widetilde{A} = \{(s_1, 0.8)(s_2, 0.6)(s_3, 0.9)(s_4, 0.1)(s_5, 0.8)\}$. Here $\widetilde{A}$ indicates that the Intelligence of s_1 is 0.8 and the rest are similar.

8. Probabilistic Metric Spaces

Definition. "A probabilistic metric space is a triple (X, F, T), where X is a nonempty set, T is a continuous t-norm, and F is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at a point $(x, y)\ X \times X$, the following conditions hold: for all x, y, z in X,

(1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$

(2) $F_{x,y}(t) = F_{y,x}(t)$

(3) $F_{x,z}(t + s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z\ X$ and $t, s \geq 0$."

9. t-norm (Triangular norm)

Schweizer and Sklar (1960) proposed the notion of "continuous t-norm".

Definition. "A t-norm is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that, for all $a, b, c \in [0, 1]$, the following four axioms are satisfied:

(1) $T(a, b) = T(b, a)$ (Commutativity)

(2) $T(a, T(b, c)) = T(T(a, b), c)$ (Associativity)

(3) $T(a, 1) = a$ (Boundary condition)

(4) $T(a, b) \leq T(a, c)$ whenever $b \leq c$ (Monotonicity)"

10. Fuzzy -Metric Space

Definition. "A fuzzy-metric space is an ordered triple $(X, M, *)$ such that X is a non-empty set, M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions, for all $y, z, w \in X$ and $s, t > 0$,

(1) $M(y, z, 0) = 0$.

(2) $M(y, z, t) = 1$ if and only if $x = y$.

(3) $M(y, z, t) = M(z, y, t)$.

(4) $M(y, z, t) * M(z, w, s) \leq M(y, w, t + s)$.

(5) $M(y, z, \cdot) : [0, \infty) \to [0, 1]$ is left continuous."

Example. Let us assume a metric-space (X, d). Let $a * b = ab$, and $M(p, q, s) = \left\{\begin{matrix} \frac{p}{q} \ if \ p \geq q \\ \frac{q}{p} \ if \ q \geq p \end{matrix}\right\}$ for all $s > 0$, then $(X, M, *)$ is said to be a FMS (i.e. fuzzy-metric space).

11. t-conorm (Triangular conorm)

Definition. "A binary operation T from domain into the unit interval $[0, 1]$ i.e.

$$T : [0, 1] \times [0, 1] \to [0, 1]$$

is called continuous t-conorm(triangular conorm) if it satisfies the following axioms:

(1) $\Diamond$ is commutative as well as associative.

(2) $\Diamond$ is continuous.

(3) $p \Diamond 1 = 1$.

(4) $p \Diamond q \leq r \Diamond s,$ whenever $p \leq r$ and $q \leq s, \forall x, y, z, w \in [0, 1]$."

12. IFMS (Intuitionistic Fuzzy-Metric Space)

Definition. "An intuitionistic fuzzy metric space (IFMS) is a 5-tuple $(X, M, N, *, \Diamond)$ such that X is a non-empty set, * is a continuous t-norm, $\Diamond$ is a continuous t-conorm M and N are fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions, for all $a, b, c \in X$ and $s, t > 0$.

(1) $M(a, b, t) + N(a, b, t) \leq 1$.

(2) $M(a, b, t) > 0$.

(3) $M(a, b, t) = 1$ if and only if $a = b$.

(4) $M(a, b, t) = M(b, a, t)$.

(5) $M(a, b, t) * M(b, c, s) \leq M(a, c, t + s)$.

(6) $M(a, b, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

(7) $N(a, b, t) > 0$.

(8) $N(a, b, t) = 0$ if and only if $a = b$.

(9) $N(a, b, t) = N(b, a, t)$.

(10) $N(a, b, t) \lozenge N(b, c, s) \geq N(a, c, t + s)$.

(11) $N(a, b, \cdot) : [0, \infty) \to [0, 1]$ is continuous".

Then (M, N) is named as an IFMS (intuitionistic fuzzy metric spaces) on the domain X."

13. M-FMS (M-Fuzzy Metric Space)

Definition. "The 3-tuple $(X, M, *)$ called a M-FMS (M-fuzzy metric space) if, a non-empty set X together with a continuous t-norm * and M is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions:

(1) $M(x, y, z, a) > 0$;

(2) $M(x, y, z, a) = 1$ if and only if $x = y = z$

(3) $M(x, y, z, a) = M(p(a, b, c), a)$; where p is the permutation function.

(4) $M(x, y, w, a) * M(w, z, z, b) \leq M(x, y, z, a + b)$

(5) $M(x, y, z, \cdot) : (0, \infty) \to [0, 1]$ is continuous,

for all $x, y, z \in X$ and $a, b > 0$."

14. Conclusions

In this paper we provide expansions of different kinds of fuzzy aspects in metric spaces and concise study of the "fixed-point theory" in various FMS (i.e. Fuzzy Metric Spaces).

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