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NEIGHBOURHOOD-PRIME LABELING OF CERTAIN CLASSES OF CAYLEY GRAPHS

K. PALANI[1], G. SUGANYA[2] and M. LALITHA KUMARI[3]

[1]Associate Professor
[2]Assistant Professor
[3]Research Scholar, Reg. No. 20212012092007
PG and Research, Department of Mathematics
A. P. C. Mahalaxmi College for Women
Thoothukudi, TN, India
Affiliated to Manonmaniam Sundaranar University
Abishekapatti, Tirunelveli-627012, TN, India
E-mail: palani@apcmcollege.ac.in
gsuganyaveni@gmail.com
lalithasat32@gmail.com

Abstract

A Cayley graph is a graph constructed from a finite group Γ and a Cayley set S of Γ. It is denoted by $Cay(\Gamma, S)$. Let $G = (V, E)$ be a graph with n vertices. A bijection $f : V(G) \to \{1, 2, 3, \ldots, n\}$ is said to be a neighbourhood-prime labeling if for every vertex $v \in V(G)$ with $\deg(v) > 1$, $\gcd\{f(u) \mid u \in N(v)\} = 1$. A graph which admits neighbourhood-prime labeling is called a neighbourhood-prime graph. This paper studies the result connecting Cayley graphs and neighbourhood-prime labeling.

1. Introduction

Cayley graph was introduced by Arthur Cayley [1] in 1878 is an important concept relating group theory and graph theory. S. K. Patel and N. P. Shrimali [7] introduced neighbourhood-prime labeling of a graph.

Definition 1.1[4]**.** A subset S of a group Γ is called a generating set for Γ, denoted by $\langle S \rangle = \Gamma$, if every element of Γ can be expressed as a finite product

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of elements in S and their inverses.

Definition 1.2 [4]. A dihedral group $D_{2n}, n \geq 3$ is a group with $2n$ elements such that it contains an element 'a' of order 2 and an element 'b' of order n with $ba = ab^{-1}$. Thus $D_{2n} = \langle a, b \mid a^2 = b^n = 1, ba = ab^{-1} \rangle$. The elements of dihedral group can be explicitly listed as $D_{2n} = \{1, b, b^2, b^3, \ldots, b^{n-1}, a, ab, ab^2, ab^3, \ldots, ab^{n-1}\}$.

The orders of the elements in the dihedral group D_{2n} are $o(1) = 1, o(ab^i) = 2$ where $0 \leq i \leq n-1$ and if n is even then $o(b^{\frac{n}{2}}) = 2$.

Definition 1.3 [6, 8]. Let Γ be a finite group with identity e and S a subset of Γ. If $e \notin S$ and $s \in S$ implies $s^{-1} \in S$, then S is called a Cayley set of Γ. The Cayley graph of Γ with respect to S is the graph whose vertices are the elements of Γ and two elements x, y of Γ are adjacent if and only if there is $s \in S$ such that $y = xs$. This graph is denoted by $Cay(\Gamma, S)$.

Remark 1.4[8]. (1) $Cay(\Gamma, S)$ is $|S|$-regular graph.

(2) $Cay(\Gamma, S)$ is connected graph if and only if $\langle S \rangle = \Gamma$.

The following facts are from [7].

Definition 1.5. Let $G = (V, E)$ be a graph with n vertices. A bijective function $f : V(G) \to \{1, 2, 3, \ldots, n\}$ is said to be neighbourhood-prime labeling, if for each vertex $v \in V(G)$, with $\deg(v) > 1$, $\gcd\{f(u) \mid u \in N(v)\} = 1$. A graph which admits neighbourhood-prime labeling is called a neighbourhood-prime graph.

Remark 1.6. A graph G in which every vertex is of degree atmost 1 is neighbourhood-prime vacuously.

Theorem 1.7. *The cycle* C_n *is neighbourhood-prime if* $n \not\equiv 2 \pmod 4$.

Theorem 1.8. *The cycle* C_n *is not neighbourhood-prime if* $n \equiv 2 \pmod 4$. *The following theorems are from* [5].

Theorem 1.9. *Let G be a graph of order n such that* $n \not\equiv 2 \pmod 4$. *If G is* *hamiltonian, then G has a neighbourhood-prime labeling.*

Theorem 1.10. *If the graph G contains a hamiltonian cycle C and a chord that forms a cycle of length $4k$ for some positive integer $k \in Z$ using only the chord and edges from C, then G is neighbourhood-prime.*

Theorem 1.11. *If G is hamiltonian and contains an odd cycle then G is neighbourhood-prime.*

Theorem 1.12. *All graphs with minimum degree at least $\frac{n}{2}$ are neighbourhood-prime.*

Theorem 1.13. *A hamiltonian graph of order n with $|E| > n\left\lfloor \frac{n-6}{8} \right\rfloor + n$ is neighbourhood-prime.*

Theorem 1.14 [3]**.** Every connected Cayley graph of a finite abelian group of order at least three is hamiltonian.

Definition 1.15 [2]**.** A group G is described as hamiltonian if and only if G is a non-abelian group such that every subgroup is normal.

Theorem 1.16 [2]**.** *Any connected Cayley graph of a finite hamiltonian group is hamiltonian.*

2. Main Results

Theorem 2.1. *$Cay(\Gamma, S)$ where $o(\Gamma) \leq 3$ is neighbourhood-prime.*

Proof. Let $o(\Gamma) \leq 3$ and S, a Cayley set of Γ.

Case (i). $o(\Gamma) = 1$.

Here $Cay(\Gamma, S) \cong K_1$ and so is neighbourhood-prime vacuously.

Case (ii). $o(\Gamma) = 2$.

Let $\Gamma = \{e, x\}$ where e is the identity element. Then $S = \Phi$ or $\{x\}$. Therefore, $cay(\Gamma, S) \cong \begin{cases} \overline{K_2} & \text{if } S = \Phi \\ K_2 & \text{if S} = \{x\} \end{cases}$. Obviously K_2 and $\overline{K_2}$ are neighbourhood-prime.

Case (iii). $o(\Gamma) = 3.$

Let $\Gamma = \{e, x, y\}$ where e is the identity element. Since the order of every element of the group divides the order of the group, every element of Γ is of order 1 or 3. Therefore, x and y are generators of Γ and so $x^2 \neq e, y^2 \neq e.$ Then $x \neq x^{-1}, y \neq y^{-1}$ and $x^2 = y, y^2 = x.$ Hence $x^{-1} = y; y^{-1} = x.$ Therefore, either $S = \Phi$ or $S = \{x, y\}$. Therefore, $cay(\Gamma, S)$ $\cong \begin{cases} \overline{K_3} & \text{if } S = \Phi \\ K_3 & \text{if } S = \{x, y\} \end{cases}$. Clearly K_3 and $\overline{K_3}$ are neighbourhood-prime.

Hence the theorem.

Observation 2.2. (1) $Cay(\Gamma, S)$ of a finite group Γ of order n with $S = \Phi$ is isomorphic to $\overline{K_n}$ and so is neighbourhood-prime vacuously.

(2) By theorem 1.12, $Cay(\Gamma, S)$ of a finite group Γ of order n with $|S| \geq \frac{n}{2}$ is neighbourhood-prime, since $Cay(\Gamma, S)$ is a regular graph of degree $|S|$.

Theorem 2.3. *Let Γ be a finite abelian group of order n. Let S be a Cayley set of Γ such that $\langle S \rangle = \Gamma$. Then, $Cay(\Gamma, S)$ is neighbourhood-prime if $n \leq 3$ or if $n \not\equiv 2 (\text{mod } 4)$ for $n > 3$.*

Proof. Let Γ be a finite abelian group of order n and S, a Cayley set of Γ such that $\langle S \rangle = \Gamma$.

Case (i). $n \leq 3$

By theorem 2.1, $Cay(\Gamma, S)$ is neighbourhood-prime.

Case (ii). $n > 3$ and $n \not\equiv 2 (\text{mod } 4)$

Since $\langle S \rangle = \Gamma$, $Cay(\Gamma, S)$ is connected. Therefore, by theorem 1.14, $Cay(\Gamma, S)$ is hamiltonian. Therefore, by theorem 1.9, $Cay(\Gamma, S)$ is neighbourhood-prime.

Hence the theorem.

Corollary 2.4. *Let* Γ *be a finite cyclic group of order* n. *Let* S *be a Cayley set of* Γ *such that* $\langle S \rangle = \Gamma$. *Then,* $Cay(\Gamma, S)$ *is neighbourhood-prime if* $n \leq 3$ *or if* $n \not\equiv 2 (\bmod 4)$ *for* $n > 3$.

Proof. Since any cyclic group is abelian, the result follows.

Theorem 2.5. *Every Cayley graph of a finite group of prime order is neighbourhood-prime.*

Proof. Let Γ be a finite group of prime order p and S, a Cayley set of Γ.

Case (i). $S = \Phi$

Here, $Cay(\Gamma, S) \cong \overline{K_p}$ and so is neighbourhood-prime.

Case (ii). $S \neq \Phi$

When $p = 2$, $Cay(\Gamma, S) \cong K_2$ and so is neighbourhood-prime.

Let $p \geq 3$. Since p is prime, every element of Γ except identity is a generator of Γ. Therefore, S is always a generating set of Γ. Further, since p is prime, Γ is cyclic and $p \not\equiv 2 (\bmod 4)$. Hence the result follows from 2.4.

Theorem 2.6. *Let* Γ *be a finite hamiltonian group of order* n. *Let* S *be a Cayley set of* Γ *such that* $\langle S \rangle = \Gamma$. *Then* $Cay(\Gamma, S)$ *is neighbourhood-prime if* $n \not\equiv 2 (\bmod 4)$.

Proof. Let Γ be a finite hamiltonian group of order n and S, a Cayley set of Γ such that $\langle S \rangle = \Gamma$. Then $Cay(\Gamma, S)$ is connected. Therefore, by theorem 1.16, $Cay(\Gamma, S)$ is hamiltonian. Hence, by theorem 1.9, $Cay(\Gamma, S)$ is neighbourhood-prime if $n \not\equiv 2 (\bmod 4)$.

Theorem 2.7. *Let* Γ *be a finite abelian group of order* n. *Let* S *be a Cayley set of* Γ *such that* $\langle S \rangle = \Gamma$ *and* $|S| > 2\left(\left\lfloor \frac{n-6}{8} \right\rfloor + 1\right)$. *Then* $Cay(\Gamma, S)$ *is neighbourhood-prime.*

Proof. Let Γ be a finite abelian group of order n and S, a Cayley set of Γ such that $\langle S \rangle = \Gamma$ and $|S| > 2\left(\left\lfloor \frac{n-6}{8} \right\rfloor + 1\right)$.

Case (i). $n \leq 3$

By theorem 2.1, $Cay(\Gamma, S)$ is neighbourhood-prime.

Case (ii). $n > 3$

Since $\langle S \rangle = \Gamma$, $Cay(\Gamma, S)$ is connected regular graph of degree $|S|$. By theorem 1.14, $Cay(\Gamma, S)$ is hamiltonian. Also, $|E| = \frac{n}{2}|S| > n\left(\left\lfloor \frac{n-6}{8} \right\rfloor + 1\right)$ $= n\left\lfloor \frac{n-6}{8} \right\rfloor + n$. Hence by theorem 1.13, $Cay(\Gamma, S)$ is neighbourhood-prime.

Corollary 2.8. *Let* Γ *be a finite cyclic group of order* n*.* Le*t* S *be a Cayley set of* Γ *such that* $\langle S \rangle = \Gamma$ *and* $|S| > 2\left(\left\lfloor \frac{n-6}{8} \right\rfloor + 1\right)$*. Then* $Cay(\Gamma, S)$ *is neighbourhood-prime.*

Proof. Since any cyclic group is abelian, the result follows.

Theorem 2.9. *Let* Γ *be a finite hamiltonian group of order* n*. Let* S *be a Cayley set of* Γ *such that* $\langle S \rangle = \Gamma$ *and* $|S| > 2\left(\left\lfloor \frac{n-6}{8} \right\rfloor + 1\right)$*. Then* $Cay(\Gamma, S)$ *is neighbourhood-prime.*

Proof. Let Γ be a finite hamiltonian group of order n and S, a Cayley set of Γ such that $\langle S \rangle = \Gamma$ and $|S| > 2\left(\left\lfloor \frac{n-6}{8} \right\rfloor + 1\right)$. Then, $Cay(\Gamma, S)$ is connected regular graph of degree $|S|$. By theorem 1.16, $Cay(\Gamma, S)$ is hamiltonian. Also, $|E| = \frac{n}{2}|S| > n\left(\left\lfloor \frac{n-6}{8} \right\rfloor + 1\right) = n\left\lfloor \frac{n-6}{8} \right\rfloor + n$. Hence by theorem 1.13, $Cay(\Gamma, S)$ is neighbourhood-prime.

Observation 2.10. (1) $Cay(Z_n, \{1, 2, 3, \ldots, n-1\})$ is isomorphic to the complete graph K_n and so by theorem 1.12, is neighbourhood-prime.

(2) $Cay(Z_{2n}, \{1, 3, 5, \ldots, 2n-1\})$ is isomorphic to the complete n-regular bipartite graph $K_{n,n}$ and so by theorem 1.12, is neighbourhood-prime.

(3) $Cay(Z_{2n}, \{n\})$ is isomorphic to nK_2 and so is neighbourhood-prime vacuously.

(4) $Cay(Z_n, \{x, x^{-1}\})$ where $n \geq 3$ and x is a generator of Z_n, is isomorphic to the cycle C_n and so by 1.7 and 1.8, is neighbourhood-prime if and only if $n \not\equiv 2 (\text{mod } 4)$.

(5) *Let S be a Cayley set of* Z_n *containing atleast one generator of* Z_n. *Then* $Cay(Z_n, S)$ *is hamiltonian. Hence, by theorem* 1.9, $Cay(Z_n, S)$ *is neighbourhood-prime if* $n \not\equiv 2 (\text{mod } 4)$.

Theorem 2.11. *Let* $n \geq 3$ *and* S *be a Cayley set of* Z_n *such that* $\{1, k\} \subseteq S$ *for some* $k \not\equiv 1 (\text{mod } 4)$. *Then* $Cay(Z_n, S)$ *is neighbourhood-prime.*

Proof. Let $n \geq 3$ and S be a Cayley set of Z_n such that $\{1, k\} \subseteq S$ for some $k \not\equiv 1 (\text{mod } 4)$. Since $1 \in S$, $Cay(Z_n, S)$ contains a hamiltonian cycle $C = (0, 1, 2, \ldots, n-1, 0)$.

Therefore, by theorem 1.9, $Cay(Z_n, S)$ is neighbourhood-prime if $k \not\equiv 2 (\text{mod } 4)$.

Suppose $k \equiv 2 (\text{mod } 4)$. Since $k \in S$, $Cay(Z_n, S)$ contains a chord connecting 0 and k.

Case (i). $k \equiv 0, 2 (\text{mod } 4)$.

Then k is even and so $C_{k+1}(0, 1, 2, \ldots, k, 0)$ is an odd cycle in $Cay(Z_n, S)$. Therefore, by theorem 1.11, $Cay(Z_n, S)$ is neighbourhood-prime.

Case (ii). $k \equiv 3 (\text{mod } 4)$.

Here $C_{k+1}(0, 1, 2, \ldots, k, 0)$ is a cycle of length $4m$ for some $m \in Z$ formed by the chord and edges from C. Therefore, by theorem 1.10, $Cay(Z_n, S)$ is neighbourhood-prime.

Hence the theorem.

Theorem 2.12. *Let* $n \geq 3$ *and* S *be a Cayley set of* Z_n *such that* $\{x, y\} \subseteq S$ *where* $\gcd(x, n) = 1$ *and* y *is even. Then* $Cay(Z_n, S)$ *is neighbourhood-prime.*

Proof. Let $n \geq 3$ and S be a Cayley set of Z_n satisfying the hypothesis of the theorem. Since $x \in S$ and $\gcd(x, n) = 1$, $\langle x \rangle = Z_n$. Then $Cay(Z_n, S)$ contains a hamiltonian cycle $C = (x, x \oplus x, \ldots, x \oplus x \oplus \ldots (n-1 \text{ times}), 0, x)$. Therefore, by theorem 1.9, $Cay(Z_n, S)$ is neighbourhood-prime if $n \not\equiv 2 (\text{mod } 4)$.

Suppose $n \equiv 2 (\text{mod } 4)$. Since $y \in S$, $Cay(Z_n, S)$ contains a chord connecting 0 and y. Further, n is even and so x is odd. Hence y is even implies, C contains a path P joining 0 and y of even length. Therefore, P together with the chord $(0, y)$ forms an odd cycle in $Cay(Z_n, S)$. Hence by theorem 1. 11, $Cay(Z_n, S)$ is neighbourhood-prime.

Theorem 2.13. *Let $n > 2k$ and $n \equiv 2k (\text{mod } 4k)$ where $k \in Z^+$. Let S be a Cayley set of Z_n such that $\langle S \rangle = Z_n$ and $2k \in S$. Then $Cay(Z_n, S)$ is neighbourhood-prime.*

Proof. Let $n > 2k$ and $n \equiv 2k (\text{mod } 4k)$ where $k \in Z^+$. Let S be a Cayley set of Z_n satisfying the hypothesis of the theorem. Since $\langle S \rangle = Z_n$, $Cay(Z_n, S)$ is connected. $(Z_n, \oplus)$ is finite and cyclic implies $(Z_n, \oplus)$ is abelian. Therefore, by theorem 1.14, $Cay(Z_n, S)$ is hamiltonian. Let $n = 4kt + 2k$ for some $t \in Z^+$. Let l be the order of the element $2k$. Since $2k \in S$, $Cay(Z_n, S)$ contains the cycle,

$$C = (2k, 2k \oplus 2k, 2k \oplus 2k \oplus 2k, \ldots, 2k \oplus 2k \oplus \ldots (l-1 \text{ times}), 2k \oplus 2k \oplus \ldots (l \text{ times}), 2k)$$

$$= (2k, 2(2k), 3(2k), \ldots, n - 2k, 0, 2k)$$

$$= (2k, 2(2k), 3(2k), \ldots, 4kt, 0, 2k)$$

$$= (2k, 2(2k), 3(2k), \ldots, 2t(2k), 0, 2k)$$

Obviously C is of odd length $2t + 1$.

Hence by theorem 1.11, $Cay(Z_n, S)$ is neighbourhood-prime.

Theorem 2.14. *Let* $n > 1$ *and* $x \neq n \in Z_{2n}$. *Let* S *be a Cayley set of* Z_{2n} *such that* $\{x, x \oplus x \oplus \ldots (2m \text{ times})\} \subseteq S$ *for some* $m \in Z^+$ *and* $2m < l$ *where* $l = o(x)$. If $\langle S \rangle = Z_{2n}$, *then* $Cay(Z_{2n}, S)$ *is neighbourhood-prime.*

Proof. Let $n > 1$ and $x \neq n \in Z_{2n}$. Let S be a Cayley set of Z_{2n} satisfying the hypothesis of the theorem. Since $\langle S \rangle = Z_{2n}$, $Cay(Z_{2n}, S)$ is connected. $(Z_{2n}, \oplus)$ is finite and cyclic implies $(Z_{2n}, \oplus)$ is abelian. Therefore, by theorem 1.14, $Cay(Z_{2n}, S)$ is hamiltonian.

Let $y = x \oplus x \oplus \ldots (2m \text{ times})$ for some $m \in Z^+$ and $2m < l$ where $l = o(x)$. Since $x \in S$, $Cay(Z_{2n}, S)$ contains the cycle, $C = (x, x \oplus x, \ldots, x \oplus x \oplus \ldots (n-1 \text{ times}), 0, x)$. Hence C contains a path P joining 0 and y of even length. Since $y \in S$, $Cay(Z_{2n}, S)$ contains an edge connecting 0 and y. Therefore, P together with the edge $(0, y)$ forms an odd cycle in $Cay(Z_{2n}, S)$. Hence by theorem 1.11, $Cay(Z_{2n}, S)$ is neighbourhood-prime.

Lemma 2.15. *Let* $x \in Z_n$ *where* n *is even and* $\gcd(x, n) = 1$. *Then* $x \oplus x \oplus x \oplus \ldots \left(\frac{n}{2} \, times\right) = \frac{n}{2}$.

Proof. Let $x \in Z_n$ where n is even. Further, $\gcd(x, n) = 1$ and so x is odd. Let $x = 2k + 1$ for some $k \in Z^+$. Therefore, $\frac{n}{2} x = \frac{n}{2}(2k+1) = nk + \frac{n}{2}$. Hence $x \oplus x \oplus x \oplus \ldots \left(\frac{n}{2} \text{ times}\right) = \frac{n}{2}$.

Theorem 2.16. *Let* $n \equiv 6 (\text{mod } 8)$ *and* S *be a Cayley set of* Z_n *such that* $\left\{x, \frac{n}{2}\right\} \subseteq S$ *where* $\gcd(x, n) = 1$. *Then* $Cay(Z_n, S)$ *is neighbourhood-prime.*

Proof. Let $n \equiv 6 (\text{mod } 8)$ and S be a Cayley set of Z_n satisfying the hypothesis of the theorem. Since $x \in S$ and $\gcd(x, n) = 1$, $\langle x \rangle = Z_n$. Then $Cay(Z_n, S)$ contains a hamiltonian cycle $C = (x, x \oplus x, \ldots, x \oplus x \oplus \ldots (n-1 \text{ times}), 0, x)$. Let $n = 8t + 6$ for some $t \in Z^+ \cup \{0\}$. Clearly, n is even.

By lemma 2.15, $x \oplus x \oplus x \oplus \ldots \left(\frac{n}{2} \text{times}\right) = \frac{n}{2}$. Hence C contains a path P joining 0 and $\frac{n}{2}$ of length $4t + 3$. Since $\frac{n}{2} \in S$, $Cay(Z_n, S)$ contains a chord connecting 0 and $\frac{n}{2}$. Therefore, P together with the chord $\left(0, \frac{n}{2}\right)$ forms a cycle of length $4(t+1)$ in $Cay(Z_n, S)$. Hence by theorem 1.10, $Cay(Z_n, S)$ is neighbourhood-prime.

Theorem 2.17. *Let $n > 3$ and $n \equiv 0 (\text{mod } 3)$. Then, $Cay\left(Z_n, \left\{\frac{n}{3}, \frac{2n}{3}\right\}\right)$ is not neighbourhood-prime.*

Proof. Let $n > 3$ and $n \equiv 0 (\text{mod } 3)$. Let $S = \left\{\frac{n}{3}, \frac{2n}{3}\right\}$. Then $Cay(Z_n, S) \cong \frac{n}{3} C_3$. Further, to label each cycle we need a minimum of two odd positive integers and so totally $\frac{2n}{3}$ odd positive integers. But there are only $\left\lceil \frac{n}{2} \right\rceil$ odd positive integers $\leq n$. Also $\left\lceil \frac{n}{2} \right\rceil < \frac{2n}{3}$ and so the neighbourhood-prime labeling is not possible. Hence $Cay(Z_n, S)$ is not neighbourhood-prime.

Observation 2.18. (1) $Cay(D_{2n}, \{ab^i\})$ where $0 \leq i \leq n-1$ and $Cay(D_{2n}, \{b^{\frac{n}{2}}\})$ where n is even, are isomorphic to nK_2 and so neighbourhood-prime vacuously.

Theorem 2.19. *Let S be a Cayley set of a dihedral group D_{2n} such that $\{b, a\} \subseteq S$. Then, $Cay(D_{2n}, S)$ is neighbourhood-prime.*

Proof. Let S be a Cayley set of a dihedral group D_{2n} satisfying the hypothesis of the theorem. Since $b \in S$ and $o(b) = n$, $Cay(D_{2n}, S)$ contains two disjoint cycles $C_1 = (b, b^2, b^3, \ldots, b^{n-1}, 1, b)$ and $C_2 = (ab, ab^2, ab^3, \ldots, ab^{n-1}, a, ab)$ of length n. Hence C_1 contains a path P_1 joining b and 1 of length $n-1$ and C_2 contains a path P_2 joining a and ab^{n-1} of length $n-1$. Since $a \in S$, $Cay(D_{2n}, S)$ contains an edge connecting 1 and a and also contains an edge connecting ab^{n-1} and $ab^{n-1}a = aab = b$. Therefore, $P_1 + (1, a) + P_2 + (ab^{n-1}, b) = (b, b^2, b^3, \ldots, b^{n-1}, 1, a, ab, ab^2, ab^3, \ldots, ab^{n-1}, b)$ is a hamiltonian cycle in $Cay(D_{2n}, S)$ of length $2n$.

Case (i). n is odd

Then C_1 is an odd cycle in $Cay(D_{2n}, S)$. Hence by theorem 1.11, $Cay(D_{2n}, S)$ is neighbourhood-prime.

Case (ii). n is even

Then $2n \equiv 0 (\mathrm{mod}\, 4)$ and so $2n \not\equiv 0 (\mathrm{mod}\, 4)$. Hence by theorem 1.9, $Cay(D_{2n}, S)$ is neighbourhood-prime.

Theorem 2.20. *Let S be a Cayley set of a dihedral group D_{2n} such that $\{b^i, ab^i\} \subseteq S$ where $1 \le i \le n-1$ and $\gcd(i, n) = 1$. Then $Cay(D_{2n}, S)$ is neighbourhood-prime.*

Proof. Let S be a Cayley set of a dihedral group D_{2n} satisfying the hypothesis of the theorem. Let $1 \le i \le n-1$. Since $b^i \in S$ and $\gcd(i, n) = 1$, $o(b^i) = n$. Then $Cay(D_{2n}, S)$ contains two disjoint cycles $C_1 = (b^i, b^{r_{2i}}, b^{r_{3i}}, \ldots, b^{r_{(n-1)i}}, 1, b^i)$ and $C_2 = (ab^i, ab^{r_{2i}}, ab^{r_{3i}}, \ldots, ab^{r_{(n-1)i}}, a, ab^i)$ where r_x = remainder of $x (\mathrm{mod}\, n)$, of length n. Hence C_1 contains a path P_1 joining b^i and 1 of length $n-1$ and C_2 contains a path P_2 joining ab^i and a of length $n-1$. Since $ab^i \in S$, $Cay(D_{2n}, S)$ contains an edge connecting 1 and *abi* and also contains an edge connecting a and $aab^i = b^i$. Therefore, $P_1 + (1, ab^i) + P_2 + (a, b^i) = (b^i, b^{r_{2i}}, b^{r_{3i}}, \ldots, b^{r_{(n-1)i}}, 1, ab^i, ab^{r_{2i}}, ab^{r_{3i}}, \ldots, ab^{r_{(n-1)i}}, a, b^i)$ is a hamiltonian cycle in $Cay(D_{2n}, S)$ of length $2n$.

Case (i). n is odd

Then C_1 is an odd cycle in $Cay(D_{2n}, S)$. Hence by theorem 1.11, $Cay(D_{2n}, S)$ is neighbourhood-prime.

Case (ii). n is even

Then $2n \equiv 0 (\bmod 4)$ and so $2n \not\equiv 0 (\bmod 4)$. Hence by theorem 1.9, $Cay(D_{2n}, S)$ is neighbourhood-prime.

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