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COMMON FIXED POINT THEOREMS IN COMPLEX PARTIAL *b*-METRIC SPACE

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Abstract

In this paper, we obtain existence and unique common fixed point theorem for four self-maps in complex partial b-metric space.

1. Introduction

The fixed point theory is one of the most important tools in many branches of science, computer science, engineering and the development of non-linear analysis. Backhtin [1] introduced the concept of b-metric spaces in 1989. Azam et al. [2] introduced the concept of complex valued metric spaces. Rao et.al [3] introduced complex valued b-metric space. P. Dhivya and M. Marudai [4] introduced the concept of complex valued partial metric space and extended the common fixed point theorems under the contraction condition of rational expression. In 2019, M. Gunaseelan [6] introduced the notion of complex valued partial b-metric space and proved existence and uniqueness of fixed point theorem.

In this paper, we prove an existence and uniqueness of common fixed point theorems using weakly compatible in complex partial b-metric space.

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2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $e_1, e_2 \in \mathbb{C}$.

Define a partial order $\leq \mathbb{C}$ as follows:

$$e_1 \leq e_2 \text{ iff } \operatorname{Re}(e_1) \leq \operatorname{Re}(e_2), \operatorname{Im}(e_1) \leq \operatorname{Im}(e_2).$$

It follows that $e_1 \leq e_2$ if one of the following conditions is satisfied.

i. $\operatorname{Re}(e_1) = \operatorname{Re}(e_2)$, $\operatorname{Im}(e_1) < \operatorname{Im}(e_2)$.

ii. $\operatorname{Re}(e_1) < \operatorname{Re}(e_2)$, $\operatorname{Im}(e_1) = \operatorname{Im}(e_2)$.

iii. $\operatorname{Re}(e_1) < \operatorname{Re}(e_2)$, $\operatorname{Im}(e_1) < \operatorname{Im}(e_2)$.

iv. $\operatorname{Re}(e_1) = \operatorname{Re}(e_2)$, $\operatorname{Im}(e_1) = \operatorname{Im}(e_2)$.

In particular, we write $e_1 \lneqq e_2$ if $e_1 \neq e_2$ and one of i, ii, iii is satisfied and we write $e_1 < e_2$ if only iii is satisfied Notice that

(a) If $0 \leq e_1 \lneqq e_2$, then $|e_1| < |e_2|$.

(b) If $e_1 \leq e_2$, and $e_2 < e_3$ then $e_1 < e_3$.

(c) If $f, g \in \mathbb{R}$ and $f \leq g$ then $fh \leq gh$, for all $h \in \mathbb{C}$.

Definition 2.1 [5]. A complex partial metric on a non-void set R is a mapping $\rho_c : R \times R \to \mathbb{C}^+$ satisfying the following conditions:

(i) $0 \leq \rho_c(u, u) \leq \rho_c(u, v)$ (small self distance)

(ii) $\rho_c(u, v) = \rho_c(v, u)$ (symmetry)

(iii) $\rho_c(u, u) = \rho_c(u, v) = \rho_c(v, v)$ iff $u = v$ (equality)

(iv) $\rho_c(u, v) \leq \rho_c(u, w) + \rho_c(w, v) - \rho_c(w, w)$ (triangularity)

for all $u, v, w \in R$. A complex partial metric spaces is a pair (R, ρ_c) such that R is a non-void set and ρ_c is a complex partial metric on R.

Definition 2.2 [6]. A complex partial b-metric on a non-void set R is a mapping $\sigma_{cb} : R \times R \to \mathbb{C}^+$ satisfying the following conditions:

(i) $0 \leq \sigma_{cb}(x, x) \leq \sigma_{cb}(x, y)$ (small self distance)

(ii) $\sigma_{cb}(x, y) = \sigma_{cb}(y, x)$ (symmetry)

(iii) $\sigma_{cb}(x, x) = \sigma_{cb}(x, y) = \sigma_{cb}(y, z)$ iff $x = y$ (equality)

(iv) $\sigma_{cb}(x, y) \leq s[\sigma_{cb}(x, z) + \rho_c(z, y) - \sigma_{cb}(z, z)]$ (triangularity)

for all $x, y, z \in R$. A complex partial b-metric space is a pair (R, σ_{cb}) such that R is a non-void set and σ_{cb} is a complex partial b-metric on. The number s is called the coefficient of (R, σ_{cb}).

Remark 2.1 [6]**.** In a complex partial b-metric space (R, σ_{cb}) if $x, y \in R$ and $\sigma_{cb}(x, y) = 0$, then $x = y$. But the converse may not be true.

Remark 2.2 [6]**.** It is clear that every complex partial metric is a complex partial b-metric space with coefficient $s = 1$ and every complex valued b-metric is a complex partial b-metric space with the same coefficient and zero self-distance. However, the converse of the fact need not hold.

Definition 2.3 [6]. Let (R, σ_{cb}) be a complex partial b-metric space with coefficient s. Let $\{x_n\}$ be any sequence in R and $x \in R$. Then

1. The sequence $\{x_n\}$ is said to be convergent w.r.to σ_{cb} and converges to x, if $\lim_{n \to \infty} \sigma_{cb}(x_n, x) = \sigma_{cb}(x, x)$

2. The sequence $\{x_n\}$ is said to be Cauchy sequence in (R, σ_{cb}) if $\lim_{n \to \infty} \sigma_{cb}(x_n, x_m)$ exists and is finite.

3. Every Cauchy sequence $\{x_n\}$ in R there exists $x \in R$ such that $\lim_{n \to \infty} \sigma_{cb}(x_n, x_m) = \lim_{n \to \infty} \sigma_{cb}(x_n, x) = \sigma_{cb}(x, x)$. Then (R, σ_{cb}) is said to be a complete complex partial b-metric space.

4. A mapping $p : R \to R$ is said to be continuous at $x_0 \in R$ if for every $\varepsilon > 0$, there exists $\delta > 0$, such that

$$P(B_{\sigma_{cb}}(\lambda_0, \delta)) \subset (B_{\sigma_{cb}}(\lambda_0, \varepsilon)).$$

Definition 2.4 [3]**.** Let R be a non-void set and let $s \geq 1$ be a given real number. A mapping $\delta : R \times R \to \mathbb{C}$ is said to be a complex valued b-metric if the following conditions are satisfied:

1. $0 \leq \delta(x, y)$ and $\delta(x, y) = 0 \Leftrightarrow x = y, \forall x, y \in R.$

2. $\delta(x, y) = \delta(y, x), \forall x, y \in R.$

3. $\delta(x, y) \leq s[\delta(x, z) + \delta(z, y)], \forall x, y, z \in R.$

The pair (x, δ) is called a complex valued b-metric space.

Definition 2.5 [6]**.** Let V and W be two self maps defined on a set R, then V and W are said to be weakly compatible if they commute at coincide points.

3. Main Results

3.1. Common Fixed Point Theorem

We prove fixed point theorem for four self-maps in complex partial b-metric space.

Theorem 3.1. *Let* (R, σ_{cb}) *be a complete complex partial b-metric space with coefficient* $s \geq 1$ *and let* C, D, I *and* J *are four self maps of* R *such that*

$$J(R) \subseteq C(R) \text{ and } I(R) \subseteq D(R) \text{ satisfying}$$

(i) $\sigma_{cb}(Iz, Jw) \leq \frac{\lambda}{s^2} p(z, w)$ *if* $s \geq 1$ *and* $\lambda \in (0, 1), \forall z, w \in R.$

where $$p(z, w) = \max\{\sigma_{cb}(Cz, Dw), \sigma_{cb}(Cz, Iz), \sigma_{cb}(Dw, Jw), \frac{1}{2}[\sigma_{cb}(Dw, Iz) + \sigma_{cb}(Cz, Jw)], \frac{\sigma_{cb}(Cz, Iz) \cdot \sigma_{cb}(Dw, Jw)}{1 + \sigma_{cb}(Cz, Dw)}\}.$$

(ii) Suppose that the pairs $\{C, I\}$ and $\{D, J\}$ are weakly compatible, then C, D, I and J have a unique common fixed point.

Proof. Let $z_0 \in R$ be arbitrary from the condition $J(R) \subseteq C(R)$ and $I(R) \subseteq D(R)$, there exists z_1, z_2 such that $w_0 = Dz_1 = Iz_0$ and $w_1 = Dz_2 = Jz_1$.

We can construct successively the sequences $\{w_n\}$ and $\{z_n\}$ in R as follows:

$$w_{2n} = DZ_{2n+1} = Iz_{2n} \text{ and } w_{2n+1} = Cz_{2n+2} = Jz_{2n+1} \tag{3.1}$$

using equation (3.1) in (i), we get

$$\sigma_{cb}(w_{2n}, w_{2n+1}) = \sigma_{cb}(Iz_{2n}, Jz_{2n+1}) \leq \frac{\lambda}{s^2} p(z_{2n}, z_{2n+1}) \tag{3.2}$$

where

$$P(z_{2n}, z_{2n+1}) = \max \{\sigma_{cb}(Cz_{2n}, Dz_{2n+1}), \sigma_{cb}(Cz_{2n}, Iz_{2n}), \sigma_{cb}(Dz_{2n+1}, Jz_{2n+1})$$

$$\frac{1}{2}[\sigma_{cb}(Dz_{2n+1}, Iz_{2n}) + \sigma_{cb}(Cz_{2n}, Jz_{2n+1})], \frac{\sigma_{cb}(Cz_{2n}, Jz_{2n}) \cdot \sigma_{cb}(Dz_{2n+1}, Jz_{2n+1})}{1+\sigma_{cb}(Cz_{2n}, Dz_{2n+1})}\}$$

$$= \max \{\sigma_{cb}(w_{2n-1}, w_{2n}), \sigma_{cb}(w_{2n-1}, w_{2n}), \sigma_{cb}(w_{2n}, w_{2n+1}),$$

$$\frac{1}{2}[\sigma_{cb}(w_{2n}, w_{2n}) + \sigma_{cb}(w_{2n-1}, w_{2n+1})], \frac{\sigma_{cb}(w_{2n-1}, w_{2n}) \cdot \sigma_{cb}(w_{2n}, w_{2n+1})}{1 + \sigma_{cb}(w_{2n-1}, w_{2n})}\}$$

$$P(z_{2n}, z_{2n+1}) = \max \{\sigma_{cb}(w_{2n-1}, w_{2n}), \sigma_{cb}(w_{2n}, w_{2n+1})\}. \tag{3.3}$$

Substitute (3.3) in equation (3.2)

$$\sigma_{cb}(w_{2n}, w_{2n+1}) \leq \frac{\lambda}{s^2}[\sigma_{cb}(w_{2n}, w_{2n+1})]$$

$$\left(1 - \frac{\lambda}{s^2}\right)\sigma_{cb}(w_{2n}, w_{2n+1}) \leq 0$$

which is a contradiction. Since $\lambda \in (0, 1)$ and $s \geq 1$.

We conclude that $\sigma_{cb}(w_{2n}, w_{2n+1}) \leq \frac{\lambda}{s^2}[\sigma_{cb}(w_{2n-1}, w_{2n})]$.

Similarly we get

$$\sigma_{cb}(w_{2n+1}, w_{2n+2}) \leq \frac{\lambda}{s^2}[\sigma_{cb}(w_{2n}, w_{2n+1})].$$

It follows that

$$\sigma_{cb}(w_n, w_{n+1}) \leq \frac{\lambda}{s^2}[\sigma_{cb}(w_{n-1}, w_n) \leq \ldots \leq \left(\frac{\lambda}{s^2}\right)^n \sigma_{cb}(w_0, w_1).$$

Which implies

$$|\sigma_{cb}(w_n, w_m)| \leq \left(\frac{\lambda}{s^2}\right) s[\sigma_{cb}(w_n, w_{n+1}) + \sigma_{cb}(w_{n+1}, w_m) - \sigma_{cb}(w_{n+1}, w_{n+1})]$$

$$\leq s\left(\frac{\lambda}{s^2}\right)[\sigma_{cb}(w_n, w_{n+1}) + \sigma_{cb}(w_{n+1}, w_m)]$$

$$\leq s\left(\frac{\lambda}{s^2}\right)[\sigma_{cb}(w_n, w_{n+1})] + s^2\left(\frac{\lambda}{s^2}\right)[\sigma_{cb}(w_{n+1}, w_{n+2}) + \sigma_{cb}(w_{n+2}, w_m)]$$

$$- s\left(\frac{\lambda}{s^2}\right)[\sigma_{cb}(w_{n+2}, w_{n+2})]$$

$$\leq s\left(\frac{\lambda}{s^2}\right)[\sigma_{cb}(w_n, w_{n+1})] + s^2\left(\frac{\lambda}{s^2}\right)[\sigma_{cb}(w_{n+1}, w_{n+2}) + s^2\left(\frac{\lambda}{s^2}\right)[\sigma_{cb}(w_{n+2}, w_m)]$$

$$\leq s\left(\frac{\lambda}{s^2}\right)^n |\sigma_{cb}(w_0, w_1)| + s^2\left(\frac{\lambda}{s^2}\right)^{n+1} |\sigma_{cb}(w_0, w_1)|.$$

$$+ \ldots + s^{m-n}\left(\frac{\lambda}{s^2}\right)^{m-n} |\sigma_{cb}(w_0, w_1)| = \sum_{i=1}^{m-n} s^i \left(\frac{\lambda}{s^2}\right)^{i+n-1} |\sigma_{cb}(w_0, w_1)|$$

$$\therefore |\sigma_{cb}(w_n, w_m)| \leq \sum_{i=1}^{m-n} s^{i+n-1}\left(\frac{\lambda}{s^2}\right)^{i+n-1} |\sigma_{cb}(w_0, w_1)|$$

$$= \sum_{i=1}^{m-n} s^t \left(\frac{\lambda}{s^2}\right)^t |\sigma_{cb}(w_0, w_1)|$$

$$\leq \sum_{i=1}^{m-n} \left(\frac{\lambda}{s^2}\right)^t |\sigma_{cb}(w_0, w_1)|$$

$$= \frac{(\lambda/s)^n}{1-(\lambda/s)} |\sigma_{cb}(w_0, w_1)|.$$

Hence

$$|\sigma_{cb}(w_n, w_{n+1})| \leq \frac{(\lambda/s)^n}{1-(\lambda/s)} |\sigma_{cb}(w_0, w_1)| \text{ as } n \to \infty.$$

Thus $\{w_n\}$ is a Cauchy sequence.

Since R is complete, so there exist some $u \in R$ such that $w_n \to u$, as $n \to \infty$ and

$$\sigma_{cb}(u, u) = \lim_{n\to\infty} \sigma_{cb}(u, w_n) = \lim_{n\to\infty} \sigma_{cb}(w_n, w_n) = 0.$$

For its sequence we have $Dz_{2n+1} \to u, Iz_{2n} \to u, Cz_{2n+1} \to u$ and $Jz_{2n} \to u$.

Since, $J(R) \subseteq C(R)$, there exist a point $v \in R$ such that $u = Cv$.

Suppose that $\sigma_{cb}(Iv, u) > 0$.

Then

$$\sigma_{cb}(Iv, u) \leq s\sigma_{cb}(Iv, Jz_{2n}) + s\sigma_{cb}(Jz_{2n}, z) - \sigma_{cb}(Jz_{2n}, Jz_{2n})$$

$$\leq s\frac{\lambda}{s^2} P[v, z_{2n}]$$

$$P(v, z_{2n}) = \max\{\sigma_{cb}(Cv, Dz_{2n}), \sigma_{cb}(Cv, Iv), \sigma_{cb}(DZ_{2n}, Jz_{2n}),$$

$$\frac{1}{2}[\sigma_{cb}(Dz_{2n}, Iv) + \sigma_{cb}(Cv, Jz_{2n})], \frac{\sigma_{cb}(Dz_{2n}, Jz_{2n}) \cdot \sigma_{cb}(Cv, Iv)}{1 + \sigma_{cb}(Cv, Dz_{2n})}\}$$

As

$$n \to \infty = \max\{\sigma_{cb}(u, u), \sigma_{cb}(u, Iv), \sigma_{cb}(u, u), \frac{1}{2}[\sigma_{cb}(u, Iv) + \sigma_{cb}(u, u)],$$

$$\frac{\sigma_{cb}(u, u) \cdot \sigma_{cb}(u, u)}{1 + \sigma_{cb}(u, u)}\}$$

$$= \sigma_{cb}(u, Iv)$$

$$|\sigma_{cb}(Iv, u)| \leq \frac{\lambda}{s^2} s |\sigma_{cb}(u, Iv)|.$$

Since, $s \geq 1$ and $\lambda \in (0, 1)$

$$|\sigma_{cb}(Iv, u)|(1 - \lambda/s) \leq 0$$

which is a contradiction.

Hence $Cv = Iv = u$.

Since $I(R) \subseteq D((R))$, there exists a point $y \in R$ such that $u = Dy$. Suppose that $\sigma_{cb}(u, Jy) > 0$.

Then $\sigma_{cb}(u, Jy) \leq \sigma_{cb}(Iv, Jy) \leq \frac{\lambda}{s^2} P(v, y)$

$$P(v, y) = \max \{\sigma_{cb}(Cv, Dy), \sigma_{cb}(Cv, Iv), \sigma_{cb}(Dy, Jy),$$

$$\frac{1}{2}[\sigma_{cb}(Dy, Iv) + \sigma_{cb}(Cv, Jy)], \frac{\sigma_{cb}(Cv, Iv) \cdot \sigma_{cb}(Dy, Jy)}{1 + \sigma_{cb}(Cv, Dy)}\}$$

As $n \to \infty$

$$P(u, v) = \max \{\sigma_{cb}(u, u), \sigma_{cb}(u, u), \sigma_{cb}(u, Jy),$$

$$\frac{1}{2}[\sigma_{cb}(y, u) + \sigma_{cb}(u, Jy)], \frac{\sigma_{cb}(u, u) \cdot \sigma_{cb}(u, Jy)}{1 + \sigma_{cb}(u, u)}\}$$

$$= \frac{\lambda}{s^2} \sigma_{cb}(u, Jy)$$

$$\sigma_{cb}(u, Jy) = \frac{\lambda}{s^2} \sigma_{cb}(u, jy)$$

$$\sigma_{cb}(u, Jy)(1 - \lambda/s^2) \leq 0$$

which is a contradiction.

Therefore $Jy = Dy = u$.

Hence $Cv = Iv = Jy = Dy = u$.

Since C and I are weakly compatible maps then $ICv = CIv$.

Therefore $Iu = Cu$.

Now we claim that u is a fixed point of I if $u \neq u$ we have

$$\sigma_{cb}(Iu, u) \leq \sigma_{cb}(Iu, Jy) \leq \frac{\lambda}{s^2} P(u, y)$$

$$P(u, y) = \max \{\sigma_{cb}(Cu, Dy), \sigma_{cb}(Cu, Iu), \sigma_{cb}(Dy, Jy),$$

$$\frac{1}{2}[\sigma_{cb}(Dy, Iu) + \sigma_{cb}(Cu, Jy)], \frac{\sigma_{cb}(Cu, Iu) \cdot \sigma_{cb}(Dy, Jy)}{1 + \sigma_{cb}(Cu, Dy)}\}$$

$$= \max\{\sigma_{cb}(Iu, u), \sigma_{cb}(Iu, Iu), \sigma_{cb}(u, u),$$

$$\frac{1}{2}[\sigma_{cb}(u, Iu) + \sigma_{cb}(Iu, u)], \frac{\sigma_{cb}(Iu, Iu) \cdot \sigma_{cb}(u, u)}{1 + \sigma_{cb}(Iu, u)}\}$$

$$= \sigma_{cb}(Iu, u)$$

$$\sigma_{cb}(Iu, u) = \frac{\lambda}{s^2}\sigma_{cb}(Iu, u)$$

$$(1 - \lambda/s^2)\sigma_{cb}(Iu, u) \le 0$$

which is a contradiction.

$$Iu = u.$$

Hence $Iu = cu = u$.

Similarly, D and J weakly compatible maps, we have $Du = Ju$. Now, we claim that u is a fixed point of J. suppose that $Ju \neq u$.

Then we have

$$\sigma_{cb}(u, Ju) \le \sigma_{cb}(Iu, Ju) \le \frac{\lambda}{s^2}P(u, u)$$

$$P(u, u) = \max\{\sigma_{cb}(Cu, Du), \sigma_{cb}(Cu, Iu), \sigma_{cb}(Du, Ju),$$

$$\frac{1}{2}[\sigma_{cb}(Du, Iu) + \sigma_{cb}(Cu, Ju)], \frac{\sigma_{cb}(Cu, Iu) \cdot \sigma_{cb}(Du, Ju)}{1 + \sigma_{cb}(Cu, Du)}\}$$

$$= \max\{\sigma_{cb}(u, Ju), \sigma_{cb}(u, u), \sigma_{cb}(Ju, Ju),$$

$$\frac{1}{2}[\sigma_{cb}(Ju, u) + \sigma_{cb}(u, Ju)], \frac{\sigma_{cb}(u, u) \cdot \sigma_{cb}(Ju, Ju)}{1 + \sigma_{cb}(u, Ju)}\}$$

$$= \sigma_{cb}(u, Ju)$$

$$\sigma_{cb}(u, Ju) \le \frac{\lambda}{s^2}\sigma_{cb}(u, Ju)$$

$$(1 - \lambda/s^2)\,\sigma_{cb}(u,\, Ju) \le 0$$

which is contradiction.

Hence $Du = Ju = u$.

Hence $Cu = Iu = Du = Ju = u$. And it follows that u is a common fixed point of C, D, I and J. Next we claim that the uniqueness of u.

Let u and v are distinct common fixed point of C, D, J and I.

Suppose not

$$\sigma_{cb}(u,\, v) \le \sigma_{cb}(Iu,\, Ju) \le \frac{\lambda}{s^2} P(u,\, v)$$

$$\begin{aligned} P(u,\, v) = \max\, \{&\sigma_{cb}(Cu,\, Dv),\, \sigma_{cb}(Cu,\, Iv),\, \sigma_{cb}(DvS,\, Jv), \\ &\frac{1}{2}[\sigma_{cb}(Dv,\, Iu) + \sigma_{cb}(Cu,\, Jv)],\, \frac{\sigma_{cb}(Cu,\, Du)\cdot\sigma_{cb}(Dv,\, Jv)}{1+\sigma_{cb}(Cu,\, Dv)}\} \\ = \max\, \{&\sigma_{cb}(u,\, v),\, \sigma_{cb}(u,\, u),\, \sigma_{cb}(v,\, v), \\ &\frac{1}{2}[\sigma_{cb}(v,\, u) + \sigma_{cb}(u,\, v)],\, \frac{\sigma_{cb}(u,\, u)\cdot\sigma_{cb}(v,\, v)}{1+\sigma_{cb}(u,\, v)}\} \\ = \sigma_{cb}&(u,\, v) \end{aligned}$$

$$\sigma_{cb}(u,\, v) \le \frac{\lambda}{s^2}\,\sigma_{cb}(u,\, v)$$

$$(1 - \lambda/s^2)\,\sigma_{cb}(u,\, v) \le 0.$$

Therefore $u = v$. Hence u is the unique common fixed point of C, D, I and J.

Corollary 3.2. *Let $(R,\, \sigma_{cb})$ be a complete complex partial b-matric space with coefficient $s \ge 1$ and let C, D, I and J are four self maps of R such that $J(R) \subseteq D(R)$ and $I(R) \subseteq D(R)$ satisfying*

(i) $\sigma_{cb}(Iz,\, Jw) \le \frac{\lambda}{s^2} P(z,\, w)$ *if* $s \ge 1$ *and* $\lambda \in (0,\, 1)$, $\forall z,\, w \in R$

$$P(z, w) = \max\{\sigma_{cb}(Cz, Dw), \sigma_{cb}(Cz, Iz), \sigma_{cb}(Dw, Jw)\}.$$

(ii) *Suppose that the pairs* $\{C, I\}$ *and* $\{D, I\}$ *are weakly compatible. Then* C, D, I *and* J *have a unique common fixed point.*

Corollary 3.3. *Let* (R, σ_{cb}) *be a complete complex partial b-matric space with coefficient* $s \geq 1$ *and let* C, D, I *and* J *are four self maps of* R *such that* $J(R) \subseteq D(R)$ *and* $I(R) \subseteq D(R)$ *satisfying*

(i)

$$\sigma_{cb}(Iz, Jw) \leq a_1\sigma_{cb}(Cz, Dw) + a_2\sigma_{cb}(Cz, Iz) + a_3\sigma_{cb}(Dw, Iw) + a_4\sigma_{cb}(Cz, Iw).$$

Where $z, w \in R$ and $a_1, a_2, a_3, a_4 \geq 0$, $a_1 + a_2 + a_3 + 2a_4 < 1$

(ii) The pairs $\{C, I\}$ and $\{D, J\}$ are weakly compatible. Then C, D, I and J have a unique common points.

Example 3.4. Let $R = \left\{-\frac{1}{2}\right\} \cup (0, 2]$ and $\sigma_{cb}(u, v) = \{\max\{u, v\}\}^2 + (1 + iv)$ where $u, v \in R$ then (R, σ_{cb}) be a complete complex partial b-matric space.

Let C, D, I and $J : R \to R$ be defined by

$$C(r) = \begin{cases} 1 \text{ if } r \in \left\{-\frac{1}{2}\right\} \cup [1, 2] \\ \frac{1}{2} \text{ if } r \in (0, 1) \end{cases}$$

$$D(r) = \begin{cases} 1 \text{ if } r \in \left\{-\frac{1}{2}\right\} \cup [1, 2] \\ \frac{1}{4} \text{ if } r \in (0, 1) \end{cases}$$

$$I(r) = \begin{cases} 1 \text{ if } r = 1 \\ \frac{4}{3} \text{ if } r \in (0, 1) \\ r - 1 \text{ if } r \in \left\{-\frac{1}{2}\right\} \cup (0, 1] \end{cases}$$

$$M(r) = \begin{cases} 1 \text{ if } r = 1 \\ \frac{5}{4} \text{ if } r \in (0, 1) \\ r^2 - 2r + 2 \text{ if } r \in \left\{-\frac{1}{2}\right\} \cup (1, 2] \end{cases}.$$

Then

$$C(R) = \left\{1, \frac{1}{2}\right\},\ D(R) = \left\{1, \frac{1}{4}\right\},\ I(R) = \left\{-\frac{3}{2}, \frac{4}{3}\right\} \cup (0, 1]\ M(R) = \left\{\frac{5}{4}\right\} \cup [1, 2]$$

and

(i) $J(R) \subseteq D(R)$ and $I(R) \subseteq D(R)$

(ii) For all $z, w \in R$ and $s \geq 0$, $\lambda \in (0, 1)$ are can verify that

$$\sigma_{cb}(Iz, Iw) \leq \frac{\lambda}{s^2} P(z, w)$$

$$P(u, v) = \max \{\sigma_{cb}(Cz, Dw), \sigma_{cb}(Cz, Iz), \sigma_{cb}(Dw, Jw),$$

$$\frac{1}{2}[\sigma_{cb}(Dw, Iz) + \sigma_{cb}(Cz, Jw)], \frac{\sigma_{cb}(Cz, Iz) \cdot \sigma_{cb}(Dw, Jw)}{1 + \sigma_{cb}(Cz, Dw)}\}$$

(iii) The pairs $\{C, I\}$ and $\{D, J\}$ are weakly compatible. Hence by theorem 3.1, 1 is a unique common fixed point of C, D, I and J.

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