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COMMON FIXED-POINT THEOREMS IN DIGITAL METRIC SPACE USING IMPLICIT RELATION

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Abstract

The aim of the present paper is to prove a generalization of the Banach contraction principle in digital metric space. Also, a common fixed-point theorem for a class Φ and φ-contractive type mapping is proved. Further a common fixed-point theorem using implicit relation of integral type in digital metric space has been proved. An example and an application for image processing in digital metric space is also given in our support.

1. Introduction

Digital topology is the study of the topological properties of images arrays. The results provide a sound mathematical basis for image processing operations such as image thinning, border following, contour filling and object counting. In metric spaces, this theory begins with Banach contraction. There are various applications of fixed point theory in Mathematics, computer science, game theory, engineering, image processing. Fixed point theory consists of many fields of mathematics such as mathematical analysis, general topology and functional analysis.

Banach contraction mapping principle was firstly given in [1]. Its structure is so simple and useful so it is used in existence problems in various fields of mathematical analysis, computer science, image processing, game theory etc. In recent time, many authors [10, 11, 15, 16] using the Banach contraction principle for important Studies.

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First Rosenfeld [17] studied and introduced the concept of digital topology. Then Kong [12] introduce the digital fundamental group of discrete objects. Boxer [3] gives the digital version of several notions from topology and [4] defined variety of digital continuous functions. Ege and Karaca [5] construct Lefschetz fixed point theory for digital image and study the fixed-point properties of digital images. They also calculate degree of the antipodal map for sphere-like digital images using fixed point properties.

This paper is organized as follows. In first part, the required background about the digital topology and fixed-point theory are given. In the next section, state and prove main result on common fixed point theorems to digital images illustrate by examples also applying fuzzy logics on image processing.

2. Motivations

Let X be a subset of $\mathbb{Z}^n$ for a positive integer n where $\mathbb{Z}^n$ is the set of lattice points in n-dimensional Euclidean space and ℓ represent an adjacency relation for the members of X. A digital image consists of (X, ℓ).

Definition 2.1 [2]. Let l, n be positive integers, $1 \leq l \leq n$ and two distinct points

$$p = (p_1, p_2, \ldots, p_n), q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$$

p and q are ℓ-adjacent if there are at most ℓ indices i such that $| p_i - q_i | = 1$ and for all other indices j such that $| p_i - q_i | \neq 1, p_j = q_j$.

There are some statements which can be obtained from definition 2.1:

• Two points p and q in $\mathbb{Z}$ are 2-adjacent if $| p - q | = 1$

Figure 1. 2-adjacent.

• Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 in each coordinate.

• Two points p and q in $\mathbb{Z}^2$ are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate (see Figure 2).

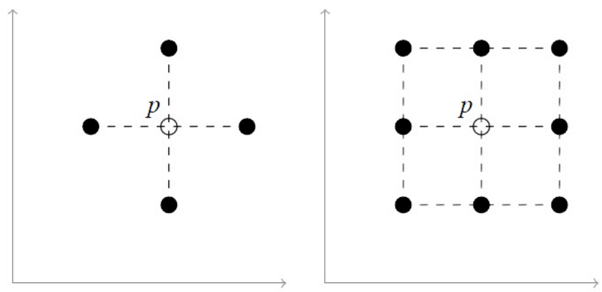


Figure 2. 4-adjacent and 8-adjacent.

• Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if they are distinct and differ by at most 1 in each coordinate.

• Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if they are 26-adjacent and differ at most two coordinates.

• Two points p and q in $\mathbb{Z}^3$ are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate (see Figure 3).

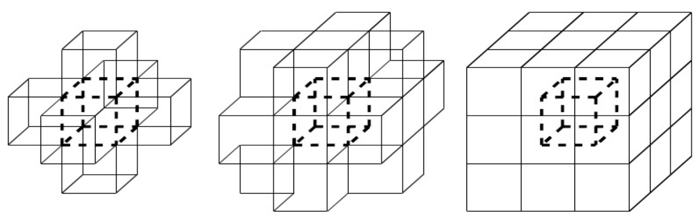


Figure 3. 6-adjacent, 18-adjacent, 26-adjacent.

A ℓ- neighbor [2] of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is ℓ- adjacent to p where $\ell \in \{2, 4, 8, 18, 26\}$ and $n \in 1, 2, 3$. The set

$$N_\ell(p) = \{q \mid q \text{ is } \ell\text{-adjacent to } p\}$$

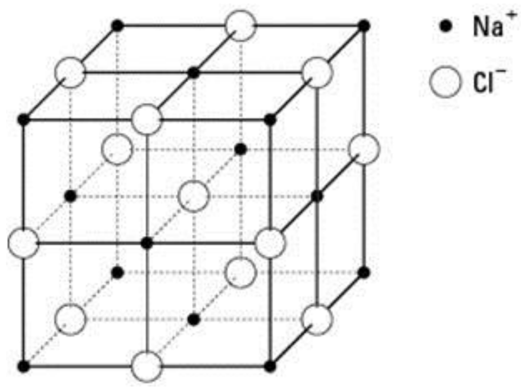
is called the ℓ- neighbourhood of p. A digital interval [3] is defined by

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$$

Where $a, b \in \mathbb{Z}$ and $a < b$.

A digital image $X \subset \mathbb{Z}^n$ is ℓ-connected [9] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_r and x_{r+1} are ℓ-neighbours where $i = 1, 2, \ldots, r - 1$.

Note that the coordination number of Na in a crystal structure of NaCl is 6 which is equal to adjacency relation in digital image of given figure



Definition 2.2 [6]. Let $(X, \ell_0) \subset \mathbb{Z}^{n_0}$, $(Y, \ell_1) \subset \mathbb{Z}^{n_1}$ be a digital image and $f : X \to Y$ be a function.

• If for every ℓ_0-connected subset U of X, $f(U)$ is ℓ_1-connected subset of Y, then f is said to be (ℓ_0, ℓ_1)-connected.

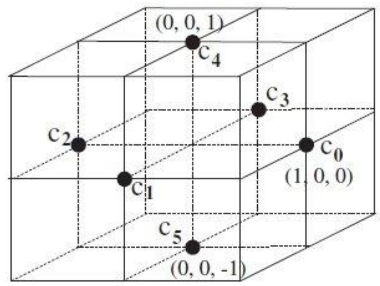
• If (ℓ_0, ℓ_1)-continuous for every ℓ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are a ℓ_1-adjacent in Y.

• If (ℓ_0, ℓ_1)-continuous, bijective and f^{-1} is (ℓ_0, ℓ_1)-continuous, then f is called (ℓ_0, ℓ_1)-isomorphism and denoted by $X \cong_{(\ell_0, \ell_1)} Y$.

Definition 2.3 [6]. Suppose $m \in Z^+$, (X, ℓ) is a digital image in $\mathbb{Z}^n$. A $(2, \ell)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ such that $f(0) = x$ and $f(m) = y$ is called digital ℓ-path from x to y. And X is called ℓ-path connected.

A simple closed ℓ-curve of $m \geq 4$ points in a digital image X is a sequence $\{f(0), f(1), \ldots, f(m-1)\}$ of images of the ℓ-path $f : [0, m-1]_{\mathbb{Z}} \to X$ such that $f(i)$ and $f(j)$ are ℓ-adjacent if and only if $j = i \pm \bmod m$.

Example 2.1 [6]. $MS\ S' = \{c_0 = (1, 1, 0), c_1 = (0, 2, 0), c_2 = (-1, 1, 0), c_3 = (0, 0, 0), c_4 = (0, 1, -1), c_5 = (0, 1, 1)\} \subset \mathbb{Z}^3$ is a minimal closed 18-surface.



Definition 2.4 [7]. Let $X \subset \mathbb{Z}^n, d$ be the Euclidean metric on $\mathbb{Z}^n \cdot (X, d)$ is metric space. Suppose (X, ℓ) is digital image with ℓ-adjacency then (X, ℓ, d) is called digital metric space.

Definition 2.5 [7]. A sequence $\{x_n\}$ of points of digital metric space (X, ℓ, d) is a Cauchy sequence if there is a $M \in \mathbb{N}$ such that, $d(x_n, x_m) < 1$ for all $n, m \in M$.

Definition 2.6 [7]. For a digital metric space (X, ℓ, d), if a sequence $\{x_n\} \subset X \subset \mathbb{Z}^n$ is a Cauchy sequence if there is a $M \in \mathbb{N}$ such that for all $n, n \in M$ we have $x_n = x_m$.

Definition 2.7 [6]. A sequence $\{x_n\}$ of points of digital metric space (X, ℓ, d) converges to a limit $a \in X$ if for all $\varepsilon > 0$, there exists $\alpha \in \mathbb{N}$ such that for all $n > \alpha$, then $d(x_n, a) < \varepsilon$.

Definition 2.8 [6]. A digital metric space (X, ℓ, d) is complete of digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X, ℓ, d) converges to a point a of (X, ℓ, d).

Now we define (E, A) property in digital metric space.

Definition 2.9. A pair of self digital mappings (A, S) in digital metric space (X, ℓ, d) is said to satisfy the property (E, A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z$ for some $z \in X$.

Definition 2.10. Two pairs of self digital mappings (A, S) and (B, T) in digital metric space (X, ℓ, d) is said to satisfy the property (E, A) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z$ for some $z \in X$.

Definition 2.11 [17]**.** Two self digital mappings f and g of a digital metric space (X, ℓ, d) are said to be occasionally weakly compatible iff there is a point $x \in X$ which is coincidence point of f and g at which f and g commute.

Notation. Let $\Phi = \{\varphi : [0, \infty) \to [0, \infty)\}$ be such that φ is increasing. $\varphi(t) < t$ for $t > 0$ and $\varphi(t) = 0$ if $t = 0$.

Definition 2.12 [14]**.** Let (X, ℓ, d) be a digital metric space. A self digital map $T : X \to X$ be a digital $-\alpha - \psi - \varphi$- contractive type mapping if there exists three functions $\alpha : X \times X \to [0, \infty)$ and $\psi, \varphi \in \Phi$ such that for all $x, y \in X$,

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \psi(d(Tx, Ty)) - \varphi(d(Tx, Ty)).$$

Definition 2.13 [14]**.** Let X be a non-empty set and $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say T is α-admissible if for all $x, y \in X$

We have $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

Definition 2.14 [6]**.** Let (X, ℓ, d) be a digital metric space and $f : (X, \ell, d) \to (X, \ell, d)$ be a self digital map. If there exists $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(f(x)\ f(y)) \leq \lambda d(x, y)$$

then f is called digital contraction map.

Proposition 2.1 [6]**.** *Every digital contraction map is continuous.*

3. Main Results

Theorem 3.1. *Let* (X, ℓ, d) *be a digital metric space and* A, B *are self digital mappings of* (X, ℓ, d). *Let* $B(X) \subset A(X)$. γ *be a right continuous real function such that*

$$\gamma(a) < a \tag{3.1}$$

if $a > 0$, *for all* $x, y \in X$,

$$d(Bx, By) \leq \gamma(d(Ax, Ay)). \tag{3.2}$$

Then A and B have a unique common fixed point.

Proof of Theorem 3.1. Since $B(X) \subset A(X)$, then there exists a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Ax_n = Bx_{n-1}.$$

Now to prove $\{x_n\}$ is Cauchy sequence. For each $\varepsilon > 0$, we can choose n_0 such that $d(x_n, x_{n+1}) = \varepsilon$ for all $n > n_0$, by using triangle inequality

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_m)$$

since $d(x_{n-1}, x_n) \to 0$ as $n \to \infty$, so we conclude that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq \varepsilon + \varepsilon + \ldots + \varepsilon(m - n) \text{ times} \\ &< \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in X and since (X, ℓ, d) is a complete digital metric space, x_n converse to a point in X.

Now define the following sequence $a_n = d(x_n, x_{n-1})$. Using inequality (3.1) and (3.2)

$$a_{n+1} = d(x_{n+1}, x_n) = d(Bx_{n-1}, Bx_{n-2}) \le \gamma(d(Ax_{n-1}, Ax_{n-2}))$$
$$\le \gamma(d(x_n, x_{n-1}))$$
$$< d(x_n, x_{n-1}) = a_n$$

for all $n \in N$. Thus the sequence a_n is decreasing and so it has a limit a. If we assume that $a > 0$, we have $a_{n+1} \le \gamma(a_n)$ since γ is right continuous, therefore $a \le \gamma(a)$ but it contradicts of (3.1).

As a result, $a_n \to 0$ as $n \to \infty$, therefore, A and B have a common fixed point.

To prove the uniqueness. Let u and v be two fixed point of A and B.

By using (3.1) and (3.2)

$$d(u, v) = d(Bu, Bv) \le \gamma(d(Au, Av)) \le \gamma(d(u, v)) \Rightarrow u = v.$$

Hence A and B have a unique common fixed point.

Example 3.1. Let $[0, 1]_Z$, $A, B : X \to X$ be given by

$$Ax = \frac{2}{3}x^2 + \frac{1}{3}, \quad Bx = \frac{2}{5}x^2 + \frac{3}{5}$$

and

$$d(x, y) = |y - x|.$$

By using inequality (3.2)

$$d(Bx, By) \le \gamma(d(Ax, Ay))$$

$$\left(\frac{2}{5}y^2 + \frac{3}{5} - \frac{2}{5}x^2 - \frac{3}{5}\right) \le \gamma\left(\frac{2}{3}y^2 + \frac{1}{3} - \frac{2}{3}x^2 - \frac{1}{3}\right)$$

$$\frac{2}{5}(y^2 - x^2) \le \gamma\left(\frac{2}{3}(y^2 - x^2)\right)$$

$$\frac{2}{5}(y^2 - x^2) < \left(\frac{2}{3}(y^2 - x^2)\right)$$

Which is hold, also A and B have a common fixed point.

Theorem 3.2. *Let* (X, ℓ, d) *be a digital metric space and* S, T, A, B *are* $\alpha - \psi - \varphi$ *type self digital mappings of* (X, ℓ, d). *Let the pairs* (A, S) *and* (B, T) *be occasionally weakly compatible. For all* $x, y \in X$,

$$\alpha(x, y)\psi(d(Ax, By)) \leq \psi(M(x, y)) - \gamma(M(x, y)). \tag{3.3}$$

Where

$$M(x, y) = \max \{d(Sx, Ty), d(By, Sx), d(Sx, Ax),$$

$$d(By, Ty), d(Ax, Ty), \left(\frac{2d(Sx, Ax)}{1 + d(By, Ty)}\right)\}.$$

Then there is a unique fixed point of S, T, A, B.

Proof of Theorem 3.2. Since A, B are α-admissible then for all $x, y \in X$

$$\alpha(x, y) \geq 1.$$

As the pairs (A, S) and (B, T) are occasionally weakly compatible, so $Sx = Ax$ and $By = Ty$. Now to prove $Ax = By$, by inequality (3.3)

$$\psi(d(Ax, By)) \leq \alpha(x, y)\psi(d(Ax, By)) \leq \psi(M(x, y)) - \varphi(M(x, y))$$

where

$$M(x, y) = \max \{d(Sx, Ty), d(By, Sx), d(Sx, Ax),$$

$$d(By, Ty), d(Ax, Ty), \left(\frac{2d(Sx, Ax)}{1 + d(By, Ty)}\right)\}$$

$$\alpha(x, y)\psi(d(Ax, By)) \leq \psi(\max \{d(Sx, Ty), d(By, Sx), d(Sx, Ax),$$

$$d(By, Ty), d(Ax, Ty), \left(\frac{2d(Sx, Ax)}{1 + d(By, Ty)}\right)\})$$

$$-\varphi(\max \{d(Sx, Ty), d(By, Sx), d(Sx, Ax),$$

$$d(By, Ty), d(Ax, Ty), \left(\frac{2d(Sx, Ax)}{1 + d(By, Ty)}\right)\})$$

$$\leq \psi(\max \{d(Ax, By), d(By, Ax), d(Ax, Ax),$$

$$d(By, By), d(Ax, By), \left(\frac{2d(Ax, Ax)}{1+d(By, By)}\right)\})$$

$$-\varphi(\max\{d(Ax, By), d(By, Ax), d(Ax, Ax),$$

$$d(By, By), d(Ax, By), \left(\frac{2d(Ax, Ax)}{1+d(By, By)}\right)\})$$

$$\leq \psi(\max\{d(Ax, By), d(By, Ax), 0, 0, d(Ax, By), 0\})$$

$$-\varphi(\max\{d(Ax, By), d(By, Ax), 0, 0, d(Ax, By), 0\})$$

$$\leq \psi(d(Ax, By)) - \varphi(d(Ax, By))$$

or

$$\psi(d(Ax, By)) \leq \psi(d(Ax, By)) - \varphi(d(Ax, By))$$

or

$$\varphi(d(Ax, By)) \leq 0$$

$$\Rightarrow d(Ax, By) = 0$$

$$\Rightarrow Ax = By.$$

Therefore $Ax = Sx = By = Ty$.

Suppose that there is another point z such that $Az = Sz$ then by inequality (3.3)

$$\psi(d(w, z)) = \psi(\alpha(Aw, Bz)) \leq \alpha(w, z)\psi(\alpha(Aw, Bz))$$

$$\leq \psi(M(w, z)) - \varphi(M(w, z))$$

$$\leq \psi(\max\{d(Sw, Tz), d(Bz, Sw), d(Sw, Aw),$$

$$d(Bz, Tz), d(Aw, Tz), \left(\frac{2d(Sw, Aw)}{1+d(Bz, Tz)}\right)\})$$

$$-\varphi(\max\{d(Sw, Tz), d(Bz, Sw), d(Sw, Aw),$$

$$d(Bz, Tz), d(Aw, Tz), \left(\frac{2d(Sw, Aw)}{1+d(Bz, Tz)}\right)\})$$

$$\leq \psi(\max\{d(w, z), d(z, w), d(w, w), d(z, z),$$

$$d(w, z), \left(\frac{2d(w, w)}{1 + d(z, z)}\right)\})$$

$$-\varphi(\max\{d(w, z), d(z, w), d(w, w),$$

$$d(z, z), d(w, z), \left(\frac{2d(w, w)}{1 + d(z, z)}\right)\})$$

$$\leq \psi(\max\{d(w, z), d(z, w), 0, 0, d(w, z), 0\})$$

$$-\varphi(\max\{d(w, z), d(z, w), 0, 0, d(w, z), 0\})$$

$$\leq \psi(d(w, z)) - \varphi(d(w, z))$$

or

$$\psi(d(w, z)) \leq \psi(d(w, z)) - \varphi(d(w, z))$$

or

$$\varphi(d(w, z)) \leq 0$$

or

$$d(w, z) = 0.$$

Therefore $w = z$. Hence z is common fixed point of A, B, S, T.

Uniqueness. Let u be another common fixed point of A, B, S, T. Then by inequality (3.3)

$$\psi(d(z, u))\psi(d(Az, Bu)) \leq \alpha(z, u)\psi(d(Az, Bu)) \leq \psi(M(z, u)) - \varphi(M(z, u))$$

$$\leq \psi(\max\{d(Sz, Tu), d(Bu, Sz), d(Sz, Az),$$

$$d(Bu, Tu), d(Az, Tu), \left(\frac{2d(Sz, Az)}{1 + d(Bu, Tu)}\right)\}$$

$$-\varphi(\max\{d(Sz, Tu), d(Bu, Sz), d(Sz, Az),$$

$$d(Bu, Tu), d(Az, Tu), \left(\frac{2d(Sz, Az)}{1 + d(Bu, Tu)}\right)\})$$

$$\leq \psi(\max\{d(z, u), d(u, z), d(z, z),$$

$$d(u, u), d(z, u), \left(\frac{2d(z, z)}{1 + d(u, u)}\right)\})$$

$$-\varphi(\max\{d(z, u), d(u, z), d(z, z),$$

$$d(u, u), d(z, u), \left(\frac{2d(z, z)}{1 + d(u, u)}\right)\})$$

$$\leq \psi(\max\{d(z, u), d(u, z), 0, 0, d(z, u), 0\})$$

$$-\varphi(\max\{d(z, u), d(u, z), 0, 0, d(z, u), 0\})$$

$$\leq \psi(d(z, u)) - \varphi(d(z, u))$$

or

$$\psi(d(z, u)) \leq \psi(d(z, u)) - \varphi(d(z, u))$$

or

$$\varphi(d(z, u)) \leq 0$$

or

$$d(z, u) = 0.$$

Therefore $z = u$. Hence z is unique common fixed point of A, B, S, T.

Example 3.2. Let $X = \{1, 2, 3, \ldots\}$ and $d(x, y) = |y - x|$ and $(X, d, 4)$ is a digital metric space in $\mathbb{Z}$ with 4-adjacency and A, B, S and T have common fixed point define by $Ax = x + 1$, $By = y + 1$, $sx = x - 1$, $Ty = y - 1$.

By inequality (3.3)

$$\psi(d(Ax, By)) \leq \alpha(x, y)\psi(d(Ax, By)) \leq \psi(M(x, y)) - \varphi(M(x, y))$$

where

$$M(x, y) = \max\{d(Sx, Ty), d(By, Sx), d(Sx, Ax),$$

$$d(By, Ty), d(Ax, Ty), \left(\frac{2d(Sx, Ax)}{1 + d(By, Ty)}\right)\}.$$

Therefore

$$M(x, y) = \max \{|y - 1 - x + 1|, |x - 1 - y + 1|, |x + 1 - x + 1|,$$

$$|y - 1 - y + 1|, |y - 1 - x - 1|, \left(\frac{2|x + 1 - x + 1|}{1 + |y - 1 - y + 1|}\right)$$

$$M(x, y) = \max \{|y - x|, |y - x - 2|, |2|, 0, |y - x|, 2\}$$

$$= |y - x|.$$

Therefore inequality (3.3) hold.

And A, B, S, T have a common fixed point.

Implicit Relation.

Let M_6 denotes the set of all real valued continuous function $\phi : [0, 1]^6 \to R$ satisfying the following conditions:

(A) $\int_0^{\phi(u, u, 0, 0, u, u)} \varphi(t) dt \leq 0$ implies $u \geq 0$.

(B) $\int_0^{\phi(u, u, 0, 0, u, 0)} \varphi(t) dt \leq 0$ implies $u \geq 0$.

(C) $\int_0^{\phi(0, u, u, u, u, u)} \varphi(t) dt \leq 0$ implies $u \geq 0$.

Example 3.3. Define $\phi : [0, 1]^6 \to R$ as

$$\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \phi_1(\max \{t_2, t_3, t_4, t_5, t_6\}),$$

where $\phi_1 : [0, 1] \to [0, 1]$ is continuous function such that $\phi_1(k) < k$ for all $k \in (0, 1)$.

Clearly ϕ satisfying conditions (A) and (B). Therefore $\phi \in M_6$.

Theorem 3.3. *Let* (X, ℓ, d) *be a digital metric space and* S, T, A, B *are self digital mappings of* (X, ℓ, d) *satisfying following conditions*:

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X) \tag{3.4}$$

$$\int_0^{\phi((d(Ax,Sx)+d(By,Ty),(d(By,Sx)+d(Ax,Ty)))),\frac{d(Sx,Ty),d(By,Sx)}{1+d(Sx,Ty)+d(By,Sx)},d(Sx,Ty),\left(\frac{1}{2}d(Ax,By)+d(Ax,Ty)\right),d(Ax,Ty)}\varphi(t)dt \le 0. \quad (3.5)$$

For all $x, y \in X$ and $\phi \in M_6$, $\varphi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, nonnegative and such that $\int_0^{\varepsilon}\varphi(t)dt > 0$ for each $\varepsilon > 0$.

Suppose that (A, S) or (B, T) satisfies property (E, A) and the pairs (A, S) and (B, T) are weakly compatible. If one of the $S(X)$, $T(X)$, $A(X)$ and $B(X)$ is closed subset of *X*, then there is a unique common fixed point of S, T, A, B.

Proof of Theorem 3.3. Suppose that (B, T) satisfies property (E, A), then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, therefore, we have $\lim_{n\to\infty} d(Bx_n, Tx_n) = 0$. Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ such that $Bx_n = Sx_n$. Hence $\lim_{n\to\infty} Sy_n = z$.

Let us show that $\lim_{n\to\infty} Ay_n = z$.

By using inequality (3.5)

$$\int_0^{\phi(d(Ay_n,Sy_n)+d(Bx_n,Tx_n)),(d(Bx_n,Sy_n)+d(Ay_n,Tx_n)),\frac{d(Sy_n,Tx_n)\cdot d(Bx_n,Sy_n))}{1+d(Sy_n,Tx_n)+d(Bx_n,Sy_n))},d(Sy_n,Tx_n),\left(\frac{1}{2}[d(Ay_n,Bx_n)+d(Ay_n,Tx_n)\right),d(Ay_n,Tx_n)}\varphi(t)dt \le 0.$$

Suppose that $\lim_{n\to\infty}\inf d(Ay_n, Bx_n) = u$. Taking limit $n \to \infty$ we get

$$\int_0^{\phi(u,u,0,0,u,u)}\varphi(t)dt \le 0.$$

And it is a contradiction of (A). so $u = 0$ and $\lim_{n\to\infty} Ay_n = z$.

Suppose $S(X)$ is closed subset of *X*, then $z = Su$ for some $u \in X$. If $z \neq Au$.

By inequality (3.5)

$$\int_0^{\phi(d(Au, Su)+d(Bx_n, Tx_n)), (d(Bx_n, Su)+d(Au, Tx_n)), \frac{d(Su, Tx_n)\cdot d(Bx_n, Su))}{1+d(Su, Tx_n)\cdot d(Bx_n, Su))}, d(Su, Tx_n), \left(\frac{1}{2}[d(Au, Bx_n)+d(Au, Tx_n)\right), d(Au, Tx_n)} \varphi(t)\, dt \le 0.$$

Taking limit $n \to \infty$, we have

$$\int_0^{\phi(d(Au, z),\, d(Au, z),\, 0,\, 0,\, d(Au, z),\, d(Au, z))} \varphi(t) dt \le 0.$$

Which is a contradiction of (A). hence $d(Au, z) = 0$ therefore $Au = z = Su$. Since $A(X) \subset T$, there exists $v \in X$ such that $z = Au = Tv$. If $z \neq Bv$.

By using inequality (3.5) we obtain

$$\int_0^{\phi(d(Au, su)+d(Bv, Tv),\, d(Bv, Su)+d(Au, Tv)),\, \frac{d(Su, Tv)\cdot d(Bv, Su)}{1+d(Su, Tv)+d(Bv, su)},\, d(Su, Tv),\, \left(\frac{1}{2}[d(Au, Bv)+d(Au, Tv)]\right),\, d(Au, Tv)} \varphi(t) dt \le 0$$

$$\int_0^{\phi\left(d(Bv, z),\, d(Bv, z),\, 0,\, 0\left(\frac{1}{2}[d(z, Bv)]\right),\, 0\right)} \varphi(t) dt \le 0.$$

Which is contradiction of (B), and therefore $Au = Su = z = Tv = Bv$.

Since the pair (A, S) is weakly compatible, we have $ASu = SAu$ i.e. $Az = Sz$. If $Az \neq z$, using (3.5)

$$\int_0^{\phi(d(Az, sz)+d(Bv, Tv),\, d(Bv, Sz)+d(Az, Tv)),\, \frac{d(Sz, Tv)\cdot d(Bv, Sz)}{1+d(Sz, Tv)+d(Bv, Sz)},\, d(Sz, Tv),\, \left(\frac{1}{2}[d(Az, Bv)+d(Az, Tv)]\right),\, d(Az, Tv)} \varphi(t) dt \le 0$$

$$\int_0^{\phi\left(0,\, 2d(z, Az),\, \frac{d(Az, z)\cdot d(z, Az)}{1+d(Az, z)+d,\, Az},\, d(Az, z),\, d(Az, z),\, d(Az, z)\right)} \varphi(t) dt \le 0.$$

Which is contradiction of (C), therefore $d(Az, z) = 0$, hence $Az = z$ or $Az = Sz = z$. Since the pair (B, T) is weakly compatible, we have $BTv = TBv$ i.e. $Bz = Tz$. If $Bz \neq z$, using (3.5)

$$\int_0^{\phi((d(Az, Sz)+d(Bz, Tz)), (d(Bz, Sz)+d(Az, Tz))), \frac{d(Sz, Tz)\cdot d(Bz, Sz)}{1+d(Sz, Tz)+d(Bz, Sz)}, d(Sz, Tz), \left(\frac{1}{2}[d(Az, Bz)+d(Az, Tz)]\right), d(Az, Tz)} \varphi(t)dt \leq 0$$

$$\int_0^{\phi(0, 2(d(Bz, z))), \frac{d(z, Bz)\cdot d(Bz, z)}{1+d(Sz, Tz)+d(Bz, Sz)}, d(z, Bz), d(Bz, z), d(Bz, z)} \varphi(t)dt \leq 0.$$

Which is again contradiction of (C) therefore $d(Bz, z) = 0$,

Hence $Az = Bz = Sz = Tz = z$.

Therefore, z is common fixed point of A, B, S, T.

To prove uniqueness of z let w be another common fixed point of A, B, S, T. Then, using (3.5)

$$\int_0^{\phi(d(Az, Sz)+d(Bw, Tw), d(Bw, Sz)+d(Az, Tw)), \frac{d(Sz, Tw)\cdot d(Bw, Sz)}{1+d(Sz, Tw)+d(Bw, Sz)}, d(Sz, Tw), \left(\frac{1}{2}[d(Az, Bw)+d(Az, Tw)]\right), d(Az, Tw)} \varphi(t)dt \leq 0$$

$$\int_0^{\phi(0, d(d(w, z)+d(z, w))), \frac{d(z, w)\cdot d(w, z)}{1+2d(z, w)}, d(z, w), d(z, w), d(z, w)} \varphi(t)dt \leq 0.$$

Which is contradiction of (C), therefore $z = w$.

Hence z is a unique common fixed point of A, B, S, T.

Corollary 3.1. *Let* (X, ℓ, d) *be a digital metric space and* S, T, A, B *are self digital mappings of* (X, ℓ, d) *satisfying following conditions*:

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X) \tag{3.6}$$

$$\phi(d(Ax, Sx) + d(By, Ty)), (d(By, Sx) + d(Ax, Ty)), \frac{d(Sx, Ty)\cdot d(By, Sx)}{1 + d(Sx, Ty) + d(By, Sx)},$$

$$d(Sx, Ty), \left(\frac{1}{2}[d(Ax, By) + d(Ax, Ty)]\right), d(Ax, Ty) \leq 0. \tag{3.7}$$

For all $x, y \in X$ *and* $\phi \in M_6$.

Suppose that (A, S) or (B, T) satisfies property (E, A) and the pairs

(A, S) and (B, T) are weakly compatible. If one of the $S(X)$, $T(X)$, $A(X)$ and $B(X)$ is closed subset of X, then there is a unique common fixed point of S, T, A, B.

Proof. If we put $\varphi(t) = 1$ in theorem 3.3, the result follows.

Applications

In this section,

1. an application of common fixed point theorem to image compression. The aim of image compression is to reduce redundant image information in the digital image. There are some problems in the storing an image. Memory data is usually too large and sometimes stored image has not more information than original image. It's known that the quality of compressed image can be poor. For this reason, we must pay attention to compress a digital image. Fixed point theorem can be used to image compression of a digital image.

2. an application of common fixed point in digital metric space of integral type can be given by an example.

In image processing, we generally rely on features specific to certain regions of the entire image. Hence we need properties of those regions because integral image is an image we get by cumulative addition on subsequent pixels in both horizontal and vertical axis.

Now, assume a matrix A of size 5×5 representing an image, as shown below.

5	4	3	8	3
3	9	1	2	6
9	6	0	5	7
7	3	6	5	9
1	2	2	8	3

And its integral form is given by

5	9	12	20	23
8	21	25	35	44
17	36	40	55	71
24	46	56	76	101
25	49	61	89	117

For just hundred operations over a 5×5 matrix, using an integral image uses about 50% less computations, imagine the amount of difference it makes for large images and more such operations.

Creation of integral image changes other sum difference operations almost $O(1)$ time complexity, thereby decreasing the number of calculations.

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