Advances and Applications in Mathematical Sciences
Volume 21, Issue 12, October 2022, Pages 7201-7216

TOTAL EDGE IRREGULARITY STRENGTH OF SOME PLANE GRAPHS

S. TERESA AROCKIAMARY and J. MARIA ANGELIN VISITHRA

Department of Mathematics
Stella Maris College, Chennai
Affiliated to the University of Madras
E-mail: drtessys70@gmail.com
jpa367@gmail.com

Abstract

Given a graph $G = (V, E)$ a labeling $\partial : V \cup E \to \{1, 2, \ldots, k\}$ is called an edge irregular total k-labeling if for every pair of distinct edges uv and xy, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. The minimum k for which G has an edge irregular total k-labeling is called the total edge irregularity strength of G. The total edge irregularity strength of G is denoted by $tes(G)$. In our current study, we took into account the plane graph and calculated its total edge irregularity strength.

1. Introduction

One of the exciting subfields of graph theory is graph labeling. Rosa first used graph labeling in 1967. With certain restrictions, graph labeling involves assigning integers to vertices, edges or both. Over 200 methods for labeling graphs have been explored in numerous research papers, resulting in the researcher's involvement during the previous 60 years. Labeled graphs are practical models with a variety of uses such as astro studies, exchange of secret messages, circuit design, radar detection etc [7], network addressing, X-ray crystallography, coding theory, rulers, radar and missile guidance, radio antenna problems, dental arch problem, database management problem, communication networks, secret sharing scheme, information security scheme [1]. Measuring irregular strength of a network

2020 Mathematics Subject Classification: 05Cxx.
Keywords: Edge irregular total k-labeling, total edge irregularity strength, plane graph.
Received September 9, 2022; Accepted September 20, 2022

mathematically is a problem of concern in communication networks. This can be achieved by establishing edge irregular total k-labeling of graph of the corresponding network [11]. For a dynamic survey of various graph labelings along with an extensive bibliography, one may refer to Gallian [6].

Baca, Jendrol', Miller and Ryan [9] introduced the total edge irregularity strength of a graph. Total edge irregularity strength has been well studied for honeycomb mesh networks [5], hexagonal networks [8], butterfly networks [2, 4], benes networks [2] and series compositions of uniform theta graphs [3], generalized uniform theta graph and the lower bound has been determined [12]. Syed Ahtshma Ul Haq Bokhary et al., [13], proved the total irregularity strength of convex polytope graphs S_n, T_n, U_n.

We now begin with some known results on $tes(G)$ and basic definitions.

Theorem 1.1 [9]. *Let G be a graph with m edges. Then* $tes(G) \geq \lceil (m+2)/3 \rceil$.

Theorem 1.2 [9]. *Let G be a graph with maximum degree Δ. Then* $tes(G) \geq \lceil [\Delta + 2]/3 \rceil$.

Definition 1.1 [6]. Given a graph $G = (V, E)$ a labeling $\partial : V \cup E \to \{1, 2, \ldots, k\}$ is called an edge irregular total k-labeling if for every pair of distinct edges uv and xy, $\partial(u) + \partial(uv) + \partial(v) \neq \partial(x) + \partial(xy) + \partial(y)$. The minimum k for which G has an edge irregular total k-labeling is called the total edge irregularity strength of G. The total edge irregularity strength of G is denoted by $tes(G)$.

In our study, we have considered some plane graphs. Our results on edge irregular total k-labeling applied to these graphs are presented in this paper. Further we have proved that a bound on tes is sharp as given in Theorem 1.1. In our discussion we call the weight of the edges as edge sums.

2. Main Results

2.1 The plane graph S_m^p

Definition 2.1 [10]. A pendant edge is attached to each vertex of the outer cycle of the convex polytope S_m to create the plane graph S_m^p. For our necessity, we call the vertices of the inner cycle as u_i, the vertices of the cycle between the inner and the alternating band of triangle as v_i, the vertices of the middle cycle as w_i, then the vertices of the outer cycle as y_i followed by the vertices of the pendant edge as z_i. The number of vertices of S_m^p is $5m$ and the number of edges of S_m^p is $9m$.

Notation.

The vertex set and edge set of S_m^p are defined as follows:

$V(S_m^p) = V(S_m) \cup \{z_i : 1 \leq i \leq m\}$ and $E(S_m^p) = E(S_m) \cup \{y_i z_i : 1 \leq i \leq m\}$.

See Figure 1.

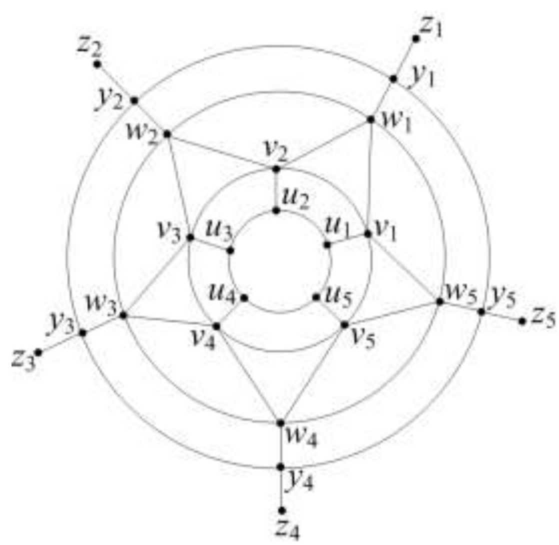

Figure 1. The plane graph S_m^p.

Theorem 2.1. *For every* $m \geq 3$ *the total edge irregularity strength of plane graph* S_m^p *is* $tes(S_m^p) = \lceil (9m+2)/3 \rceil = 3m+1$.

Proof. The vertices and edges of S_m^p are traversed in the anticlockwise direction. First we label the vertices of the inner cycle, the vertices of the cycle between the inner and the middle cycle, the vertices of the middle cycle which is connected with the alternating triangle, then the vertices of the outer cycle followed by the vertices of the pendant edge. The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input. The graph of plane graph S_m^p, $m \geq 3$.

Algorithm.

Step 1. For $1 \leq i \leq m$

$f(u_i) = 1$

$f(v_i) = m + 1$

$f(w_i) = 2m$

$f(y_i) = 3m$

$f(z_i) = 3m + 1.$

Step 2. $f(u_i u_{i+1}) = i, 1 \leq i \leq m - 1$

$f(u_m u_1) = m.$

Thus the edge sums are $3, 4, \ldots, m + 2.$

Step 3. $f(u_i v_i) = i, 1 \leq i \leq m$

Thus the edge sums are from $m + 3$ to $2m + 2.$

Step 4. $f(v_i v_{i+1}) = i, 1 \leq i \leq m - 1$

$f(v_m v_1) = m.$

Thus the edge sums are from $2m + 3$ to $3m + 2.$

Step 5. $f(v_i w_i) = 2i\, f(v_{i+1} w_i) = 2i + 1, 1 \leq i \leq m.$

Thus the edge sums are $3m + 3$ to $5m + 2.$

Step 6. $f(w_i w_{i+1}) = m + 2 + i, 1 \leq i \leq m - 1$

$f(w_m w_1) = 2m + 2.$

Thus the edge sums are $5m + 3$ to $6m + 2$.

Step 7. $f(w_i y_i) = m + 2 + i, 1 \leq i \leq m.$

Thus the edge sums are $6m + 3$ to $7m + 2$.

Step 8. $f(y_i y_{i+1}) = m + 2 + i, 1 \leq i \leq m - 1$

$f(y_m y_1) = 2m + 2.$

Thus the edge sums are $7m + 3$ to $8m + 2$.

Step 9. $f(y_i z_i) = 2m + 1 + i, 1 \leq i \leq m.$

Thus the edge sums are $8m + 3$ to $9m + 2$.

Output. $tes(S_m^p) = \lceil (9m + 2)/3 \rceil = 3m + 1.$

Proof of Correctness. In the inner cycle from the above stepwise procedure $f(u_i) = 1, 1 \leq i \leq m$ $f(u_i u_{i+1}) = i, 1 \leq i \leq m - 1$ we have the edge sums $u_i u_{i+1}$ and $u_m u_1$ consecutive since the labels received by the edges are consecutive.

In the cycle between the inner and middle cycle from the above stepwise procedure $f(u_i) = 1, f(v_i) = m + 1, 1 \leq i \leq m$ $f(u_i v_i) = i, 1 \leq i \leq m$ we have the edge sums $u_i v_i$ consecutive since the labels received by the edges are consecutive.

In the alternating band of triangle between v_i's and the middle cycle from the above stepwise procedure $f(v_i) = m + 1, f(w_i) = 2m, 1 \leq i \leq m$ $f(v_i v_{i+1}) = i, 1 \leq i \leq m$ we have the edge sums $v_i v_{i+1} = i$ consecutive since the labels received by the edges are consecutive.

In the middle cycle from the above stepwise procedure $f(w_i) = 2m,$ $1 \leq i \leq m$ $f(w_i w_{i+1}) = m + 2 + i, 1 \leq i \leq m - 1$ we have the edge sums $w_i w_{i+1}$ and $w_m w_1$ consecutive since the labels received by the edges are consecutive.

In the edge between the middle and the outer cycle from the above stepwise procedure $f(w_i) = 2m, f(y_i) = 3m, 1 \leq i \leq m - 1$ $f(w_i y_i) = m + 2 + i,$ $1 \leq i \leq m$ we have the edge sums $w_i y_i$ consecutive since the labels received by the edges are consecutive.

In the outer cycle from the above stepwise procedure $f(y_i) = 3m$, $1 \leq i \leq m$ $f(y_i y_{i+1}) = m + 2 + i$, $1 \leq i \leq m - 1$ we have the edge sums $y_i y_{i+1}$ and $y_m y_1$ consecutive since the labels received by the edges are consecutive.

In the pendant edge connected with the outer cycle from the above stepwise procedure $f(z_i) = 3m + 1$, $1 \leq i \leq m$ $f(y_i z_i) = 2m + 1 + i$, $1 \leq i \leq m$ we have the edge sums $y_i z_i$ consecutive since the labels received by the edges are consecutive. Hence the above aforementioned edge sums are distinct. Thus S_m^p is total edge k-irregular. An illustration of $tes(S_5^p)$ is shown in Figure 2.

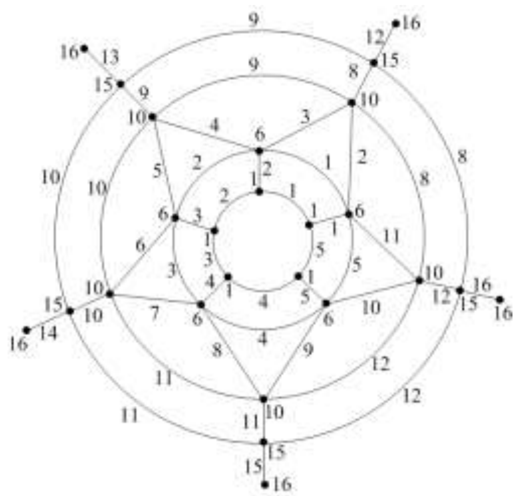

Figure 2. $tes(S_5^p) = 16$.

2.2 The plane graph T_m^p

Definition 2.2 [10]. A pendant edge is attached to each vertex of the outer cycle of the convex polytope T_m to create the plane graph T_m^p. The number of vertices of T_m^p is $5m$ and the number of edges of T_m^p is $9m$.

Notation. The vertex set and edge set of T_m^p are defined as follows:

$V(T_m^p) = V(T_m) \cup \{z_i, 1 \leq i \leq m\}$ and $E(T_m^p) = E(T_m) \cup \{y_i z_i, 1 \leq i \leq m\}$.

See Figure 3.

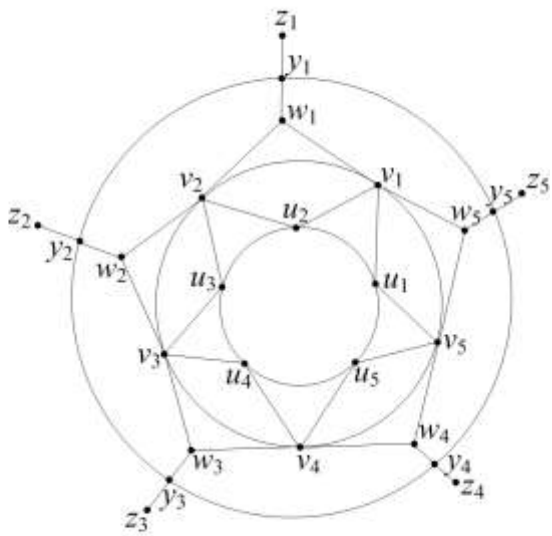

Figure 3. The plane graph T_m^p.

Theorem 2.2. *For every* $m \geq 3$ *the total edge irregularity strength of plane graph* T_m^p *is* $tes(T_m^p) = \lceil (9m+2)/3 \rceil = 3m+1$.

Proof. The vertices and edges of T_m^p are traversed in the anticlockwise direction. First we label the vertices of the inner cycle, vertices of the middle cycle, the vertices of an alternating band of triangle then the vertices of the outer cycle followed by the vertices of the pendant edge. The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input. The graph of plane graph T_m^p for $m \geq 3$.

Algorithm.

Step 1. For $1 \leq i \leq m$

$f(u_i) = 1$

$f(v_i) = m + 1$

$f(w_i) = 3m - 1$

$f(y_i) = 3m$

$f(z_i) = 3m + 1.$

Step 2. $f(u_i u_{i+1}) = i, 1 \leq i \leq m - 1$

$f(u_m u_1) = m.$

Thus the edge sums are $3, 4, \ldots, m + 2.$

Step 3. $f(u_i v_i) = 2i - 1, 1 \leq i \leq m$

$f(u_{i+1} v_i) = 2i, 1 \leq i \leq m - 1$

$f(u_1 v_m) = 2m.$

Thus the edge sums are $m + 3, \ldots, 3m + 2.$

Step 4. $f(v_i v_{i+1}) = m + i, 1 \leq i \leq m - 1$

$f(v_m v_1) = 2m.$

Thus the edge sums are $3m + 3$ to $4m + 2.$

Step 5. $f(v_i w_i) = 2i + 1, 1 \leq i \leq m$

$f(v_{i+1} w_i) = 2i + 2, 1 \leq i \leq m - 1$

$f(v_1 w_m) = 2m + 2.$

Thus the edge sums are from $4m + 3$ to $6m + 2.$

Step 6. $f(w_i y_i) = i + 3, 1 \leq i \leq m.$

Thus the edge sums are $6m + 3$ to $7m + 2.$

Step 7. $f(y_i y_{i+1}) = m + 2 + i, 1 \leq i \leq m - 1$

$f(y_m y_1) = 2m + 2.$

Thus the edge sums are $7m + 3$ to $8m + 2.$

Step 8. $f(y_i z_i) = m + i + 1, 1 \leq i \leq m$

Thus the edge sums are $8m + 3$ to $9m + 2.$

Output. $tes(T_m^p) = \lceil (9m + 2)/3 \rceil = 3m + 1.$

Proof of Correctness. In the inner cycle from the above stepwise procedure $f(u_i) = 1, 1 \leq i \leq m$ $f(u_i u_{i+1}) = i, 1 \leq i \leq m-1$ we have the edge sums $u_i u_{i+1}$ and $u_m u_1$ consecutive since the labels received by the edges are consecutive.

In the alternating band between the inner and the middle cycle from the above stepwise procedure $f(u_i) = 1, f(v_i) = m+1, 1 \leq i \leq m$ $f(u_i v_i) = 2i - 1$, $1 \leq i \leq m, f(u_{i+1} v_i) = 2i, 1 \leq i \leq m-1, 1 \leq i \leq m-1$ we have the edge sums $u_i v_i, u_{i+1} v_i$ and $u_1 v_m$ consecutive since the labels received by the edges are consecutive.

In the middle cycle from the above stepwise procedure $f(v_i) = m+1$, $1 \leq i \leq m$ $f(v_i v_{i+1}) = m + i, 1 \leq i \leq m-1$ we have the edge sums $v_i v_{i+1}$ and $v_m v_1$ consecutive since the labels received by the edges are consecutive.

In the alternating band of triangle between the middle cycle and w_i from the above stepwise procedure $f(v_i) = m+1, f(w_i) = 3m-1, 1 \leq i \leq m$ $f(v_i w_i) = 2i - 1, 1 \leq i \leq m, f(v_{i+1} w_i) = 2i + 2, 1 \leq i \leq m-1$ we have the edge sums $v_i w_i, v_{i+1} w_i$ and $v_1 w_m$ consecutive since the labels received by the edges are consecutive.

In the edge between the from the w_i's and the outer cycle from the above stepwise procedure $f(w_i) = 3m-1, f(y_i) = 3m, 1 \leq i \leq m$ $f(w_i y_i) = i + 3$, $1 \leq i \leq m$ we have the edge sums $w_i y_i$ consecutive since the labels received by the edges are consecutive.

In the outer cycle from the above stepwise procedure $f(y_i) = 3m, 1 \leq i \leq m$ $f(y_i y_{i+1}) = m + 2 + i, 1 \leq i \leq m-1$ we have the edge sums $y_i y_{i+1}$ and $y_m y_1$ consecutive since the labels received by the edges are consecutive.

In the pendant edge between the outer cycle and z_i's from the above stepwise procedure $f(y_i) = 3m, f(z_i) = 3m+1, 1 \leq i \leq m$ $f(y_i z_i) = m + i + 1$, $1 \leq i \leq m-1$ we have the edge sums $y_i z_i$ consecutive since the labels received by the edges are consecutive. Hence the aforementioned stepwise process are all different. Thus T_m^p is total edge k-irregular. An illustration of $tes(T_5^p)$ is shown in Figure 4.

2.3 The plane graph U_m^p

Definition 2.3 [10]**.** A pendant edge is attached to each vertex of the outer cycle of the convex polytope U_m to create the plane graph U_m^p. The number of vertices of U_m^p are $6m$ and the number of edges of U_m^p is $9m$.

Notation. The vertex set and edge set of U_m^p are defined as follows:

$$V(U_m^p) = V(U_m) \cup \{z_i,\ 1 \le i \le m\} \text{ and } E(U_m^p) = E(U_m) \cup \{y_i z_i,\ 1 \le i \le m\}.$$

See Figure 5.

Theorem 2.3. *For every $m \ge 3$ the total edge irregularity strength of plane graph U_m^p is $tes(U_m^p) = \lceil (9m+2)/3 \rceil = 3m + 1$.*

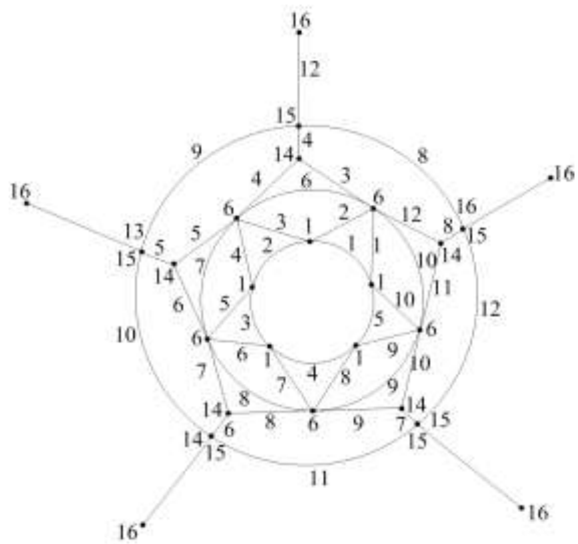

Figure 4. $tes(T_5^p) = 16$.

Proof. The vertices and edges of U_m^p are traversed in the anticlockwise direction. First we label the vertices of the inner cycle, the vertices of the middle cycle, the vertices between the edge $x_i x_{i+1}$ and one vertex between $x_1 x_m$ then by the vertices of the outer cycle followed by the vertices of the pendant edge. The edges are also labeled in the same sequence so that the edge sums are consecutive.

Input. The graph of convex polytope U_m^p for $m \geq 3$.

Algorithm.

Step 1. For $1 \leq i \leq m$

$f(u_i) = 1$

$f(v_i) = m + 1$

$f(w_i) = 2m$

$f(x_i) = 2m + 2$

$f(y_i) = 3m$

$f(z_i) = 3m + 1.$

Step 2. $f(u_i u_{i+1}) = i, 1 \leq i \leq m - 1$

$f(u_m u_1) = m.$

Thus the edge sums are $3, 4, \ldots, m + 2.$

Step 3.

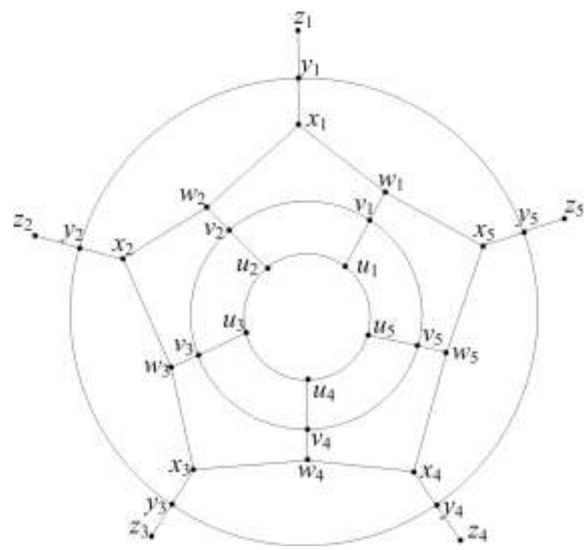

Figure 5. The plane graph U_m^p.

$f(u_iv_i) = 2i - 1, 1 \leq i \leq m.$

Thus the edge sums are $m + 3$ to $2m + 2$.

Step 4. $f(v_iv_{i+1}) = i, 1 \leq i \leq m - 1$

$f(v_mv_1) = m.$

Thus the edge sums are from $2m + 3$ to $3m + 2$.

Step 5. $f(v_iw_i) = i + 1, 1 \leq i \leq m.$

Thus the edge sums are from $3m + 3$ to $4m + 2$.

Step 6. $f(x_iw_i) = 2i - 1, 1 \leq i \leq m$

$f(x_iw_{i+1}) = 2i, 1 \leq i \leq m - 1$

$f(x_mw_1) = 2m.$

Thus the edge sums are from $4m + 3$ to $6m + 2$.

Step 7. $f(x_iy_i) = m + i, 1 \leq i \leq m$

Thus the edge sums $6m + 3$ to $7m + 2$.

Step 8. $f(y_i y_{i+1}) = m + i + 2, 1 \leq i \leq m - 1$

$f(y_m y_1) = 2m + 2.$

Thus the edge sums are $7m + 3$ to $8m + 2.$

Step 9. For $1 \leq i \leq m$

$f(y_i z_i) = 2m + i + 1, 1 \leq i \leq m.$

Thus the edge sums are $8m + 3$ to $9m + 2.$

Output. $tes(U_m^p) = \lceil (9m + 2)/3 \rceil = 3m + 1.$

Proof of Correctness. In the inner cycle from the above stepwise procedure $f(u_i) = 1, 1 \leq i \leq m$ $f(u_i u_{i+1}) = i, 1 \leq i \leq m - 1$ we have the edge sums $u_i u_{i+1}$ and $u_m u_1$ consecutive since the labels received by the edges are consecutive.

In the edge between the inner and the middle cycle from the above stepwise procedure $f(u_i) = 1, f(v_i) = m + 1, 1 \leq i \leq m$ $f(u_i v_i) = 2i - 1, 1 \leq i \leq m - 1$ we have the edge sums $u_i v_i$ consecutive since the labels received by the edges are consecutive.

In the middle cycle from the above stepwise procedure $f(v_i) = m + 1, 1 \leq i \leq m$ $f(v_i v_{i+1}) = i, 1 \leq i \leq m - 1$ we have the edge sums $v_i v_{i+1}$ and $v_m v_1$ consecutive since the labels received by the edges are consecutive.

In the edge between the middle cycle and w_i's from the above stepwise procedure $f(v_i) = m + 1, f(w_i) = 2m, 1 \leq i \leq m$ $f(v_i w_i) = i + 1, 1 \leq i \leq m - 1$ we have the edge sums $v_i w_i$ consecutive since the labels received by the edges are consecutive.

In the edge connecting w_i's and x_i's from the above stepwise procedure $f(w_i) = 2m, f(x_i) = 2m + 2, 1 \leq i \leq m$ $f(x_i w_i) = 2i - 1, 1 \leq i \leq m, f(x_i w_{i+1}) = 2i, 1 \leq i \leq m - 1$ we have the edge sums $x_i w_i, x_i w_{i+1}, x_m w_1$ consecutive since the labels received by the edges are consecutive.

In the edge between the x_i's and the outer cycle from the above stepwise procedure $f(x_i) = 2m + 2$, $f(y_i) = 3m$, $1 \leq i \leq m$ $f(x_i y_i) = m + i$, $1 \leq i \leq m - 1$ we have the edge sums $x_i y_i$ consecutive since the labels received by the edges are consecutive.

In the outer cycle from the above stepwise procedure $f(y_i) = 3m$, $1 \leq i \leq m$ $f(y_i y_{i+1}) = m + i + 2$, $1 \leq i \leq m - 1$ we have the edge sums $y_i y_{i+1}$ and $y_m y_1$ consecutive since the labels received by the edges are consecutive.

In the pendant edge between the outer cycle and z_i's from the above stepwise procedure $f(y_i) = 3m$, $f(z_i) = 3m + 1$, $1 \leq i \leq m$ $f(y_i z_i) = 2m + i + 1$, $1 \leq i \leq m - 1$ we have the edge sums $y_i z_i$ consecutive since the labels received by the edges are consecutive. Hence the aforementioned stepwise process are all different. Thus U_m^p is total edge k-irregular. An Illustration of $tes(U_5^p)$ is shown in Figure 6.

Conclusion

We have demonstrated in this study that the plane graphs S_m^p, T_m^p, U_m^p satisfies edge irregular total k-labeling and have obtained its total edge irregularity strength. Our investigation will be expanded to other interconnection networks and applications of edge irregular total k-labeling.

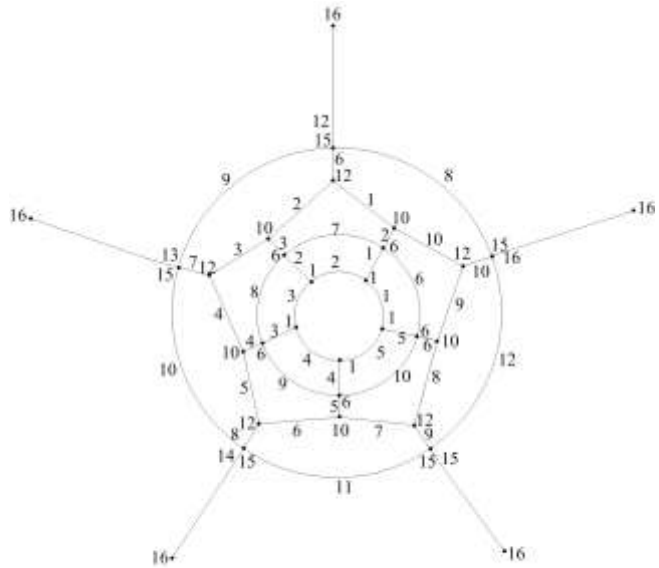

Figure 6. $tes(U_5^p) = 16$.

References

[1] Dedas Mishra, Gobind Mohanty and Subarna Bhattacharjee, (k, d)-Graceful labeling of some new families of graphs derived from cyclic firecrackers, International Journal of Scientific Research in Mathematical and Statistical Sciences 9(3) (2022), 35-42.

[2] Indra Rajasingh, Bharati Rajan and S. Teresa Arockiamary, Irregular total labeling of butterfly and benes networks, Informatics Engineering and Information Science (2011), 284-293.

[3] Indra Rajasingh and S. Teresa Arockiamary, Total Edge Irregularity Strength of Series Parallel Graphs, International Journal of Pure and Applied Mathematics 99(1) (2015), 11-21.

[4] Indra Rajasingh, Bharati Rajan and S. Teresa Arockiamary, Total Edge Irregularity Strength of Butterfly Networks, International Journal of Computer Applications 49(3) (2012).

[5] Indra Rajasingh and S. Teresa Arockiamary, Total Edge Irregularity Strength of Honeycomb Torus Networks, Global Journal of Pure and Applied Mathematics 13(4) (2017).

[6] J. A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, 22nd edition, 2020.

[7] J. Lisy Bennet and S. Chandra Kumar, A Study on Graph Labeling Problems, Studies in Indian Place Names 40(60) (2020).

[8] J. Quadras and S. Teresa Arockiamary, Total Edge Irregularity Strength of Hexagonal Networks, Journal of Combinatorial Mathematics and Combinatorial Computing (2015), 131-138.

[9] Martin Baca, Stanislav Jendrol, Mirka Miller and Joseph Ryan, On Irregular Total Labellings, Discrete Mathematics (2007), 1378-1388.

[10] Muhammad Imran, Syed Ahtshma Ul Haq Bokhary and A. Q. Baig, On the Metric Dimension of Convex Polytopes, AKCE International Journal of Graphs and Combinatorics 10(3) (2013), 295-307.

[11] R. Jegan, P. Vijayakumar and K. Thirusangu, On Total Edge Irregularity Strength of Certain Classes of Extended Duplicate Graphs, Journal of Algebraic Statistics 13(2) (2022), 634-647.

[12] R. W. Putra and Y. Susanti, The Total Edge Irregularity Strength of Uniform Theta Graphs, IOP Conf. Series: Journal of Physics, 2018.

[13] Syed Ahtshma Ul Haq Bokhary, Muhammad Imran and Usman Ali, On the Total Irregularity Strength of Convex Polytope Graphs, Proyecciones Journal of Mathematics 40(5) 1267-1277.