Advances and Applications in Mathematical Sciences
Volume 21, Issue 1, November 2021, Pages 213-224

ULAM-HYERS STABILITY OF A HELMHOLTZ NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

A. PONMANA SELVAN, R. MURALI and K. VARADHARAJAN

Department of Mathematics
Sri Sairam Institute of Technology
Sairam College Road, Sai Leo Nagar
West Tambaram, Chennai-600044
Tamil Nadu, India
E-mail: selvaharry@yahoo.com

Department of Mathematics
Sacred Heart College (Autonomous)
Tirupattur - 635 601
Tirupattur Dt., Tamil Nadu, India
E-mail: shcrmurali@yahoo.co.in

Department of Mathematics
Kings Engineering College
Sriperumbudur, Chennai - 602 117
Tamil Nadu, India
E-mail: varadharajan@kingsedu.ac.in

Abstract

The purpose this paper is to establish the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the Damped Helmholtz Nonlinear differential equations with initial conditions. Moreover, the Ulam stability constants for this Damped Helmholtz equations are obtained.

1. Introduction and Preliminaries

A classical question in the theory of functional equation is the following : "when is it true that a function which approximately satisfies a functional

2020 Mathematics Subject Classification: 39A99, 39A30, 26D10, 26D15, 39B82.

Keywords: Ulam-Hyers Stability, Ulam-Hyers-Rassias stability, Helmholtz Nonlinear differential equation, initial conditions.

Received June 2, 2020; December Accepted 14, 2020

equation g must be close to an exact solution of g?" If the problem accepts a solution, we say that the equation g is stable.

The Hyers-Ulam stability problem was introduced by S. M. Ulam [1] in 1940. He discussed the number of important unsolved problems in a talk concerning the study of stability problems for various functional equations. Among such problems, a problem concerning the stability of functional equations "Give conditions for a order mapping near approximately linear mapping to exist". D. H. Hyers [2] found the partial solution of the Ulam's problem on Banach spaces in 1941. Further more, in 1978, Th. M. Rassias [3] formulated and proved the special case of Hyers Theorem. For a decades, many researchers have extended the theory of Hyers-Ulam stability to other functional equations, and generalized the Hyers result in different directions (See for example, [4, 5, 6, 7, 8]).

Recently, Hyers-Ulam stability of differential equation has been given attention. Obloza [9, 10] was the first author, who introduced the Hyers-Ulam stability of linear differential equation and connections between Hyers and Lyapunov stability of the ordinary differential equation. Alsina and Ger [11], continued and investigated the Hyers-Ulam stability of the differential $g' = g$ and this result was generalized by Miura [12], Takahasi [13] and Jung [14, 15, 16, 17, 18] they proved the Hyers-Ulam stability of linear differential equation.

Thereafter, The theory of Hyers-Ulam stability of differential equations was developed in a series of papers [19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33] and the investigation is ongoing.

Now a days, only some authors are discussing the Hyers-Ulam stability of Nonlinear differential equations (for example [34, 35, 36, 37, 38, 39]). Motivated by the above ideas, our main intention is to obtain the Ulam-Hyers-Rassias stability and Ulam-Hyers stability of the Damped Helmholtz oscillator. The Helmholtz oscillator is nonlinear differential equation of second order, which takes the form

$$x'' + \mu x' + x - x^2 = F\cos(wt).$$

Here, we study the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the Damped Helmholtz equations of the form

$$x'' + \mu x' + x - x^2 = 0 \tag{1}$$

and

$$x'' + \mu x' + x - x^2 = F\cos(wt) \tag{2}$$

with initial conditions

$$x(a) = x'(a) = 0 \tag{3}$$

where $x \in C^2(I)$, $I = [a, b] \subseteq \mathbb{R}$ and $\mu, F,$ and w are positive constants.

Now, we provide the Deffnitions of the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability of the differential equations (1) and (2) with initial conditions (3).

Definition 1. We say that the nonlinear differential equation (1) has the Ulam-Hyers stability, if there exists a constant $K > 0$ with the following properties: For every $\epsilon > 0$, there exists $x \in C^2[a, b]$ satisfies the inequality

$$| x'' + \mu x' + x - x^2 | \leq \epsilon$$

with initial condition (3), then there exists some $y \in C^2[a, b]$ satisfies the differential equation $y'' + \mu y' + y - y^2 = 0$ with $y(a) = y'(a) = 0$ such that $| x(t) - y(t) | \leq K\epsilon$. We call such K as a Ulam-Hyers stability constant for the differential equation (1).

Definition 2. We say that the nonlinear differential equation (2) has the Ulam-Hyers stability, if there exists a constant $K > 0$ with the following conditions: For every $\epsilon > 0$, $x \in C^2[a, b]$, satisfies the inequality

$$| x'' + \mu x' + x - x^2 - F\cos(wt) | \leq \epsilon$$

with initial condition (3), then there exists a $y \in C^2[a, b]$ satisfying the differential equation $y'' + \mu y' + y - y^2 = F\cos(wt)$ with initial conditions $y(a) = y'(a) = 0$ such that $| x(t) - y(t) | \leq K\epsilon$. We call such K as a Ulam-Hyers stability constant for the differential equation (2).

Definition 3. We say that the nonlinear differential equation (1) has the Ulam-Hyers-Rassias stability, if there exists a constant $K > 0$ with the following properties: For every $\epsilon > 0$ and $x \in C^2[a, b]$, if there exists $\phi : [0, \infty) \to [0, \infty)$ satisfying

$$| x'' + \mu x' + x - x^2 | \leq \phi(t)\epsilon$$

with initial condition (3), then there exists a solution $y \in C^2[a, b]$ satisfying the differential equation $y'' + \mu y' + y - y^2 = 0$ with $y(a) = y'(a) = 0$ such that $| x(t) - y(t) | \leq K\phi(t)\epsilon$. We call such K as a Ulam-Hyers-Rassias stability constant for (1).

Definition 4. We say that the nonlinear differential equation (2) has the Ulam-Hyers-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$ and $x \in C^2[a, b]$, if there exists $\phi : [0, \infty) \to [0, \infty)$ satisfying the inequality

$$| x'' + \mu x' + x - x^2 - F\cos(wt) | \leq \phi(t)\epsilon$$

with initial condition (3), then there exists $y \in C^2[a, b]$ satisfies the differential equation $y'' + \mu y' + y - y^2 = F\cos(wt)$ with $y(a) = y'(a) = 0$ such that

$$| x(t) - y(t) | \leq K\phi(t)\epsilon.$$

We call such K as a Ulam-Hyers-Rassias stability constant for (2).

2. Hyers-Ulam Stability

In this section, we prove the Ulam-Hyers stability of the nonlinear differential equation (1) and (2) with (3).

Theorem 5. *Suppose that* μ, F *and* w *are positive constants and* $x \in C^2[a, b]$ *such that* $| x'(t) | \leq | x(t) |$ *and which satisfying the following in equation*

$$| x''(t) + \mu x'(t) + x(t) - x(t)^2 | \leq \epsilon$$

with initial condition (3), *then the nonlinear differential equation* (1) *has Ulam-Hyers stability.*

Proof. For every $\epsilon > 0$ there exists $x : [a, b] \to C$ be a twice continuously differentiable function such that $|x'(t)| \le |x(t)|$ and satisfies the inequality

$$|x''(t) + \mu x'(t) + x(t) - x(t)^2| \le \epsilon, \tag{4}$$

with initial condition (3). Let $M = \max_{t \in I} |x(t)|$. From the inequality (4), we have

$$-\epsilon \le x''(t) + \mu x'(t) + x(t) - x(t)^2 \le \epsilon. \tag{5}$$

Multiplying the above inequality (5) by $x'(t)$ and then integrating a to t, we get

$$-\epsilon \int_a^t x'(t)dt \le \int_a^t x''(t)x'(t)dt + \mu \int_a^t x'(t)^2 dt + \int_a^t x(t)x'(t)dt - \int_a^t x(t)^2 x'(t)dt$$

$$\le \epsilon \int_a^t x'(t)dt$$

$$-6\epsilon \int_a^t x'(t)dt \le 3x'(t)^2 + 6\mu \int_a^t x'(t)^2 dt + 3x^2 - 2x^3 \le 6\epsilon \int_a^t x'(t)dt$$

$$3x^2 - 2x^3 + 6\mu \int_a^t x'(t)^2 dt \le 6\epsilon \int_a^t x'(t)dt.$$

From which we get that

$$x^2 \le 6\epsilon \int_a^t x'(t)dt + 6\mu \int_a^t x'(t)^2 dt$$

$$M^2 \le 6\epsilon M + 6\mu M^2 (b - a)$$

$$M \le \frac{6\epsilon}{1 - \ell}, \text{ where, } \ell = 6\mu(b - a).$$

Hence $|x(t)| \le K\epsilon$. for all $t \in [a, b]$, where $K = \frac{6}{1 - \ell}$. Obviously,

$y(t) \equiv 0$ is a solution of (1) with initial condition (3) such that $|x(t) - y(t)| \leq K\epsilon$.

In the next theorem, we are going to prove the Ulam-Hyers stability of the nonlinear differential equation (2) with initial conditions (3).

Theorem 6. *Suppose that* μ, F *and* w *are positive constants and* $x \in C^2[a, b]$ *such that* $|x'(t)| \leq |x(t)|$ *and satisfies the inequality*

$$|x''(t) + \mu x'(t) + x(t) - x(t)^2 - \text{F}\cos(wt)| \leq \epsilon$$

with initial condition (3). *Then the differential equation* (2) *has Ulam-Hyers stability*.

Proof. For every $\epsilon > 0$, there exists $x : [a, b] \to C$, a twice continuously differentiable function such that $|x'(t)| \leq |x(t)|$ and satisfies the inequality

$$|x''(t) + \mu x'(t) + x(t) - x(t)^2 - F\cos(wt)| \leq \epsilon, \tag{6}$$

with initial condition (3). Define $M = \max\{|x(t)| : t \in I\}$. From the inequality (6), we have

$$-\epsilon \leq x''(t) + \mu x'(t) + x(t) - x(t)^2 - F\cos(wt) \leq \epsilon. \tag{7}$$

Multiplying the above inequality (7) by $x'(t)$ and then integrating, we get

$$-\epsilon\int_a^t x'(t)dt \leq \int_a^t x''(t)x'(t)dt + \mu\int_a^t x'(t)^2 dt + \int_a^t x(t)x'(t)dt - \int_a^t x(t)^2 x'(t)dt$$

$$-\int_a^t F\cos(wt)x'(t)dt \leq \epsilon\int_a^t x'(t)dt.$$

$$-6\epsilon\int_a^t x'(t)dt \leq 3x'(t)^2 + 6\mu\int_a^t x'(t)^2 dt + 3x^2 - 2x^3 - 6\int_a^t F\cos(wt)x'(t)dt$$

$$\leq 6\epsilon\int_a^t x'(t)dt$$

$$3x^2 - 2x^3 + 6\mu\int_a^t x'(t)^2 dt \leq 6\epsilon\int_a^t x'(t)dt + 6\int_a^t F\cos(wt)x'(t)dt$$

From which we obtain

$$x^2 \le 6\epsilon \int_a^t x'(t)dt + 6\mu \int_a^t x'(t)^2 dt + 6\int_a^t F\cos(wt)x'(t)dt$$

$$M^2 \le 6\epsilon M + 6\mu M^2(b-a) + 6CM(b-a)$$

$$M \le \frac{6(\epsilon + C(b-a))}{1-\ell}, \text{ where, } \ell = 6\mu(b-a).$$

Hence $|x(t)| \le K(\epsilon)$, for all $t \in [a, b]$, where $K(\epsilon) = \frac{6(\epsilon + C(b-a))}{1-\ell}$. Obviously, $y(t) \equiv 0$ is a solution of (2) with initial condition (3) such that $|x(t) - y(t)| \le K\epsilon$.

3. Hyers-Ulam-Rassias Stability

Theorem 7. *Assume that* μ, F *and* w *are positive constants and* $x \in C^2[a, b]$ *satisfies the inequality* $|x'(t)| \le |x(t)|$ *and if there is a function* $\phi : [0, \infty) \to [0, \infty)$ *such that* $|x''(t) + \mu x'(t) + x(t) - x(t)^2| \le \phi(t)\epsilon$ *with initial condition* (3), *then the nonlinear differential equation* (1) *has Ulam-Hyers-Rassias stability*.

Proof. For each $\epsilon > 0$, there exists a twice continuously differentiable function $x : [a, b] \to C$ such that $|x'(t)| \le |x(t)|$ if there is a function $\phi : [0, \infty) \to [0, \infty)$ satisfying the inequality

$$|x''(t) + \mu x'(t) + x(t) - x(t)^2| \le \phi(t)\epsilon, \tag{8}$$

with initial condition (3). Take $M = \max_{t \in I} |x(t)|$. From the inequality (8), we have

$$-\phi(t)\epsilon \le x''(t) + \mu x'(t) + x(t) - x(t)^2 \le \phi(t)\epsilon. \tag{9}$$

Multiplying the above in equation (9) by $x'(t)$ and then integrating it, we get

$$-\epsilon \int_a^t \phi(t)x'(t)dt \le \int_a^t x''(t)x'(t)dt + \mu \int_a^t x'(t)^2 dt + \int_a^t x(t)x'(t)dt$$

$$-\int_a^t x(t)^2 x'(t)dt \leq \epsilon \int_a^t \phi(t)x'(t)dt$$

$$-6\epsilon \int_a^t \phi(t)x'(t)dt \leq 3x'(t)^2 + 6\mu \int_a^t x'(t)^2 dt + 3x^2 - 2x^3 \leq 6\epsilon \int_a^t \phi(t)x'(t)dt$$

$$3x^2 - 2x^3 + 6\mu \int_a^t x'(t)^2 dt \leq 6\epsilon \int_a^t \phi(t)x'(t)dt.$$

From which we obtain

$$x^2 \leq 6\epsilon \int_a^t \phi(t)x'(t)dt + 6\mu \int_a^t x'(t)^2 dt$$

$$M^2 \leq 6M\phi(t)\epsilon + 6\mu M^2(b-a)$$

$$M \leq \frac{6\phi(t)\epsilon}{1-\ell}, \text{ where, } \ell = 6\mu(b-a).$$

Hence $|x(t)| \leq K\phi(t)\epsilon$, for all $t \in [a, b]$, where $K = \frac{6}{1-\ell}$. Obviously, $y(t) \equiv 0$ is a solution of (1) with initial condition (3) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon$.

Theorem 8. *Suppose that* μ, F *and* w *are positive constants and* $x \in C^2[a, b]$ *such that* $|x'(t)| \leq |x(t)|$ *and satisfies the inequality*

$$|x''(t) + \mu x'(t) + x(t) - x(t)^2 - F\cos(wt)| \leq \phi(t)\epsilon$$

with (3), *then the nonlinear differential equation* (2) *has Ulam-Hyers-Rassias stability.*

Proof. For every $\epsilon > 0$, there exists a $x \in C^2([a, b])$ such that $|x'(t)| \leq |x(t)|$ if there is a function $\phi : [0, \infty) \to [0, \infty)$ satisfies

$$|x''(t) + \mu x'(t) + x(t) - x(t)^2 - F\cos(wt)| \leq \phi(t)\epsilon, \tag{10}$$

with initial condition (3). Let $M = \max_{t \in I} |x(t)|$. From the inequality (10), we have

$$-\phi(t)\epsilon \le x''(t) + \mu x'(t) + x(t) - x(t)^2 - F\cos(wt) \le \phi(t)\epsilon. \tag{11}$$

Multiplying the above inequality (11) by $x'(t)$ and then integrating a to t, we obtain

$$-\epsilon\int_a^t \phi(t)x'(t)dt \le \int_a^t x''(t)x'(t)dt + \mu\int_a^t x'(t)^2 dt + \int_a^t x(t)x'(t)dt$$

$$-\int_a^t x(t)^2 x'(t)dt - \int_a^t F\cos(wt)x'(t)dt \le \epsilon\int_a^t \phi(t)x'(t)dt$$

$$-6\epsilon\int_a^t \phi(t)x'(t)dt \le 3x'(t)^2 + 6\mu\int_a^t x'(t)^2 dt + 3x^2 - 2x^3 - 6\int_a^t F\cos(wt)x'(t)dt$$

$$\le 6\epsilon\int_a^t \phi(t)x'(t)dt$$

$$3x^2 - 2x^3 + 6\mu\int_a^t x'(t)^2 dt - 6\int_a^t F\cos(wt)x'(t)dt \le 6\epsilon\int_a^t \phi(t)x'(t)dt.$$

From which we get that

$$x^2 \le 6\epsilon\int_a^t \phi(t)x'(t)dt + 6\mu\int_a^t x'(t)^2 dt + 6\int_a^t F\cos(wt)x'(t)dt$$

$$M^2 \le 6\phi(t)\epsilon M + 6\mu M^2(b-a) + 6CM(b-a)$$

$$M \le \frac{6(\phi(t)\epsilon + C(b-a))}{1-\ell}, \text{ where, } \ell = 6\mu(b-a)$$

Hence $|x(t)| \le K(\epsilon)\phi(t)$, for all $t \in [a, b]$, where

$$K(\epsilon)\phi(t) = \frac{6(\phi(t)\epsilon + C(b-a))}{1-\ell}.$$

Obviously $y(t) \equiv 0$ is a solution of (2) with equation (3) such that

$$|x(t) - y(t)| \le K(\epsilon)\phi(t).$$

This completes the proof. □

Conclusion

The Definition of Hyers-Ulam stability and Hyers-Ulam-Rassias stability have applicable significance since it means that if one is studying the Hyers-Ulam stable system then one does not have to reach the exact solution. (Which usually is quite difficult or time consuming). This is quite useful in many applications, for example Numerical Analysis, Optimization, Biology, and Economics etc., where finding the exact solution is quite difficult. Finally, in this paper, we studied the Ulam-Hyers stability and Ulam-Hyers-Rassias stability of the Damped Helmholtz Nonlinear differential equations with initial conditions. Moreover, the Ulam stability constants for this Damped Helmholtz equations are obtained. It will be very useful to the readers to apply more problems.

References

[1] S. M. Ulam, Problem in Modern Mathematics, Chapter IV, Science Editors, Willey, New York, (1960).

[2] D. H. Hyers, On the Stability of a Linear functional equation, Proc. Natl. Acad. Sci. USA 27 (1941), 222-224.

[3] Th. M. Rassias, On the stability of the linear mappings in Banach Spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[4] Lu. Gang, Xie. Jun, Liu. Qi and Jin. Yuanfeng, Hyers-Ulam stability of derivations in fuzzy Banach space, J. Nonlinear Sci. Appl. 9(12) (2016), 5970-5979.

[5] A. Bodaghi, Quintic Functional equations in Non-Archimedean Normed spaces, Journal of Mathematical Extension 9(1) (2015), 51-63.

[6] Iz. Iddine El-Fassi and Samir Kabbaj, On the generalized orthogonal stability of mixed type additive-cubic functional equations in modular spaces, Tbilisi Mathematical journal 9(1) (2016), 231-243.

[7] M. Nazarianpoor, J. M. Rassias and G. H. Sadeghi, Solution and stability of Quattuorvigintic Functional equation in Intuitionistic Fuzzy Normed spaces, Iranian Journal of Fuzzy system 15(4) (2018), 13-30.

[8] E. Elqorachi and Th. M. Rassias, Generalized Hyers-Ulam stability of Trigonometric Functional equations, Mathematics (MDPI) 6(83) (2018), 11 pages.

[9] M. Obloza, Hyers stability of the linear differential equation, Rockznik Nauk-Dydakt, Prace Math. 13 (1993), 259-270.

[10] M. Obloza, Connection between Hyers and Lyapunov stability of the ordinary differential equations, Rockznik Nauk-Dydakt, Prace Math. 14 (1997), 141-146.

[11] C. Alsina, R. Ger, On Some inequalities and stability results related to the exponential function, Journal of Inequalities Appl. 2 (1998), 373-380.

[12] T. Miura, S. M. Jung, S. E. Takahasi, Hyers-Ulam-Rassias stability of the Banach space

valued linear differential equation $y' = \lambda y$, J. Korean Math. Soc. 41 (2004), 995-1005.

[13] S. E. Takahasi, T. Miura, S. Miyajima, On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \alpha y$ Bulletin Korean Math. Soc. 39 (2002), 309-315.

[14] S. M. Jung, Hyers-Ulam stability of linear differential equations of first order (II), Appl. Math. Lett. 19 (2006) 854-858.

[15] S. M. Jung, Hyers-Ulam stability of linear differential equations of first order (III), J. Math. Anal. Appl. 311 (2005), 139-146.

[16] S. M. Jung, Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients, J. Math. Anal. Appl. 320 (2006), 549-561.

[17] S. M. Jung, Approximate solution of a Linear Differential Equation of Third Order, Bull. of the Malaysian Math. Sciences Soc. (2) 35(4) (2012), 1063-1073.

[18] S. M. Jung, Hyers-Ulam stability of linear differential equation of first order, Appl. Math. Lett. 17 (2004), 1135-1140.

[19] Q. H. Alqifiary and J. K. Miljanovic, Note on the stability of System of Differential equations $\dot{x}(t) = f(t, x(t))$, Gen. Math. Notes 20(1) (2014), 27-33.

[20] G. Wang, M. Zhou and L. Sun, Hyers-Ulam stability of linear differential equations of first order, Appl. Math. Lett. 21 (2008), 1024-1028.

[21] Y. Li and Y. Shen, Hyers-Ulam stability of linear differential equations of second order, Appl. Math. Lett. 23 (2010), 306-309.

[22] J. Xue, Hyers-Ulam stability of linear differential equations of second order with constant coefficient, Italian Journal of Pure and Applied Mathematics 32 (2014), 419-424.

[23] T. Li, A. Zada and S. Faisal, Hyers-Ulam stability of nth order linear differential equations, Journal of Nonlinear Science and Applications 9 (2016), 2070-2075.

[24] K. Ravi, R. Murali and A. Ponmana Selvan, Hyers-Ulam stability of nth order linear differential equation with initial and boundary condition, Asian Journal of Mathematics and Computer Research 11(3) (2016), 201-207.

[25] R. Murali and A. Ponmana Selvan, On the Generalized Hyers-Ulam Stability of Linear Ordinary Differential Equations of Higher Order, International Journal of Pure and Applied Mathematics 117(12) (2017), 317-326.

[26] M. Onitsuka and T. Shoji, Hyers-Ulam stability of first order homogeneous linear differential equations with a real valued coefficients, Appl. Math. Lett. 63 (2017), 102-108.

[27] R. Murali and A. Ponmana Selvan, Hyers-Ulam Stability of nth order Differential Equation, Contemporary Studies in Discrete Mathematics 2(1) (2018), 45-50.

[28] R. Murali and A. Ponmana Selvan, Hyers-Ulam-Rassias Stability for the Linear Ordinary Differential Equation of Third order, Kragujevac Journal of Mathematics 42(4) (2018), 579-590.

[29] R. Murali and A. Ponmana Selvan, Hyers-Ulam stability of a Free and Forced Vibrations, Kragujevac Journal of Mathematics 44(2) (2020), 299-312.

[30] R. Murali and A. Ponmana Selvan, Hyers-Ulam Stability of Linear Differential Equation, Computational Intelligence, Cyber Security and Computational Models, Models and Techniques for Intelligent Systems and Automation, Communications in Computer and Information Science, vol 844. Springer, Singapore (2018), 183-192.

[31] R. Murali and A. Ponmana Selvan, Hyers-Ulam Stability of nth order linear differential equation, Proyecciones: Journal of Mathematics (Revista de Matematica) (Antofagasta online) 38(3) (2019), 553-566.

[32] R. Murali and A. Ponmana Selvan, Fourier Transforms and Ulam Stabilities of Linear Differential Equations, Chapter-12, Frontiers in Functional Equations and Analytic Inequalities, Springer Nature, Switzerland, AG (2019), 195-217.

[33] Murali Ramdoss, Ponmana Selvan-Arumugam and Choonkil Park, Ulam stability of linear differential equations using Fourier transform, AIMS Mathematics 5(2) (2019), 766-780.

[34] M. N. Qarawani, Hyers-Ulam Stability of a Generalized Second-Order Nonlinear Differential Equation, Applied Mathematics 3 (2012), 1857-1861.

[35] M. N. Qarawani, On Hyers-Ulam stability for Nonlinear Differential equations of nth order, International Journal of Analysis and Applications 1(2) (2013), 71-78.

[36] J. Huang, S. M. Jung, Y. Li, On Hyers-Ulam stability of nonlinear differential equations, Bull. Korean Math. Soc. 52 (2015), 685-697.

[37] K. Ravi, R. Murali, A. Ponmana Selvan and R. Veerasivaji, Hyers-Ulam Stability of nth Order Non-Linear Differential Equations with initial Conditions, International Journal of Mathematics And its Applications 4(2-C) (2016), 121-132.

[38] I. Fakunle and P. O. Arawomo, Hyers-Ulam stability of certain class of Nonlinear second order differential equations, Global Journal of Pure and Applied Mathematics 14(8) (2018), 1029-1039.

[39] I. Fakunle and P. O. Arawomo, On Hyers-Ulam stability of Nonlinear second order ordinary and Functional differential equations, International Journal of Differential Equations and Applications 17(1) (2018), 77-88.