Advances and Applications in Mathematical Sciences
Volume 20, Issue 8, June 2021, Pages 1559-1569

ON THE ENTIRE RANDIC INDEX OF GRAPHS

A. SALEH and ISMAIL NACI CANGUL

Department of Mathematics
Faculty of Science
University of Jeddah
Jeddah, Saudi Arabia
E-mail: asaleh1@uj.edu.sa

Department of Mathematics
Bursa Uludag University
Bursa, 16059 Turkey
E-mail: cangul@uludag.edu.tr

Abstract

Topological indices are graph invariants computed by the distance or degree of vertices of the molecular graph. In chemical graph theory, topological indices have been successfully used in describing the structures and predicting certain physicochemical properties of chemical compounds. The Randic index is one of the classical graph-based molecular structure descriptors that has found countless applications in chemistry and pharmacology. The mathematical background of this index is also well elaborated.

The Randic index of a graph G is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{\deg(u)\deg(v)}},$$

where $E(G)$ is the set of edges and $\deg(u), \deg(v)$ are the degrees of the vertices u and v, respectively. In this research, we introduce the entire Randic index of graph. Exact values of this index for some graph families are obtained and some important properties of this new index are established.

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Topological graph indices have been in the center of interest related to the

2010 Mathematics Subject Classification: 05C40, 05C90.
Keywords: entire Randic index, graph index.
Received May 23, 2020; August 27, 2020

applications of graph theory. Randic Index is one of the most frequently used topological indices defined by Randic, [1], in 1975. It is a way of branching a carbon skeleton and has been correlated with a number of chemical properties. The branching index was renamed as the molecular connectivity index and is often referred to as the Randic Index.

2. Some General Results on $R^{\varepsilon}(G)$

After the classical versions of the topological graph indices are defined, several variants of them are also defined afterwards such as multiplicative, modified, generalized, inverse, etc. versions. Recently, entire versions of some topological indices have been defined and investigated due to the fact that in chemical applications, both the vertices representing atoms and edges representing chemical bonds are important, see e.g. [2] for entire Zagreb indices and [3] for the entire ABC index of graphs.

Definition 2.1. Let $G = (V, E)$ be a simple graph and let $B(G) = \{\{x, y\} : \{x, y\} \subseteq V(G) \cup E(G)$ and the elements x and y either adjacent or incident to each other$\}$.

Then the entire Randic index of G is defined by

$$R^{\varepsilon}(G) = \sum_{\{x, y\} \in B(G)} \frac{1}{\sqrt{\deg(x)\deg(y)}}.$$

The following are obvious results which will be needed:

Observation 2.2. Let G be a graph with m edges and let its first Zagreb index be $M_1(G)$. Then

$$|B(G)| = 2m + \frac{M_1(G)}{2}.$$

Observation 2.3. For any graph G, we have

$$R^{\varepsilon}(G) = R(G) + R_e(G) + \sum_{v \text{ incident to } e} \frac{1}{\sqrt{\deg(v)\deg(e)}}$$

Theorem 2.4. *For a k-regular graph G on n vertices with* $k \geq 2$, *we have*

$$R^{\varepsilon}(G) = \left(\frac{k+2}{4} + \sqrt{\frac{k}{2k-2}}\right)n.$$

Proof. Let G be a k-regular graph on n vertices with $k \geq 2$. Then G has $\frac{kn}{2}$ edges and $L(G)$ has

$$\frac{1}{2}\sum_{v \in V(G)} (\deg(v))^2 - q = \frac{nk^2}{2} - \frac{nk}{2} = \frac{kn(k-1)}{2}$$

edges. Therefore,

$$R^{\varepsilon}(G) = \frac{kn}{2}\frac{1}{\sqrt{k^2}} + \frac{kn(k-1)}{2}\frac{1}{\sqrt{(2k-2)^2}} + kn\frac{1}{\sqrt{k(2k-2)}}$$

$$= \frac{(k+2)}{4}n + n\sqrt{\frac{k}{2k-2}} = \left(\frac{k+2}{4} + \sqrt{\frac{k}{2k-2}}\right)n.$$

The following are immediate results of the above:

Theorem 2.5. *For a complete graph K_n with $n \geq 3$ vertices, we have*

$$R^{\varepsilon}(K_n) = \frac{n(n+1)}{4} + n\sqrt{\frac{n-1}{2n-4}}.$$

Theorem 2.6. *For a cycle C_n with $n \geq 3$ vertices, we have $R^{\varepsilon}(C_n) = 2n$.*

Theorem 2.7. *For any path P_n with $n \geq 4$ vertices, we have*

$$R^{\varepsilon}(P_n) = \frac{4n-9}{2} + 3\sqrt{2}.$$

Proof. Let P_n be a path with $n \geq 4$ vertices labeling as $v_1, v_2, \ldots, v_n$ and $n-1$ edges labeling as $e_1, e_2, \ldots, e_{n-1}$. we have the vertices v_1 and v_n are of degree one, similarly, the edges e_1 and e_{n-1} are of degree one and the other vertices and edges are of degree two. Therefore,

$$R^{\varepsilon}(P_n) = \frac{1}{\sqrt{4}}(n-3) + \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{4}}(n-4)$$

$$+\frac{2}{\sqrt{2}}+2(n-3)\frac{1}{\sqrt{4}}+\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{1}}$$

$$=\frac{4n-9}{2}+3\sqrt{2}.$$

Theorem 2.8. *For a complete bipartite graph* $K_{a,b}$, *where* $a+b\geq 3$

$$R^{\varepsilon}(K_{a,b})=\frac{ab}{2}+\sqrt{ab}+\frac{a\sqrt{b}+b\sqrt{a}}{\sqrt{a+b-2}}.$$

Proof. Let the vertices of $K_{a,b}$ be labelled as $v_1, v_2, \ldots, v_a$, $v_{a+1}, v_{a+2}, \ldots, v_b$. Then

$$R^{\varepsilon}(K_{a,\,b})=ab\frac{1}{\sqrt{ab}}+\left(\frac{1}{2}(a^2b+b^2a)\right)\frac{1}{\sqrt{(a+b-2)^2}}$$

$$+ab\frac{1}{\sqrt{a(a+b-2)}}+ab\frac{1}{\sqrt{b(a+b-2)}}$$

$$=\frac{ab}{2}+\sqrt{ab}+\frac{a\sqrt{b}+b\sqrt{a}}{a+b-2}.$$

A graph consisting of r paths joining two vertices is called an r-bridge graph denoted by $Q(k_1, \ldots, k_r)$, where $k_1, \ldots, k_r$ are the lengths of these r paths. Clearly an r-bridge graph is a generalized polygon tree.

Lemma 2.9. *Let G be an r-bridge graph* $Q(k_1, k_2, \ldots, k_r)$. *Then*

$$R(G)=\sqrt{2r}-r+\frac{1}{2}\sum_{i=1}k_i.$$

Proof. Let G be an r-bridge graph $Q(k_1, k_2, \ldots, k_r)$. Then

$$R(G)=(k_1-2)\frac{1}{\sqrt{4}}+(k_2-2)\frac{1}{\sqrt{4}}+,\ldots,+(k_r-2)\frac{1}{\sqrt{4}}+2\frac{r}{\sqrt{2r}}$$

$$=\sqrt{2r}-r+\frac{1}{2}\sum\nolimits_{i=1}^{r}k_i.$$

This implies the following result:

Lemma 2.10. *Let G be an r-bridge graph* $Q(k_1, k_2, \ldots, k_r)$. *Then*

$$R_e(G) = \sqrt{2r} - \frac{2r+1}{2} + \frac{1}{2}\sum_{i=1}^{r} k_i.$$

Proof. Let G be an r-bridge graph $Q(k_1, k_2, \ldots, k_r)$. Then

$$R_e(G) = (k_1 - 3)\frac{1}{\sqrt{4}} + (k_2 - 3)\frac{1}{\sqrt{4}} +, \ldots, + (k_r - 3)\frac{1}{\sqrt{4}} + 2\frac{r}{\sqrt{2r}} + \frac{r(r-1)}{2}\frac{1}{\sqrt{r^2}}$$

$$= \sqrt{2r} - \frac{2r+1}{2} + \frac{1}{2}\sum_{i=1}^{r} k_i.$$

Hence we can obtain the entire Randic index of an r-bridge graph as follows:

Theorem 2.11. *Let G be an r-bridge graph* $Q(k_1, k_2, \ldots, k_r)$. *Then*

$$R^{\varepsilon}(G) = 3\sqrt{2r} - \frac{8r-3}{2} + 2\sum_{i=1}^{r} k_i.$$

Proof. Let G be an r-bridge graph $Q(k_1, k_2, \ldots, k_r)$. Then by Observation 2.3, we have

$$R^{\varepsilon}(G) = R(G) + R_e(G) + \sum_{v \text{ incident to } e} \frac{1}{\sqrt{\deg(v)\deg(e)}}.$$

It is easy to see that

$$\sum_{v \text{ incident to } e} \frac{1}{\sqrt{\deg(v)\deg(e)}} = 2(k_1 - 2)\frac{1}{\sqrt{4}} + 2(k_2 - 2)\frac{1}{\sqrt{4}} +, \ldots,$$

$$+ 2(k_1 - 2)\frac{1}{\sqrt{4}} + 2\frac{r}{\sqrt{2r}} + 2 = \sqrt{2r} - 2r + 2 + \sum_{i=1}^{r} k_i.$$

So by Lemma 2.9 and Lemma 2.10, we get

$$R^{\varepsilon}(G) = 3\sqrt{2r} - \frac{8r-3}{2} + 2\sum_{i=1}^{r} k_i.$$

We need to define some sets to using them in the proof of some results. Let $G = (V, E)$ we define:

$V_{a,b}(G) = \{u, v : uv \in E(G), \deg(u) = a \text{ and } \deg(v) = b\}$,

$E_{a,b}(G) = \{e, f : e$ and f are adjacent edges in G, $\deg(e) = a$ and $\deg(f) = b\}$.

$B_{a,b}(G) = \{v, f : v$ and f are incident in G and G, $\deg(v) = a$ and $\deg(f) = b\}$.

Proposition 2.12. *Let G be a wheel graph W_n with $n+1$ vertices. Then*

$$R^{\varepsilon}(G) = \frac{n}{3} + \sqrt{\frac{n}{3}} + \frac{n}{4} + \frac{n}{\sqrt{n+1}} + \frac{n(n-1)}{2(n+1)} + \frac{n}{\sqrt{3}} + \frac{n}{\sqrt{3(n+1)}} + \sqrt{\frac{n}{n+1}}.$$

Proof. Let G be a wheel graph W_n with $n+1$ vertices. Then $|V_{3,3}(G)| =$

$|V_{3,n}(G)| = |E_{4,4}(G)| = n$, $|E_{4,n+1}(G)| = 2n$, $|E_{n+1,n+1}(G)| = \frac{n(n-1)}{2}$, $B_{3,4}(G) = 2n$ and $B_{3,n+1}(G) = B_{n,n+1}(G) = n$. Then by Observation 2.3, we get

$$R^{\varepsilon}(G) = n\frac{1}{\sqrt{9}} + n\frac{1}{\sqrt{3n}} + \frac{n}{\sqrt{16}} + 2n\frac{1}{\sqrt{4(n+1)}} + \frac{n(n-1)}{2}\frac{1}{\sqrt{(n+1)^2}}$$

$$+2n\frac{1}{\sqrt{12}} + n\frac{1}{\sqrt{3(n+1)}} + n\frac{1}{\sqrt{n(n+1)}}$$

$$= \frac{n}{3} + \sqrt{\frac{n}{3}} + \frac{n}{4} + \frac{n}{\sqrt{n+1}} + \frac{n(n-1)}{2(n+1)}$$

$$+ \frac{n}{\sqrt{3}} + \frac{n}{\sqrt{3(n+1)}} + \sqrt{\frac{n}{n+1}}$$

which gives the required result.

The Cartesian product $G_1 \square G_2$ of two graphs G_1 and G_2, where $V(G_1)$, $E(G_1)$ and $V(G_2)$, $E(G_2)$ are the sets of vertices and edges of G_1 and

G_2, respectively, has the vertex set $V(G_1) \times V(G_2)$ and two vertices (u, u') and (v, v') are connected by an edge if and only if either $(u = v$ and $u'v' \in E(G_2))$ or $(u' = v'$ and $uv \in E(G_1))$ [4]. We can now search for the entire Randic index of the Cartesian product of two graphs:

Lemma 2.13. *Let s, t be positive integer $s \geq 4$ and let $G \cong P_t \square P_s$ be the Cartesian product graph of two path graphs P_t and P_s. Then*

$$R(G) = \frac{1}{12}((4\sqrt{3} - 7)(s + t) + 6st + 16(\sqrt{6} - \sqrt{3}) - 12).$$

Proof. Let s, t be positive integers such that $s, t \geq 4$ and $G \cong P_t \square P_s$. Then, we have $|V_{2,3}(G)| = 8, |V_{3,3}(G)| = 2s + 2t - 12, |V_{3,4}(G)| = 2s + 2t - 8$ and $|V_{4,4}(G)| = (s - 3)(t - 2) + (s - 2)(t - 3)$. Therefore,

$$R(G) = 8\sqrt{\frac{1}{6}} + (2s + 2t - 12)\sqrt{\frac{1}{9}} + (2s + 2t - 8)\sqrt{\frac{1}{12}} + (2st - 5s - 5t + 12)\sqrt{\frac{1}{16}}.$$

Hence the result is obtained.

Lemma 2.14. *If $s, t \geq$ and $G \cong P_t \square P_s$ then*

$$R_e(G) = \frac{4}{3} + \frac{8}{\sqrt{12}} + \frac{2s + 2t - 16}{4} + \frac{8}{\sqrt{15}} + \frac{4s + 4t - 24}{\sqrt{20}} + \frac{4}{\sqrt{25}}$$
$$+ \frac{6s + 6t - 32}{\sqrt{30}} + \frac{6st - 18s - 18t + 52}{6}$$

Proof. Let $s, t \geq 4$ and let $G \cong P_t \square P_s$. So it is not difficult to see that $|E_{3,3}(G)| = 4, |E_{3,4}(G)| = 8, |E_{4,4}(G)| = 2(s - 4) + 2(t - 4) = 2s + 2t - 16,$ $|E_{3,5}(G)| = 8, |E_{4,5}(G)| = 4(s - 3) + 4(t - 3) = 4s + 4t - 24, |E_{5,5}(G)| = 4,$ $|E_{6,5}(G)| = 4(s-3) + 4(t-3) + 2(s-2) + 2(t-2) = 6s + 6t - 32$ and $|E_{6,6}(G)| =$ $(s - 4)(t - 2) + 4(s - 4)(t - 3) + 4(t - 3) + (s - 2)(t - 4) = 6st - 18s + 52 - 18t$ and then

$$R_e(G) = \frac{4}{\sqrt{9}} + \frac{8}{\sqrt{12}} + \frac{2s + 2t - 16}{\sqrt{16}} + \frac{8}{\sqrt{15}} + \frac{4s + 4t - 24}{\sqrt{20}} + \frac{4}{\sqrt{25}}$$

$$+\frac{6s+6t-32}{\sqrt{30}}+\frac{6st-18s-18t+52}{\sqrt{36}}$$

and hence the required result is obtained.

Theorem 2.15. *Let* s, t *be positive integer* $s \geq 4$ *and* $G \cong P_t \square P_s$. *Then*

$$R^{\varepsilon}(G)=\left(\frac{16}{\sqrt{6}}+4+\frac{8}{\sqrt{12}}+\frac{8}{\sqrt{15}}+\frac{4}{5}\right)+\frac{4s+4t-24}{\sqrt{12}}+\frac{2s+2t-8}{\sqrt{15}}+\frac{2s+2t-8}{\sqrt{20}}$$

$$+\frac{4st-10s-10t+24}{\sqrt{24}}+\frac{2s+2t-16}{4}+\frac{4s+4t-24}{\sqrt{20}}+\frac{6s+6t-32}{\sqrt{30}}$$

$$+\frac{6st-18s-18t+52}{6}+\frac{(2s+2t-12)}{3}+\frac{(2s+2t-8)}{\sqrt{12}}+\frac{(2st-5s-5t+12)}{4}$$

Proof. Let s, t be any positive integers such that $s, t \geq 4$ and $G \cong P_t \square P_s$. By Observation 2.3, we have

$$R^{\varepsilon}(G)=R(G)+R_e(G)+\sum_{v \text{ incident to } e}\sqrt{\frac{1}{\deg(v)\deg(e)}}.$$

To get $\sum_{v\,\text{incident}\,\text{to}\,e}\frac{1}{\sqrt{\deg(v)\deg(e)}}$, we have:

$|B_{3,2}(G)|=8, |B_{3,3}(G)|=8, |B_{4,3}(G)|=4(s-3)+4(t-3)=4s+4t-24,$

$|B_{5,3}|=2(s-2)+2(t-2)=2s+2t-8$, $|B_{5,4}|=2s+2t-8$ and

$|B_{6,4}|=2(s-3)(t-2)+2(t-3)(s-2)=4st-10s+24-10t$. Then

$$\sum_{v\,\text{incident}\,\text{to}\,e}\frac{1}{\sqrt{\deg(v)\deg(e)}}=\frac{8}{\sqrt{6}}+\frac{8}{\sqrt{9}}+\frac{4s+4t-24}{\sqrt{12}}+\frac{2s+2t-8}{\sqrt{15}}$$

$$+\frac{2s+2t-8}{\sqrt{20}}+\frac{4st-10s-10t+24}{\sqrt{24}}. \tag{2.1}$$

Then by Lemma 2.13, Lemma 2.14 and Equation 2.1, we get

$$R^{\varepsilon}(G)=\left(\frac{16}{\sqrt{6}}+4+\frac{8}{\sqrt{12}}+\frac{8}{\sqrt{15}}+\frac{4}{5}\right)+\frac{4s+4t-24}{\sqrt{12}}+\frac{2s+2t-8}{\sqrt{15}}+\frac{2s+2t-8}{\sqrt{20}}$$

$$+\frac{4st-10s-10t+24}{\sqrt{24}}+\frac{2s+2t-16}{4}+\frac{4s+4t-24}{\sqrt{20}}+\frac{6s+6t-32}{\sqrt{30}}$$

$$+\frac{6st-18s-18t+52}{6}+\frac{(2s+2t-12)}{3}+\frac{(2s+2t-8)}{\sqrt{12}}+\frac{(2st-5s-5t+12)}{4}$$

Lemma 2.16. *For positive integers $s \geq 4$ and $t \geq 3$, if $G \cong P_t \square P_s$, then*

$$R(G)=\left(\frac{2}{3}+\frac{1}{\sqrt{3}}-\frac{5}{4}\right)t+\frac{1}{2}st.$$

Proof. Let $s \geq 4$ and $t \geq 3$, and let $G \cong P_s \square C_t$, We have $|V_{3,3}(G)| = 2t$, and $|V_{3,4}(G)| = 2t$ $|V_{4,4}(G)| = t(s-2)+t(s-3) = t(2s-5)$.

Then $R(G) = 2t\frac{1}{\sqrt{9}}+2t\frac{1}{\sqrt{12}}+\frac{t(2s-5)}{\sqrt{16}}$.

Therefore the required result is attained.

Lemma 2.17. *For positive integers $s \geq 4$ and $t \geq 3$, if $G \cong P_s \square C_t$, then*

$$R_e(G)=\left(\frac{2}{\sqrt{5}}+\frac{6}{\sqrt{30}}-\frac{5}{2}\right)t+ts.$$

Proof. Let $s \geq 4$ and $t \geq 3$. If $G \cong P_s \square C_t$, it is easy to see that $|E_{4,4}(G)| = 2t$, $|E_{4,5}(G)| = 4t$, $|E_{5,6}| = 6t$ and $|E_{6,6}| = t(s-2)+t(s-4)+4t(s-3) = 6ts-18t$, then $R(G) =$. Therefore,

$$R^e(G)=2t\frac{1}{\sqrt{16}}+4t\frac{1}{\sqrt{20}}+6t\frac{1}{\sqrt{30}}+(6ts-18t)\frac{1}{\sqrt{36}}.$$

Hence,

$$R_e(G)=\left(\frac{2}{\sqrt{5}}+\frac{6}{\sqrt{30}}-\frac{5}{2}\right)t+ts.$$

Theorem 2.18. *Let $G \cong P_s \square C_t$, where $s \geq 4$ and $t \geq 3$. Then*

$$R^{\varepsilon}(G)=\left(\frac{3\sqrt{5}+3\sqrt{3}+2}{\sqrt{15}}+\frac{6}{\sqrt{30}}-\frac{37}{12}\right)t+\left(\frac{3}{2}+\frac{2}{\sqrt{6}}\right)ts.$$

Proof. Let s, t be two positive integers such that $s \geq 4$ and $t \geq 3$. If $G \cong P_s \square C_t$, then by Observation 2.3, we have

$$R^{\varepsilon}(G) = R(G) + R_e(G) + \sum_{v \text{ incident to } e} \frac{1}{\sqrt{\deg(v)\deg(e)}}.$$

To get $\sum_{v\, incidenttoe} \frac{1}{\sqrt{\deg(v)\deg(e)}}$, we have $|B_{3,4}(G)| = 4t, |B_{3,5}(G)| = 2t$, $|B_{4,5}(G)| = 2t$ and $|B_{4,6}| = 2t(s-2) + 2t(s-3) = 4ts - 10t$. Therefore

$$\sum_{v \text{ incident to } e} \frac{1}{\sqrt{\deg(v)\deg(e)}} = \frac{4t}{\sqrt{12}} + \frac{2t}{\sqrt{15}} + \frac{2t}{\sqrt{20}} + \frac{4ts - 10t}{\sqrt{24}}$$

$$= \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{15}} - \frac{5}{\sqrt{6}}\right) + \frac{2ts}{\sqrt{6}}. \tag{2.2}$$

Then by Lemma 2.16, Lemma 2.17 and Equation 2.2, we get

$$R^{\varepsilon}(G) = \left(\frac{3\sqrt{5} + 3\sqrt{3} + 2}{\sqrt{15}} + \frac{6}{\sqrt{30}} - \frac{37}{12}\right)t + \left(\frac{3}{2} + \frac{2}{\sqrt{6}}\right)ts.$$

Theorem 2.19. *For any graph G with n vertices and m edges, we have*

$$R^{\varepsilon}(G) \leq \sqrt{(2m + \frac{M_1(G)}{2})M_2^{\varepsilon *}(G)},$$

where $M_2^{\varepsilon *}(G) = \sum_{\{x,\, y\} \in B(G)} \frac{1}{\deg(x) \cdot \deg(y)}$ *is the second modified entire Zagreb index and* $M_1(G)$ *is the first Zagreb index of G.*

Proof. Let G be a graph with n vertices and m edges. Then we have

$$R^{\varepsilon}(G) = \sum_{\{x,\, y\} \in B(G)} \frac{1}{\sqrt{\deg(x)\deg(y)}}.$$

By using Cauchy-Schwarz inequality, we get

$$R^{\varepsilon}(G) = \sum_{\{x,\, y\} \in B(G)} \sqrt{\frac{1}{\deg(x)\deg(y)}}$$

$$\leq \sqrt{|B(G)| \sum_{\{x,\, y\} \in B(G)} \frac{1}{\deg(x)\deg(y)}}$$

$$= \sqrt{|\ B(G)\ | M_2^{\varepsilon *}(G)}$$

$$= \sqrt{(2m + \frac{M_1(G)}{2}) M_2^{\varepsilon *}(G)}.$$

3. Conclusions

Graphs are frequently used in modeling molecules. Topological graph indices are the mathematical invariants which help to determine some structural and physico-chemical properties of molecules by just some mathematical methods. For more than seven decades, more than 3000 indices have been defined and applied to obtain several properties. Randic index is one of the oldest indices and several variants of it have been defined. In this paper, using the recent idea of entire indices, the authors introduce the entire Randic index of graphs and study this new index for several graphs and graph operations. As open problems, this method can be applied to the class of degree based indices and several composition graphs and derived graphs.

References

[1] M. Randic, Characterization of molecular branching, J. Am. Chem. Soc. 97 23 (1975), 6609-6615.

[2] A. Alwardi, A. Alqesmah, R. Rangarajan and Cangul, In entire Zagreb indices of graphs, Discrete Math Algorithm Appl. 10(3) 1850037(16 pages) (2018).

[3] A. Alqesmah, A. Alwardi, Cangul, In on the entire ABC index of graphs, Proceedings of the Jangjeon Mathematical Society 23(1) (2020), 39-51.

[4] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.