Advances and Applications in Mathematical Sciences **ISSN 0974-6803**
Volume 22, Issue 6, April 2023, Pages 1335-1352

HOPF BIFURCATION AND SENSITIVITY ANALYSIS FOR MEASLES INFECTION MODEL WITH STRENGTH NUMBER

M. RADHA and P. V. JEYAKARTHIKEYAN

Assistant Professor
Department of Mathematics
SRM Institute of Science and Technology
Ramapuram-600 089, Tamil Nadu, India
E-mail: radham2@srmist.edu.in

Associate Professor
Department of Mechanical Engineering
SRM Institute of Science and Technology
Kattankulathur-603 203, Tamil Nadu, India
E-mail: jeyakarp@srmist.edu.in

Abstract

In this article, the model of measles transmission is proposed with nonlinear saturated incidence terms on a susceptible population and a time delay depicting the latent period of the infection. The system can be destabilized by a delay in the transmission period and the periodic solution can be obtained through Hopf bifurcation, while the delay is chosen as the bifurcating parameter. The non-linear incidence rate preserves to turn out the complex dynamics models and transform the models more realistic and convenient. The article also examines the essential parameters such as the recruitment rate of susceptible people using sensitivity indices analysis. The numerical simulation reveals that the recruitment rate can change the system dynamics from a limit cycle to a stable focus as its value decreases using MATLAB.

Introduction

Measles can be transmitted directly through contact with an infected person or throat secretions or through the coughing and sneezing of infected individuals. It was estimated that 122,000 children died of measles by the

2020 Mathematics Subject Classification: 34A34, 34D20, 34E05.
Keywords: nonlinear saturated incidence, time delay, Hopf bifurcation.
Received July 7, 2022; Accepted December 22, 2022

World Health Organization (WHO) in 2012. It is an extremely transmitted infectious disease by person-to-person mode with an attack rate of over 90% between susceptible persons and infected persons [1, 2]. It is a viral infection that affects the immune system. If an outbreak has an epidemic, it is important to control the epidemic's social effects by inhibitory effects and different restrictions on treatment approaches. In 1987, Liu et al., [3] proposed the nonlinear incidence rate. If two types of incidence which are saturated with susceptible or infection used in the epidemiological model $\beta SI(1+\alpha S)$, $\beta SI/(1+\alpha I)$ [4, 11].

The current dynamics of state variables can be applied in many real life systems, especially in many biological phenomena, depend not only upon the current process state but also upon the past values of the state variables called delay [5]. The models qualitative behavior can be changed since the system will be destabilized by a delay and therefore periodic solutions can be obtained through a Hopf bifurcation [6, 7]. Momoh A et al., [8] investigated the measles model by including an exposed class and analyzing the stability. Recently the work was extended by Muhammad Farman by using numerical solutions using the Laplace Adomian decomposition method and non-integer time-fractional derivatives. In this article, Momoh A et al. measles model reinvestigated by incorporating the time delay effect on a saturated incidence rate.

The mathematical model

The time delay effect on a saturated incidence rate with saturated terms on the susceptible of the measles model is formulated as in equation (1). Let Λ_1 is the recruitment rate or immigration rate, β_1 is the transmission rate, death rate, ε_1 developing infectivity rate, λ_1 measles therapy rate, γ_1 rate of recovery, rate that quantifies the inhibitory effect. A susceptible person is presumed to be exposed in the time $t-\tau(\tau>0)$ and become infectious, sometimes after adequate contact with an infective person. Schematic diagram is shown in figure 1.

$$\frac{dS(t)}{dt} = \Lambda_1 - \frac{\beta_1 S(t-\tau)I(t-\tau)}{1+m_1 S(t-\tau)} - \mu_1 S(t),$$

$$\frac{dE(t)}{dt} = \frac{\beta_1 S(t-\tau)I(t-\tau)}{1+m_1 S(t-\tau)} - (\mu_1 + \varepsilon_1 + \lambda_1)E(t),$$

$$\frac{dI(t)}{dt} = \varepsilon_1 E(t) - (\mu_1 + \varepsilon_1 + \lambda_1)I(t),$$

$$\frac{dI(t)}{dt} = \gamma_1 I(t) + \lambda_1 E(t) - \mu_1 R(t). \tag{1}$$

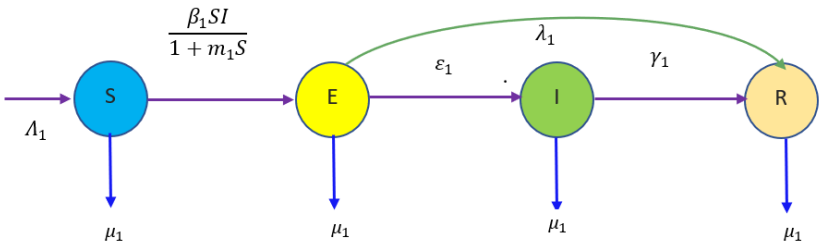


Figure 1. Biological representation of the parameters and mechanisms of the model (1).

Boundedness and positivity

The primary conditions of the system (1) are defined by $S(\theta) = \phi_1(\theta)$, $E(\theta) = \phi_2(\theta)$, $I(\theta) = \phi_3(\theta)$ and $R(\theta) = \phi_4(\theta)$, $\phi_i(\theta) \geq 0$, for all $\theta \in [-\tau, 0]$ and $\phi_i(0) > 0 (i = 1, 2, 3, 4)$ here $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)T \in C$ where $C = C([-\tau, 0], R_+^4)$, the continuous function in Banach space mapping the interval $[-\tau, 0]$ into R_+^4. It can be seen that every system solution is defined at $[0, +\infty)$ and remains positive for all. Likewise it can be seen that $E(t)$, $I(t)$, and $R(t)$ are all positive for $t > 0$. System (1) is examined biologically in the feasible are

$$\Omega = \{S(t), E(t), I(t), R(t)/S(t) > 0, E(t) \geq 0, I(t) \geq 0, R(t) > 0; S(t) + E(t) + I(t) + R(t) \leq \frac{\Lambda_1}{\mu_1}\}.$$

Reproduction number and equilibrium points:

To compute the basic reproduction number R_0 by next generation matrix method.

$$R_0 = \rho(FV^{-1}) = \frac{\beta_1\Lambda_1\varepsilon_1}{(\mu_1+\gamma_1)(\mu_1+\lambda_1+\varepsilon_1)(\mu_1+m_1\Lambda_1)}.$$

The system (1) has following disease free equilibrium $E_0 = \left(\frac{\Lambda_1}{\mu_1}, 0, 0, 0\right)$.

Endemic equilibrium $E_1(S_*, E_*, I_*, R_*)$ here

$$S_* = \frac{1}{\mu_1}\left[\Lambda_1 - \frac{(\mu_1+\varepsilon_1+\lambda_1)(\mu_1+\gamma_1)}{\varepsilon_1} I_*\right]$$

$$E_* = \frac{(\mu_1+\gamma_1)}{\varepsilon_1} I_*,\ R_* = \frac{\gamma_1}{\mu_1} + \frac{(\mu_1+\gamma_1)}{\varepsilon_1\mu_1} I_* \tag{2}$$

$$I_* = \frac{p_3\varepsilon_1}{a_1-a_2}\Lambda_1 - \frac{a_1\mu_1}{p_3\varepsilon_1\beta_1 - m_1a_1}$$

with

$$a_1 = p_1p_2p_3,\ a_2 = \gamma_1\varepsilon_1\mu_1p_1 = \mu_1\varepsilon_1 + \lambda_1p_2 = \mu_1p_3 = \mu_1+\gamma_1.$$

Strength Number. In the last decades, the concept of reproductive has been employed intensively in epidemiological modeling as it has been recognized as a useful mathematical formula to evaluate reproduction in some given infections disease. As the theory suggested one will find two components F_A and V, then $(F_AV^{-1} - \lambda I) = 0$ will be used to reproduce the reproductive number. At the disease-free equilibrium point, $\det(F_AV^{-1} - \lambda I) = 0$ leads to $A_0 = \frac{-2\beta_1\Lambda_1\varepsilon_1}{(\mu_1+m_1\Lambda_1)^2(\mu_1+\lambda_1\varepsilon_1)(\mu_1+\gamma_1)} < 0$, means there is no strength.

Stability analysis of E_0 equilibrium (Disease-free):

If $R_0 < 1$, the delayed system (disease-free) E_0 equilibrium is completely stable on $\tau_1 \geq 0$.

Proof. Jacobian matrix of the equation (1) at E_0, the determinant $J(E_0)$ gives the eigenvalues $\lambda_1 = -\mu_1,\ \lambda_2 = -\lambda_2 = -\mu_1\lambda_3,\ \lambda_4$

$$f_1(\lambda) = \lambda^2 + \lambda(2\mu_1+\varepsilon_1+\gamma_1) + (\mu_1+\varepsilon_1)(\mu_1+\gamma_1)(1 - R_0e^{\lambda\tau}) = 0 \tag{3}$$

Put $\tau = 0$ $f_1(\lambda) = \lambda^2 + \lambda(2\mu_1 + \varepsilon_1 + \gamma_1) + (\mu_1 + \varepsilon_1)(\mu_1 + \gamma_1)(1 - R_0) = 0.$

If $R_0 > 1,$ then the equations have other roots are negative real parts and E_0 is locally asymptotically stable for $\tau = 0.$ If $\tau > 0,$ then equation (3) has purely imaginary roots. Substitute $\lambda = i\omega(\omega_1 > 0)$ in equation (3), $F(\omega) = \omega^4 + \omega^2((2\mu_1 + \varepsilon_1 + \gamma_1)^2 - 2(\mu_1 + \varepsilon_1)(\mu_1 + \gamma_1)) + (\mu_1 + \varepsilon_1)^2(\mu_1 + \gamma_1)^2$ $(1 - R_0^2) = 0.$

If $R_0 < 1$ then $(\mu_1 + \varepsilon_1)^2(\mu_1 + \gamma_1)^2 - (\mu_1 + \varepsilon_1)^2(\mu_1 - \gamma_1)^2 R_0^2 > 0.$ Then E_0 is locally asymptotically stable for $\tau \geq 0$

If $R_0 < 1,$ then $(\mu_1 + \varepsilon_1)(\mu_1 + \gamma_1) < (\mu_1 + \varepsilon_1)(\mu_1 - \gamma_1)R_0$ and the equation (3) has a positive root. Moreover $\frac{\partial F}{\partial \omega} = 4\omega^3((2\mu_1 + \varepsilon_1 + \gamma_1)^2$ $-2(\mu_1 + \varepsilon_1)(\mu_1 + \gamma_1)) > 0$ and by several authors like such as cook and Van den Driessche and Freedman and Kuang [9, 10] E_0 is unstable for $\tau \geq 0.$

Local stability of the endemic equilibrium and Hopf bifurcation

In many epidemic models, bifurcation behavior exists. Such behavior is generally deleterious to biological systems and often causes a disease to spread gradually or in certain regions suddenly.

Then the system gets the linearization, let consider $S_1(t) = S(t) - S_*,$ $E_1(t) = E(t) - E_*,$ $I_1(t) = I(t) - I_*,$ $R_1(t) = R(t) - R_*,$ then

$$\frac{dS_1(t)}{dt} = a_1 S_1(t) + b_1 S_1(t - \tau) + c_1 I_1(t - \tau),$$

$$\frac{dE_1(t)}{dt} = a_2 S_1(t - \tau) + b_2 E_1(t) + c_2 I_1(t - \tau),$$

$$\frac{dI_1(t)}{dt} = a_3 E_1(t) + b_3 I_1(t),$$

$$\frac{dR_1(t)}{dt} = a_4 E(t) + b_4 I(t) + c_4 R(t), \tag{4}$$

here $a_1 = -\mu_1$, $b_1 = \dfrac{-\beta_1 I}{(1+m_1 S)^2}$, $c_1 = -\dfrac{-\beta_1 I}{(1+m_1 S)}$, $a_2 = \dfrac{\beta_1 I}{(1+m_1 S)^2}$,

$b_2 = -(\mu_1 + \varepsilon_1 + \lambda_1)$, $c_2 = \dfrac{\beta_1 I}{(1+m_1 S)^2}$, $a_3 = \varepsilon_1$, $b_3 = -(\mu_1 + \gamma_1)$, γ_1, $c_4 = -\mu_1$,

$a_4 = \lambda_1$. Thus the characteristic equation of the above system (4) can be obtained as

$$\lambda_4 + \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0 + e^{-\lambda\tau}(B_3\lambda^3 + \lambda B_2 + \lambda B_1 + B_0) + e^{-2\lambda\tau}(C_0 + C_1\lambda) = 0, \tag{5}$$

Here

$A_0 = a_2 a_1 b_3 c_4$, $A_1 = -[a_1 a_1 b_3 + a_1 a_2 c_4 + a_1 c_4 b_3 + b_3 a_2 c_4]$,

$A_2 = -[b_3 c_4 + a_2 b_3 + a_2 c_4 + a_1 c_4 b_3 + b_3 a_1 a_2]$, $A_3 = -[a_1 + a_2 + b_3 + c_4]$,

$B_0 = [a_2 b_1 b_3 c_4 - a_1 c_2 a_3 c_4]$, $B_1 = [-a_1 c_2 a_3 - a_2 b_1 b_3 - a_2 b_1 c_4 - b_1 b_3 c_4]$,

$B_2 = a_2 b_1 + b_1 c_4 + b_1 b_2 + a_3 c_2$, $B_3 = C_1 = -c_1 b_2 a_3 - b_1 c_2 b_3$,

$C_0 = -c_2 b_1 a_3 c_4 + b_2 c_1 a_3 c_4$

Case I. $\tau = 0$

Equation (5) becomes,

$$\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0, \tag{6}$$

$$\left.\begin{aligned} &\det_1 = A_3 > 0, \qquad \det_2 = \begin{vmatrix} A_3 & 1 \\ A_1 & A_3 \end{vmatrix} > 0 \\ &\det_3 = \begin{vmatrix} A_3 & 1 & 0 \\ A_1 & A_2 & A_3 \\ 0 & A_0 & A_1 \end{vmatrix} > 0, \quad \det_4 = \begin{vmatrix} A_3 & 1 & 0 & 0 \\ A_1 & A_2 & A_3 & 1 \\ 0 & A_0 & A_1 & A_1 \\ 0 & 0 & 0 & A_0 \end{vmatrix} > 0 \end{aligned}\right\}. \tag{7}$$

If the conditions (7) hold, then E_1 (endemic equilibrium) is locally asymptotically stable for $\tau = 0$ by using Routh-Hurwitz criterion.

Case II. $\tau = 0$.

For $\tau = 0$, equation (5) becomes

$$(\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0) + (B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0)e^{-2\lambda\tau}(c_0 + c_1\lambda) = 0. \quad (8)$$

Multiplying equation (8) by $e^{\tau\lambda}$ then it becomes,

$$(\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)e^{\lambda\tau} + (B_3\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0) + e^{\lambda\tau}(c_0 + c_1\lambda) = 0. \quad (9)$$

Let $\lambda = i\omega(\omega > 0)$ be the root of equation (9), then we have

$$\begin{cases} M_1 \cos\tau\omega + \omega + M_2 \sin\tau\omega M_5 \\ M_3 \sin\tau\omega - M_4 \cos\tau\omega M_6 \end{cases},$$

$M_1 = \omega^4 - A_2\omega^2 + A_0$, $M_2 = A_3\omega^3 - A_1\omega$, $M_3 = \omega$, $M_3 = \omega^4 - A_2\omega^2 - A_0$,

$M_4 = A_3\omega^3 - A_1\omega$, $M_5 = B_2\omega^2 - B_0$, $M_6 = B_3\omega^3 - B_1\omega$.

It follows that, $\cos\tau\omega = \dfrac{h_6\omega^6 + h_4\omega^4 + h_2\omega^2 + h_0}{\omega^8 + g_6\omega^6 + g_4\omega^4 + g_2\omega^2 + g_0}$,

$\sin\tau\omega = \dfrac{h_7\omega^7 + h_5\omega^5 + h_3\omega^3 + h_1\omega}{\omega^8 + g_6\omega^6 + g_4\omega^4 + g_2\omega^2 + g_0}$,

with

$g_0 = A_0^2$, $g_2 = A_1^2 - 2A_2A_0$, $g_4 = A_2^2 + 2A_0 - 2A_1A_3$,

$g_6 = A_3^2 - 2A_2$, $h_0 = -B_0A_0$, $h_1 = (A_1 + 1)B_0 - (A_0 + 1)B_1$,

$h_3 = A_0B_3 + B_3 + A_2B_1 - B_2A_1 - B_2 - A_3B_0$, h_5

$= A_3B_2 - B_3A_2 - B_1$, $h_7 = B_3$.

Then,

$$w^{16} + e_7w^{14} + e_6w^{12} + e_5w^{10} + e_8w^8 + e_3w^6 + e_2w^4 + e_1w^2 + e_0 = 0, \quad (10)$$

with $e_0 = g_0^2 - h_0^2$, $e_1 = 2g_0g_2 - 2h_0h_2 - h_1^2$, $e_2 = g_2^2 + 2g_0g_4 - 2h_0h_4 - h_2^2 - 2h_1h_3$,

$e_3 = 2g_0g_6 + 2g_2g_4 - 2h_0h_6 - h_3^2 - 2h_1h_5 - 2h_2h_4$, $e_4 = g_4^2 + 2g_2g_6 + 2g_0 - 2h_2h_6 - h_4^2 - 2h_1h_7 - 2h_3h_5$,

$e_5 = 2g_4g_6 + 2g_2 - 2h_4h_6 - h_5^2 - 2h_3h_7$, $e_6 = g_6^2 - h_5^2 + 2g_4$, $e_7 = 2g_6 - h_7^2$.

Let $w^2 = v$, then the equation (2.10) becomes $v^8 + e_7v^7 + e_6v^6 + e_5w^5 + e_4w^4 + e_3w^3 + e_2w^2 + e_1w^1 + e_0 = 0$,

It must be proved that the condition for Hopf bifurcation (C_1), "equation (11) has at least one real positive root".

If this condition (C_1) is preserved, then equation (11) has such a positive equilibrium. Equation (10) has a simple imaginary pair of roots $\pm iw = \pm i\sqrt{\nu}$ related to the critical value of time delay and is given by,

$$\tau_m^{(n)} = \frac{1}{\omega_m}\cos^{-1}\left[\frac{h_6\omega_1^6 + h_4\omega_1^4 + h_2\omega_1^2 + h_0}{\omega_1^8 + g_6\omega_1^6 + g_4\omega_1^4 + g_2\omega_1^2 + g_0} + 2n\pi\right],$$

$m = 1, 2, 3$, $n = 0, 1, 2, \ldots$

Here,

$$\tau_1 = \min\{\tau_m^{(n)}\},\ \tau_1 = \frac{1}{\omega_1}\cos^{-1}\left[\frac{h_6\omega_1^6 + h_4\omega_1^4 + h_2\omega_1^2 + h_0}{\omega_1^8 + g_6\omega_1^6 + g_4\omega_1^4 + g_2\omega_1^2 + g_0} + 2n\pi\right]$$

Differentiate both sides of equation (9) with respect to τ_1 and note that λ is a function of τ_1,

$$\left[\frac{d\lambda}{d\tau_1}\right]^{-1} = \frac{f_0(\lambda) + f_1(\lambda)e^{\lambda\tau_1}}{f_3(\lambda)e^{-\lambda\tau_1} - f_4(\lambda)e^{\lambda\tau_1}} - \frac{\tau_1}{\lambda},$$

Here $f_0(\lambda) = 3B_3\lambda^2 + 2B_2\lambda + B_1$, $f_1(\lambda) = 4\lambda^3 + 3A_3\lambda^2 + 2A_2\lambda$

$+ A_1$, $f_3(\lambda) = \lambda^2 C_1 + \lambda C_0$, $\phi_4(\lambda) = \lambda^5 + \lambda^4 A_3 + \lambda^3 A_2 + \lambda^2 A_1 + \lambda A_0$.

$$sign\left\{\frac{d\,\mathrm{Re}\,\lambda}{d\tau_1}\right\}^{-1}_{\lambda=iw_1} = sign\left\{\mathrm{Re}\left[\frac{f_0(\lambda) + f_1(\lambda)e^{\lambda\tau_1}}{f_3(\lambda)e^{-\lambda\tau_1} - f_4(\lambda)e^{\lambda\tau_1}} - \frac{\tau_1}{\lambda}\right]\right\}_{\lambda=iw_1}$$

Now

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda=iw_1} = \frac{P_{2R}Q_{2R} + P_{1I}Q_{2I}}{Q_{2R}^2 + Q_{2I}^2},$$

where

$$P_{2R} = [2\omega_1^2 - 4\omega_1^4]\cos\omega_1\tau_1 + [A_1\omega_1 - 3\omega_1^3 A_3]\sin\omega_1\tau_1 + 2B_2\omega_1^2,$$

$$P_{2I} = [(\omega_1 - 3\omega_1^3 A_3]\cos\omega_1\tau_1 + [4\omega_1^4 - 2\omega_1^2 A_2]\sin\omega_1\tau_1 + 3B_3\omega_1^2,$$

$$Q_{2R} = [\omega_1^6 - \omega_1^4 A_2 + A_0\omega_1^2]\cos\omega_1\tau_1 + [\omega_1^5 A_3 - \omega_1^3 A_1]\sin\omega_1\tau_1,$$

$$Q_{2I} = [A_3\omega_1^5 - A_1\omega_1^3]\cos\omega_1\tau_1 + [\omega_1^6 - \omega_1^4 A_2 + A_0\omega_1^2]\sin\omega_1\tau_1.$$

It is clear that if the condition $(C_1)P_{2R}Q_{2R} + P_{2I}Q_{2I} \neq 0$ satisfies, then we have $\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda=iw_1} \neq 0$

If (C_2) holds, then the requirement of transversality is fulfilled. Based on the theorem in Hopf bifurcation, then it has the subsequent Preposition 1 for the system (1).

Preposition 1. *The asymptotical stability of* (*endemic equilibrium*) E_1 *is obtained when the conditions* $(C_1) - (C_2)$ *are held for* $\tau \in [0, \tau_1)$. *System* (3) *endures a Hopf bifurcation at* $E_1(S_*, E_*, I_*, R_*)$ *at* $\tau = \tau_1$ *and a periodical solutions family bifurcate from* $E_1(S_*, E_*, I_*, R_*)$ *near* $\tau = \tau_1$.

Sensitivity parameters analysis:

The essential parameter must be identified which might be a critical threshold for disease management.

$$\frac{\partial R_0}{\partial \varepsilon_1} = \frac{\beta_1 \Lambda_1 \mu_1}{(\mu_1 + m_1\Lambda_1)(\mu_1 + \varepsilon_1\lambda_1)^2(\mu_1 + \gamma_1)},$$

$$\frac{\partial R_0}{\partial \beta_1} = \frac{\varepsilon_1 \Lambda_1}{(\mu_1 + m_1\Lambda_1)(\mu_1 + \varepsilon_1\lambda_1)(\mu_1 + \gamma_1)},$$

$$\frac{\partial R_0}{\partial \Lambda_1} = \frac{\beta_1 \Lambda_1 \mu_1}{(\mu_1 + m_1\Lambda_1)(\mu_1 + \varepsilon_1\lambda_1)^2(\mu_1 + \gamma_1)},$$

$$\frac{\partial R_0}{\partial \beta_1} = \frac{\varepsilon_1 \Lambda_1}{(\mu_1 + m_1\Lambda_1)^2(\mu_1 + \varepsilon_1\lambda_1)(\mu_1 + \gamma_1)}.$$

It is determined that all partial derivatives are positive, and that increasing any of the above factors increases the basic reproductive number R_0. The proportionate reaction to the proportional perturbation is used to estimate the elasticity.

$$E_{\varepsilon_1} = \frac{\varepsilon_1}{R_0}\frac{\partial R_0}{\partial \varepsilon_1} = \frac{\mu_1 + \lambda_1}{(\mu_1 + \varepsilon_1 + \lambda_1)} = 0.97,\ E_{\beta_1} = \frac{\beta_1}{R_0}\frac{\partial R_0}{\partial \beta_1} = 1,$$

$$E_{\Lambda_1} = \frac{\Lambda_1}{R_0}\frac{\partial R_0}{\partial \Lambda_1} = \frac{\mu_1}{(\mu_1 + m_1\Lambda_1)} = 0.67.$$

$$E_{m_1} = \frac{m_1}{R_0}\frac{\partial R_0}{\partial m_1} = -\frac{m_1}{(\mu_1 + m_1\Lambda_1)} = -0.32.$$

From the above E_{ε_1}, E_{β_1}, E_{Λ_1} expressions, it seems that they are positive and E_{m_1} is negative. This indicates that increasing the ε_1, β_1 and Λ_1 tends to increase the value of R_0 and increasing the value of m_1 tends to decrease the value of R_0. The smallest alteration in these parameters can result in a large variation in R_0.

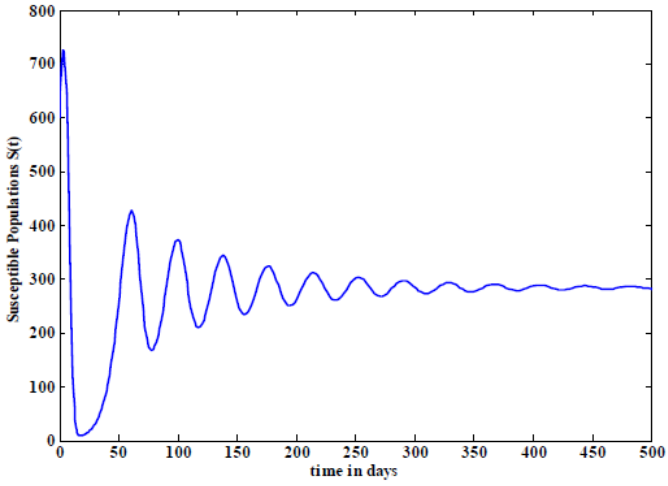
Numerical Simulation

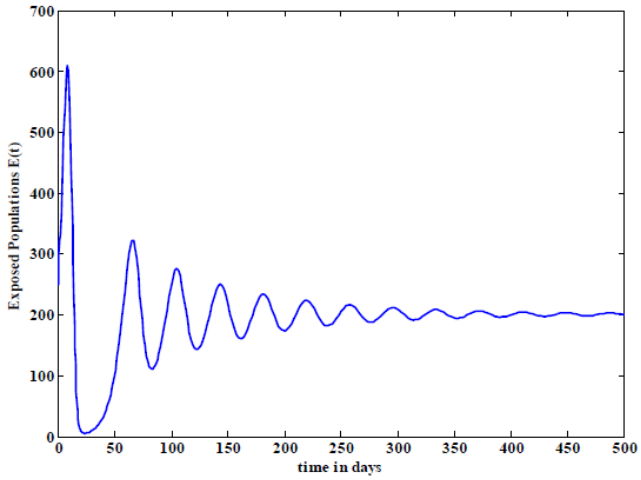
Analytical studies cannot be carried out in the outbreak model without numerical results verification. Integrate the system (1) numerically with the standard MATLAB. The parameters value used are $\Lambda_1 = 0.32$, $\mu_1 = 0.2$,

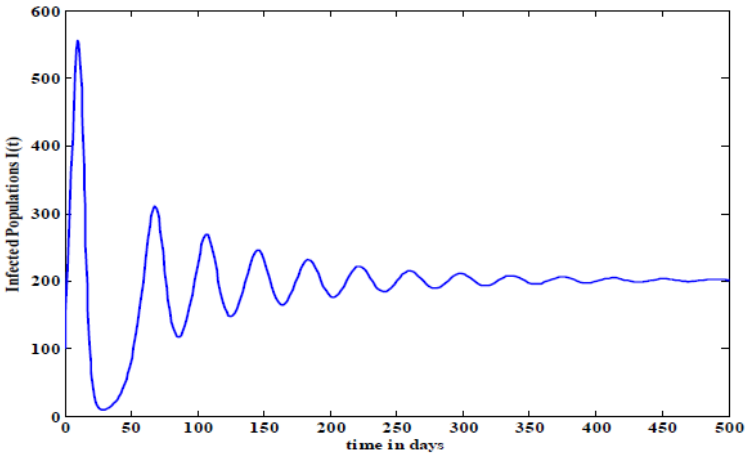
$\varepsilon_1 = 0.01$, $\lambda_1 = 0.2$, $m_1 = 0.3$, $\lambda_1 = 0.25$, $\beta_1 = 0.01$ [16].

The time delay τ (varied) demonstrates the dynamic behavior of a model and its Hopf bifurcation phenomenon. For the parameter set with $\beta_1 = 0.001$, E_0 is locally asymptotically stable with $R_0 < 1$ but for $\beta_1 = 0.01$, there exists endemic equilibrium. This is because the number of initial contacts (β) is very small and the basic reproductive number less than unity, while the basic reproductive number is more than unity as β increases.

Let consider the initial conditions be $S(0) = 600$, $E(0) = 250$, $I(0) = 100$, $R(0) = 50 - \tau_1 \le \theta \le 0$ used for all numerical simulations. Using the parameter values, the disease-free equilibrium is given by $E_0 = (50, 0, 0, 0)$, if $R_0 < 1$ and the and the endemic equilibrium (282.9, 201.2, 201.8, 453.7) if $R_0 < 1$. For $\tau = 0$, $\det_1 = 1.01 > 0$, $\det_2 = 1.011 > 0$, $\det_3 = 9.3x10^{-5} > 0$, $\det_4 = 7.06x10^{-8} > 0$ are obtained and the conditions (7) holds which provides the result that the locally asymptotically stability of endemic equilibrium. From Proposition 1, the delay moderately increases from zero, at that point E_1 is locally asymptotically stable for $\tau \in [0, \tau_1)$ which is outlined in figure 2. In any case, the delay crosses the critical value $\tau_1 = 4.7698$, the endemic equilibrium losses stability and endure Hopf bifurcation which is illustrated in Figure 3.




Exposed Populations E(t)
time in days
0
100
200
300
400
500
600
700
0
50
100
150
200
250
300
350
400
450
500


Infected Populations I(t)
time in days
0
100
200
300
400
500
600
0
50
100
150
200
250
300
350
400
450
500

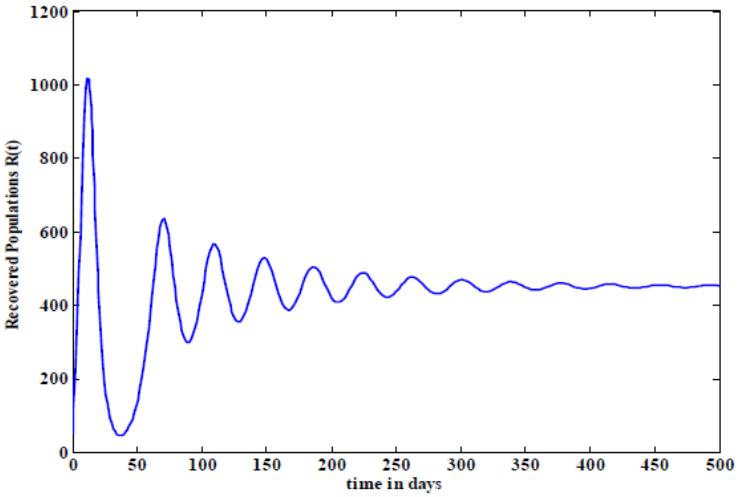
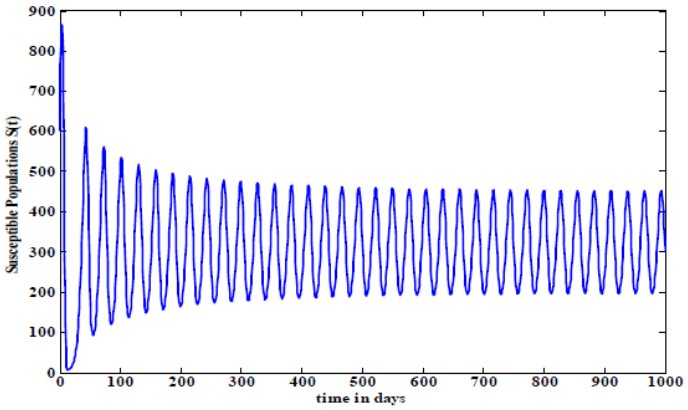


Figure 2. The simulation curves for susceptible, infected and recovered populations at $\tau < \tau_1$.

If the condition is satisfied $\tau < \tau_1 = 4.7698$ then the stability of susceptible, infected and recovered populations in the endemic equilibrium is ensured. It is found that, when $\tau < \tau_1$, the populations approach endemic equilibrium and arrives at a locally asymptotically stable. It reveals that the disease can be controlled effectively.



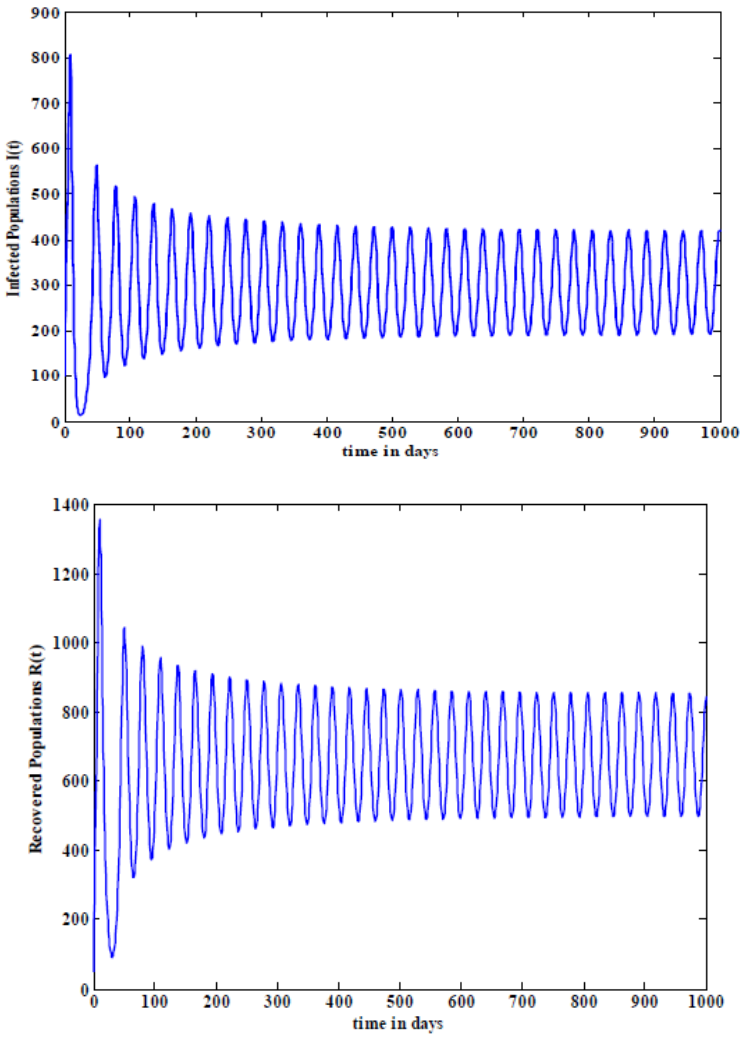
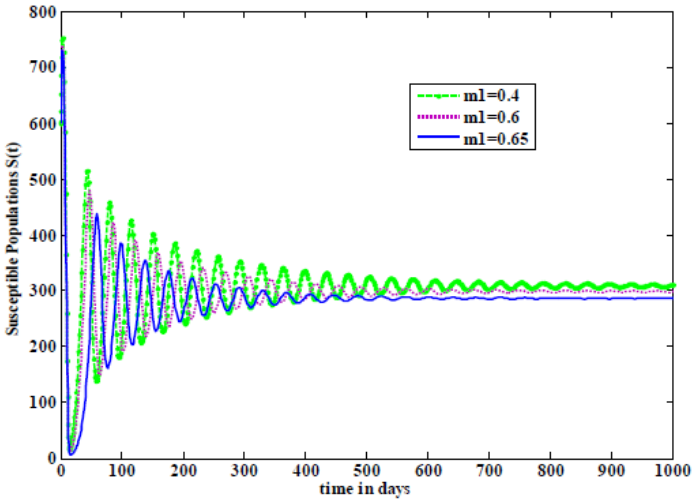
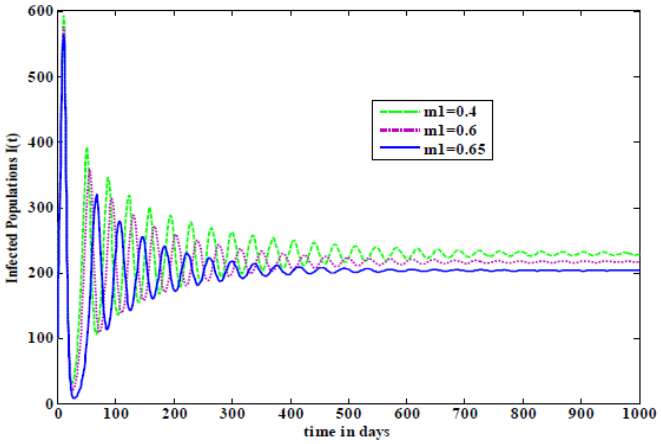


Figure 3. The simulation curves for susceptible, infected and recovered population at $\tau < \tau_1$.

It is observed that endemic equilibrium point attains unstable and periodic solution occurs at Hopf bifurcation. Moreover, our numerical simulations also specify that the value of inhibitory effect increases which states that endemic equilibrium remain stable even if $\tau < \tau_1$, observed in Figure 4.


m1=0.4
m1=0.6
m1=0.65
Susceptible Populations S(t)
time in days
0
100
200
300
400
500
600
700
800
900
1000


m1=0.4
m1=0.6
m1=0.65
Infected Populations I(t)
time in days
0
100
200
300
400
500
600
700
800
900
1000

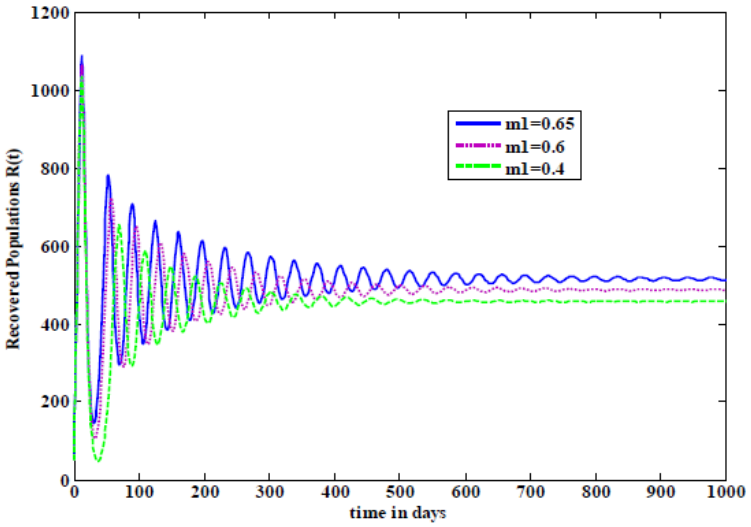


Figure 5. The simulation curves for susceptible, infected and recovered populations at different values inhibitory effect.

The figure 5 represents the different values of inhibitory effect rate $m_1 = 0.4, 0.6, 0.65,$ then the endemic equilibrium of susceptible populations preserves stability even $\tau < \tau_1$, if increases in the inhibitory effect rate can be shifted the system dynamics from limit cycle to stable focus, as seen in the diagrams above. Eventually, maximizing the inhibitory effect results in a rise in the value of the critical delay.

Results and Discussion

This study has the following outcomes are:

(i) Both the disease-free and endemic equilibrium of this model are ascertained. It is pointed out that, when R_0 it is lower than one, the delayed model becomes diseases free and locally asymptotically stable. The endemic equilibrium is locally asymptotically stable when R_0 values more than one, for all, $\tau \geq 0$ in the absence of delay.

(ii) This article analyzes the effect on the qualitative dynamics of delay and the inhibitory effect of the susceptible individuals. Numerical simulations prove the stability of equilibrium and show that the value of

delay unsustainable for the disease is 4.76. If the delay crosses the critical value τ_0 the endemic equilibrium is impacted by a Hopf bifurcation which means that the system is unstable and cannot control the disease. The time delay can affect the quantitative behavior of the model as a means of destabilizing the equilibrium and therefore occurrence of the Hopf bifurcation.

(iii) The sensitivity of the main parameters is calculated by numerical simulation and the analytical findings are validated. Depending on the sensitivity analysis value, lowering the recruitment rate can shift the system's dynamics from a limit cycle to a stable focus. The non-linear incidence rate preserves to turn out the complex dynamics models and transform the models more realistic and convenient.

Conclusion

In this study, the effect of the delay on the model dynamics system is explored in addition to a detailed study of the model, includes the positivity of solutions, local stability of equilibrium and the conditions associated with Hopf bifurcation". Further, the system can be destabilized by a delay in the transmission period and the periodic solution can be obtained through Hopf bifurcation, while the delay is chosen as the bifurcating parameter. Significant steps may be done to lower the number of persons susceptible to infection by boosting their immunity, quarantining infected persons, and limiting their contact with susceptible people. Hence, this study provides with greater transmission rates the infected population will increase, while the infected population will decrease with an increasing inhibitory value. Based on these impacts, societal inhibitions to control the disease must be implemented.

References

[1] L. J. S. Allen, M. A. Jones and C. F. Martin, A discrete-time model with vaccination for a measles epidemic, Mathematical Biosciences 105(1) (1991), 111-131.

[2] S. Edward, E. Kitengeso Raymond, T. Kiria Gabriel, F. Nestory, G. Mwema Godfrey and P. Mafarasa Arbogast, A mathematical model for control and elimination of the transmission dynamics of measles, Applied and Computational Mathematics 4 (2015), 396-408.

[3] W. M. Liu, H. W. Hethcote, S. Levin, A dynamical behavior of epidemiological models with nonlinear incidence rates, J. Math. Biol. 25 (1987), 359-380.

[4] J. Zhang, J. Jia and X. Song, Analysis of an SEIR epidemic model with saturated incidence and saturated treatment function, The Scientific World Journal (2014), 1-11.

[5] T. Zhang and Z. Teng, Pulse vaccination delayed SEIRS epidemic model with Saturation incidence, Applied Mathematical Modeling 32 (2008), 1403-1416.

[6] Z. Jiang and J. Wei, Stability and bifurcation analysis in a delayed SIR model, Chaos Soliton Fractional 35 (2008), 609-619.

[7] J. Zhang, Z. Jin and J. Yan and G. Sun, Stability and Hopf bifurcation in a delayed competition System, Nonlinear Analysis 70(2) (2009), 658-670.

[8] A. Momoh, M. Ibrahim, I. Uwanta and S. Manga, Mathematical model for control of measles epidemiology, International Journal of Pure and Applied Mathematics 87 (2013), 707-717.

[9] M. Farman, M. U. Saleem, A. L. Ahmad and M. O. Ahmad, Analysis and numerical solution of SEIR epidemic model of measles with non-integer time fractional derivatives by using Laplace Adomian decomposition method, Ain Shams Engineering Journal 9 (2018), 3391-3397.

[10] H. I. Freedman and Y. Kuang, Stability switches in linear scalar neutral delay equations, Funkcialaj Ekvacioj 34(1) (1991), 187-209.

[11] L. Q. Gao and H. W. Hethcote, Disease transmission models with density dependent demographics, J. Math. Biol. 30 (1992), 717.