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TWO MODULO THREE GRACEFUL LABELING OF SOME GRAPHS

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Abstract

A function f is called a two modulo three graceful labeling of a graph G if $f : V(G) \to \{2, 5, 8, 11, 14, \ldots, 3q+8\}$ is injective and the induced function $f^* : E(G) \to \{3, 6, 9, 12, \ldots, 3q\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is bijective. A graph which admits two modulo three graceful labeling is called two modulo three graceful graph. In this paper, we have proved comb graph $P_m \odot K_1$ the connected graph $P_m + mS_n$, for $n = 2, 3, 4,$ star graph S_n are two modulo three graceful graphs.

1. Introduction

The graph $G(p, q)$ considered here will be finite, undirected, and simple graph with p vertices and q edges. $V(G)$ and $E(G)$ will denote the vertex set and edge set of the graph. For all detailed survey of graph labeling we refer to J. A. Gallian [5]. For all other standard terminology and notations we follow F. Haray [2].

Sekar [2002] introduced one modulo three graceful labeling. Ramachandran and Sekar [2013] introduced the concept of one modulo N graceful where N is a positive integer and showed that the graphs like Paths,

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Caterpillars, Stars and S2, Lobsters, Banana trees and Rooted tree of height two, cycles and stars, every regular bamboo tree, coconut tree, the cycle related graphs, Ladder, Subdivision of ladder, the Super subdivision of ladder, Complete bipartite graph $K_{m,\, n}$, the crowns, armed crowns and chain of even cycles etc. are satisfied one modulo N graceful labeling. Velmurgan and Ramachandran [5] defined the M modulo N graceful labeling of path and star [4].

In this paper, we investigate the two modulo three graceful labeling of some graphs such as Comb graph $P_n \odot K_1$ the connected graph $P_m + mS_n$ for $n = 2, 3, 4$ and star S_n.

2. Basic Definitions

Definition 2.1. A function f is called a graceful labeling of a graph G if $f : V(G) \to \{0, 1, 2, 3, \ldots, q\}$ is injective and the induced function $f^* : E(G) \to \{1, 2, 3, \ldots, q\}$ defined as $f^*(e = uv) = | f(u) - f(v) |$ is bijective. This type of graph labeling first introduced by Rosa in 1967 as a β-valuation [1], later on Solomon W. Golomb called as graceful labeling [1].

Definition 2.2. A graph G is said to be M modulo N graceful labeling [where N is a positive integer and M is defined to be 1 to N] if there is a function f from the vertex set of G to $\{0, M, N, (N + M), 2N, (2N + M), \ldots, N(q-1), N(q-1) + M\}$ in such a way that

(i) f is $1-1$.

(ii) f induces a bijection f^* from the edge set of G to $\{M, N + M, 2N + M, \ldots, N(q-1) + M\}$ where $f^*(uv) = | f(u) - f(v) | G, u, v \in \forall$.

Definition 2.3. A function f is called a two modulo three graceful labeling of a graph G if $f : V(G) \to \{2, 5, 8, 11, 14, \ldots, 3q + 8\}$ is injective and the induced function $f^* : E(G) \to \{3, 6, 9, 12, 15, \ldots, 3q\}$ defined as $f^*(uv) = | f(u) - f(v) |$ is bijective. A graph which admits two modulo three graceful labeling is called two modulo three graceful graph.

Example 2.1.

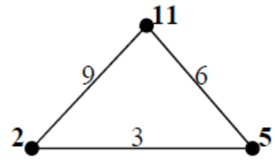

Figure 1. Two modulo three graceful labeling of C_3.

Definition 2.4. The graph obtained by joining a single pendant edge to each vertex of a path is called as comb graph.

Definition 2.5. A star S_n of order n, sometimes simply known as an "n-star" is a tree on n nodes with one vertex having vertex degree $n-1$ and the other $n-1$ vertices having vertex degree 1.

3. Main Results

Theorem 3.1. *The comb graph* $P_m \odot K_1$ *is two modulo three graceful graph for* $m \geq 3$.

Proof.

Case (i): When m is odd.

Then the graph is $P_{2r+3} \odot K_1$ when $r = 0, 1, 2, \ldots, 4n+5$.

Let $\{v_1, v_2, \ldots, v_{4n+6}\}$ be the vertices and $\{e_1, e_2, \ldots, e_{3n}\}$ be the edges.

We define the vertex labeling

$f : V(P_{2r+3} \odot K_1) \to \{2, 5, 8, \ldots, 3q+8\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \ldots, 4n+6.$

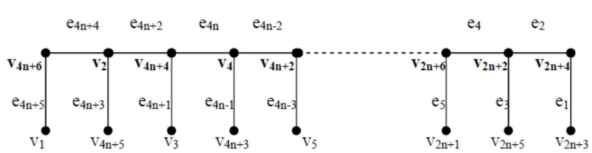

Figure 2. Two modulo three graceful labeling of $P_m \odot K_1$ when m is odd.

We define the edge labeling

$f^{*}: E(P_{2r+3} \odot K_1) \to \{3, 6, 9, 12, \ldots, 3q\}$ as

$f(e_i) = 3i; i = 1, 2, 3, \ldots, 4n+5.$

Case (ii): When m is even.

Then the graph is $P_{2r} \odot K_1$ when $r = 1, 2, 3, \ldots, 4n+4$.

Let $\{v_1, v_2, \ldots, v_{4n+4}\}$ be the vertices and $\{e_1, e_2, \ldots, e_{3n}\}$ be the edges.

We define the vertex labeling

$f: V(P_{2r} \odot K_1) \to \{2, 5, 8, \ldots, 3q+8\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \ldots, 4n+4.$

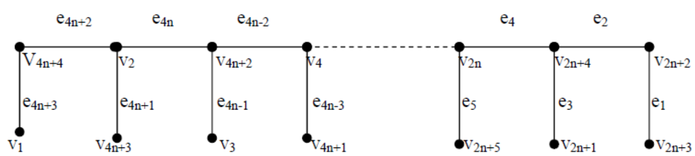

Figure 3. Two modulo three graceful labeling of $P_m \odot K_1$ when m is even.

We define the edge labeling

$f^{*}: E(P_{2r} \odot K_1) \to \{3, 6, 9, 12, \ldots, 3q\}$ as

$f(e_i) = 3i; i = 1, 2, 3, \ldots, 4n+3.$

Hence, the graph G if $f: V(G) \to \{2, 5, 8, \ldots, 3q+8\}$ is injective and the induced function $f^{*}: E(G) \to \{3, 6, 9, 12, \ldots, 3q\}$ defined as $f^{*}(u) = |f(u) - f(v)|$ is bijective.

Hence in both the cases $P_m \odot K_1$ admits two modulo three graceful labeling. Hence the graph $P_m \odot K_1$ is two modulo three graceful graph.

Example 3.1. The comb graph $P_9 \odot K_1$ and $P_{10} \odot K_1$ is two modulo three graceful graph.

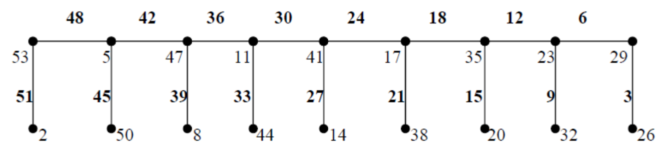

Figure 4. Two modulo three graceful labeling of $P_9 \odot K_1$.

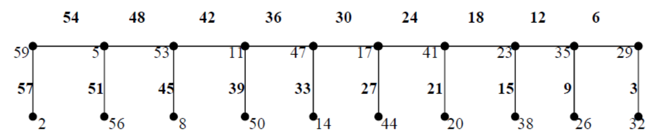

Figure 5. Two modulo three graceful labeling of $P_{10} \odot K_1$.

Theorem 3.2. *The connected graph* $P_m + mS_2$ *is two modulo three graceful graph.*

Proof.

Case (i): When m is odd.

Then the graph is $P_{2r+1} + (2r+1)S_2$ when $r = 0, 1, 2, \ldots, 4n+5$.

Let $\{v_1, v_2, \ldots, v_{6n+3}\}$ be the vertices and $\{e_1, e_2, \ldots, e_{3n}\}$ be the edges.

We define the vertex labeling

$$f : V(P_{2r+1} + (2r+1)S_2) \to \{2, 5, 8, \ldots, 3q+8\}$$

as

$$f(v_i) = 3i - 1;\ i = 1, 2, 3, \ldots, 6n+3.$$

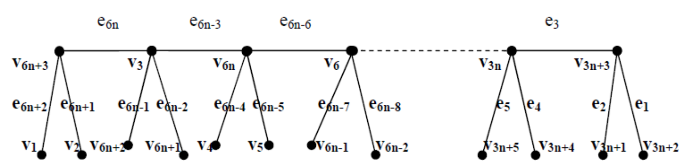

Figure 6. Two modulo three graceful labeling of $P_m + mS_2$ when m is odd.

Hence the induced edge labeling

$$f^* : E(P_{2r+1} + (2r+1)S_2) \to \{3, 6, 9, 12, \ldots, 3q\}$$

will be defined as

$f(e_i) = 3i; i = 1, 2, 3, \ldots, 6n+1.$

Case (ii): When m is even.

Then the graph is $P_{2r} + 2rS_2$ when $r = 1, 2, 3, 6n$.

Let $\{v_1, v_2, \ldots, v_{6n}\}$ be the vertices and $\{e_1, e_2, \ldots, e_{3n}\}$ be the edges.

We define the vertex labeling

$f : V(P_{2r} + 2rS_2) \rightarrow \{2, 5, 8, \ldots, 3q+8\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \ldots, 6n.$

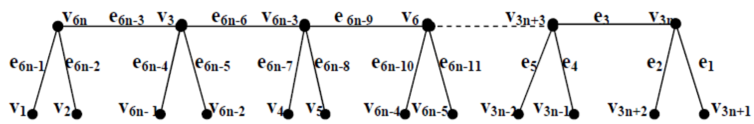

Figure 7. Two modulo three graceful labeling of $P_m + mS_2$ when m is even.

We define the edge labeling

$f^* : E(P_{2r} + 2rs_2) \rightarrow \{3, 6, 9, 12, \ldots, 3q\}$ as

$f(e_i) = 3i;\ i = 1, 2, 3, \ldots, 6n-1.$

Hence, the graph $f : V(G) \rightarrow \{2, 5, 8, \ldots, 3q+8\}$ is injective and the induced function $f^* : E(G) \rightarrow \{3, 6, 9, 12, \ldots, 3q\}$ defined as $f^*(u) = |f(u) - f(v)|$ is bijective.

Hence in both the cases $P_m + mS_2$ admits two modulo three graceful labeling. Hence the graph $P_m + mS_2$ is two modulo three graceful graph.

Example 3.2. The connected graph $P_7 + 7S_2$ and $P_8 + 8S_2$ is two modulo three graceful graphs.

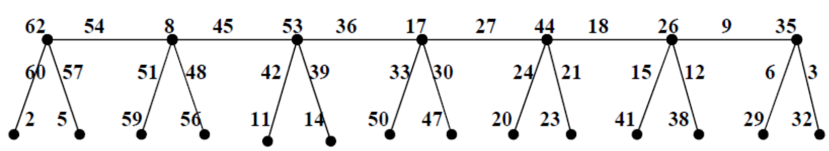

Figure 8. Two modulo three graceful labeling of $P_7 + 7S_2$.

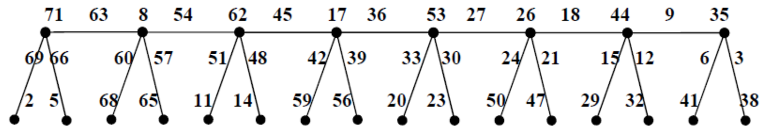

Figure 9. Two modulo three graceful labeling of $P_8 + 8S_2$.

Theorem 3.3. *The connected graph* $P_m + mS_3$ *is two modulo three graceful labeling.*

Proof.

Case (i): When m is odd.

Then the graph is $P_{2r+1} + (2r+1)S_3$ when $r = 0, 1, 2, \ldots, 8n+3$.

Let $\{v_1, v_2, \ldots, v_{8n+4}\}$ be the vertices and the edges are $\{e_1, e_2, \ldots, e_{3n}\}$.

We define the vertex labeling

$f : V(P_{2r+1} + (2r+1)S_3) \to \{2, 5, 8, \ldots, 3q+8\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \ldots, 8n+4.$

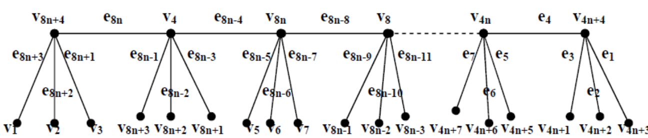

Figure 10. Two modulo three graceful labeling of $P_m + mS_3$ when m is odd.

We define the edge labeling

$f^* : E(P_{2r+1} + (2r+1)S_3) \to \{3, 6, 9, 12, \ldots, 3q\}$

as

$f(e_i) = 3i; i = 1, 2, 3, \ldots, 8n+3.$

Case (ii): When m is even.

Then the graph is $P_{2r} + 2rS_3$ when $r = 1, 2, 3, \ldots, 8n-1$.

Let $\{v_1, v_2, \ldots, v_{8n}\}$ be the vertices and $\{e_1, e_2, \ldots, e_{3n}\}$ be the edges.

We define the vertex labeling

$f : V(P_{2r} + 2rS_3) \to \{2, 5, 8, \ldots, 3q + 8\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \ldots, 8n - 1.$

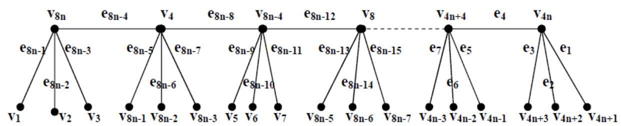

Figure 11. Two modulo three graceful labeling of $P_m + mS_3$ when m is even.

We define the edge labeling

$f^* : E(P_{2r} + 2rS_3) \to \{3, 6, 9, 12, \ldots, 3q\}$ as

$f(e_i) = 3i; i = 1, 2, 3, \ldots, 8n - 1.$

Hence, the graph G if $f : V(G) \to \{3, 6, 9, 12, \ldots, 3q + 8\}$ is injective and the induced function $f^* : E(G) \to \{3, 6, 9, 12, \ldots, 3q\}$ defined as $f^*(u) = |f(u) - f(v)|$ is bijective.

Hence both cases $P_m + mS_3$ admits two modulo three graceful labeling. Hence the graph $P_m + mS_3$ is two modulo three graceful graph.

Example 3.3. The connected graph $P_5 + 5S_3$ and $P_6 + 6S_3$ is two modulo three graceful labeling.

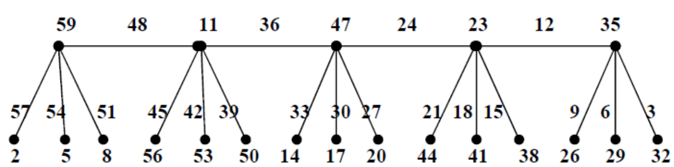

Figure 12. Two modulo three graceful labeling of $P_6 + 6S_3$.

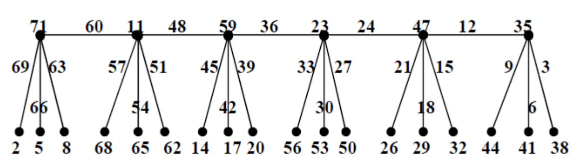

Figure 13. Two modulo three graceful labeling of $P_6 + 6S_3$.

Theorem 3.4. *The connected graph* $P_m + mS_4$ *is two modulo three graceful labeling.*

Proof.

Case (i): When m is odd.

Then the graph is $P_{2r+1} + (2r+1)S_4$ when $r = 0, 1, 2, \ldots, 10n + 4$.

Let $\{v_1, v_2, \ldots, v_{10n+5}\}$ be the vertices and $\{e_1, e_2, \ldots, e_{3n}\}$ be the edges.

We define the vertex labeling

$f : V(P_{2r+1} + (2r+1)S_4) \to \{2, 5, 8, \ldots, 3q + 8\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \ldots, 10n + 5.$

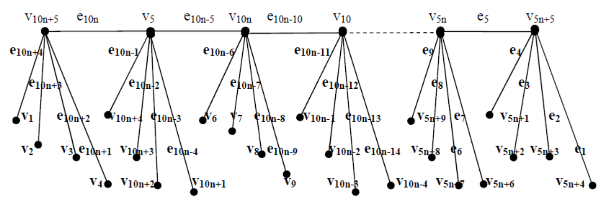

Figure 14. Two modulo three graceful labeling of $P_m + mS_4$ when m is odd.

Hence the induced edge labeling

$f^* : E(P_{2r+1} + (2r+1)S_4) \to \{3, 6, 9, \ldots, 3q\}$

will be defined as

$f(e_i) = 3i; i = 1, 2, 3, \ldots, 10n + 4.\}$

Case (ii): When m is even.

Then the graph is $P_{2r} + 2rS_4$ when $r = 1, 2, 3, \ldots, 10n - 1$.

Let $\{v_1, v_2, \ldots, v_{10n}\}$ be the vertices and $\{e_1, e_2, \ldots, e_{3n}\}$ be the edges.

We define the vertex labeling

$f : V(P_{2r}) \to \{2, 5, 8, \ldots, 3q + 8\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \ldots, 10n - 1.$

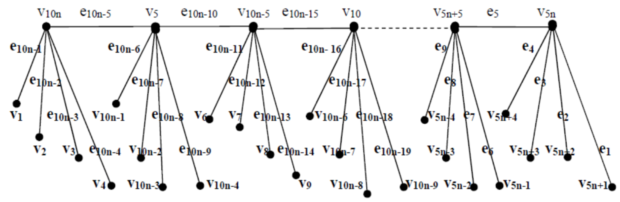

Figure 15. Two modulo three graceful labeling of $P_m + mS_4$ when m is even.

We define the edge labeling

$f^* : E(P_{2r} + 2rS_4) \to \{3, 6, 9, 12, \dots, 3q\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \dots, 10n - 1.$

Hence, the graph G if $f : V(G) \to \{2, 5, 8, \dots, 3q + 8\}$ is injective and the induced function $f^* : E(G) \to \{3, 6, 9, 12, \dots, 3q\}$ defined as $f^*(u) = | f(u) - f(v) |$ is bijective.

Hence both cases $P_m + mS_4$ admits two modulo three graceful labeling. Hence the graph $P_m + mS_4$ is two modulo three graceful graph.

Example 3.4. The connected graph $P_5 + 5S_4$ and $P_6 + 6S_4$ is two modulo three graceful graphs.

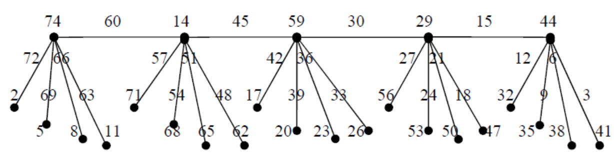

Figure 16. Two modulo three graceful labeling of $P_5 + 5S_4$.

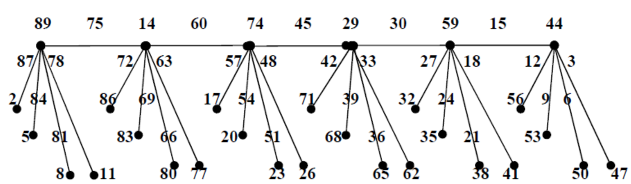

Figure 17. Two modulo three graceful labeling of $P_6 + 6S_4$.

Theorem 3.5. *The connected graph $P_m + mS_n$ is two modulo three graceful labeling.*

Proof.

Case (i): When m is odd.

Then the graph is $P_{2r+1} + (2r+1)S_n$ when $r = 0, 1, 2, \ldots, m(2n+2)+n$.

Let $\{v_1, v_2, \ldots, v_{m(2n+2)+n+1}\}$ be the vertices and $\{e_1, e_2, \ldots, e_{m(2n+2)+n+1}\}$ be the edges.

We define the vertex labeling

$f : V(P_{2r+1} + (2r+1)S_n) \to \{2, 5, 8, \ldots, 3q+8\}$ as

$f(vi) = 3i - 1; i = 1, 2, 3, \ldots, m(2n+2)+n+1.$

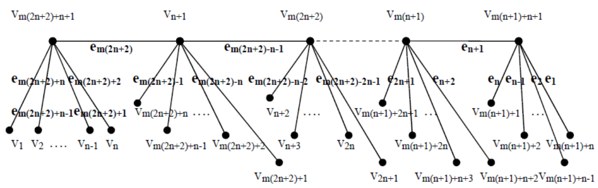

Figure 18. Two modulo three graceful labeling of $P_m + mS_n$ when m is odd.

Hence the induced edge labeling

$f^* : E(P_{2r+1} + (2r+1)S_n) \to \{3, 6, 9, \ldots, 3q\}$

will be defined as

$f(vi) = 3i; i = 1, 2, 3, \ldots, m(2n+2)+n.$

Case (ii): When m is even.

Then the graph is $P_{2r} + 2rS_n$ when $r = 1, 2, 3, \ldots, m(2n+2)$.

Let $\{v_1, v_2, \ldots, v_{m(2n+2)}\}$ be the vertices and $\{e_1 e_2, \ldots, e_{m(2n+2)-1}\}$ be the edges.

We define the vertex labeling

$f : V(P_{2r} + 2rS_n) \to \{2, 5, 8, \ldots, 3q+8\}$ as

$f(v_i) = 3i - 1; i = 1, 2, 3, \ldots, m(2n+2).$

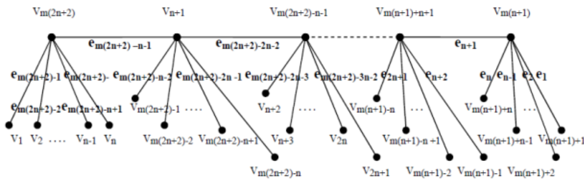

Figure 19. Two modulo three graceful labeling of $P_m + mS_n$ when m is even.

We define the edge labeling

$f^* : E(P_{2r} + 2rS_n) \to \{3, 6, 9, 12, \ldots, 3q\}$ as

$f(e_i) = 3i; i = 1, 2, 3, \ldots, m(2n+2)-1.$

Hence, the graph G if $V(G) \to \{2, 5, 8, \ldots, 3q+8\}$ is injective and the induced function $f^* : E(G) \to \{3, 6, 9, 12, \ldots, 3q\}$ defined as $f^*(u) = |f(u) - f(v)|$ is bijective.

Hence both cases $P_m + mS_n$ admits two modulo three graceful labeling. Hence the graph $P_m + mS_n$ is two modulo three graceful graph.

Theorem 3.6. *The star* S_n *is two modulo three graceful labeling.*

Proof. The star has n vertices denoted by $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ be and $n-1$ edges denoted by $\{e_1, e_2, e_{n-1}\}$.

We define the vertex labeling

$f : V(S_n) \to \{2, 5, 8, \ldots, 3q+8\}$ as

$f(v_i) = 3i - 1; = 1, 2, 3, \ldots, n.$

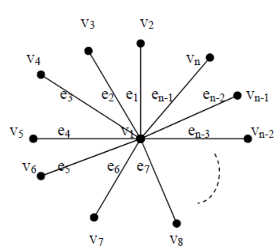

Figure 20. Two modulo three graceful labeling of S_n

Hence the induced edge labeling

$f^* : E(S_n) \to \{3, 6, 9, \ldots, 3q\}$

will be defined as

$f(e_i) = 3i; i = 1, 2, 3, \ldots, n-1.$

Hence, the graph G if $V(G) \to \{2, 5, 8, \ldots, 3q+8\}$ is injective and the induced function $f^* : E(G) \to \{3, 6, 9, 12, \ldots, 3q\}$ defined as $f^*(u) = |f(u) - f(v)|$ is bijective.

Hence the graph S_n admits two modulo three graceful labeling. Hence the graph S_n is two modulo three graceful graph.

Example 3.5. The star graph S_9 is two modulo three graceful graph.

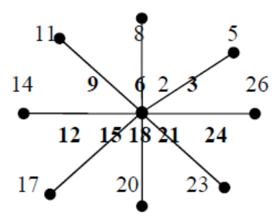

Figure 21. Two modulo three graceful labeling of S_9.

4. Conclusion

In this paper we have proved that comb graph $P_m \odot K_1$, the connected graph $P_m + mS_n$ for $n = 2, 3, 4,$ star S_n two modulo three graceful graphs.

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