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AN $M/G/1$ RETRIAL QUEUE WITH SECOND OPTIONAL SERVICE AND MULTIPLE WORKING VACATION

K. SANTHI

Department of Mathematics
PS PT MGR Govt. Arts and Science College
Sirkali-609108, India
E-mail:santhimano3169@gmail.com

Abstract

In this paper, we study a single arrival retrial queue with second optional service and exponentially distributed multiple working vacation. The server gives service to clients, one by one, on a FCFS basis. Soon after fulfillment of his administration the client may select the second optional service with probability or the client may leave the framework without taking the second optional service with probability $q(p+q=1)$. After consummation of client's service if there is no client in the circle the server may take a numerous working vacation. Using supplementary variable method, we obtain the probability generating function for the number of customers in the orbit. Some particular cases are discussed.

1. Introduction

Retrial queueing systems are described by the feature that the arriving customers who find the server busy join the retrial orbit to try their requests again. Retrial queues are widely and successfully used as Mathematical models of several Computer systems and telecommunication networks. Choi et al. [4] analysed an $M/M/1$ retrial queue with general retrial times. Martin and Gomez-Corral [9] considered an $M/M/1$ retrial queue with linear control policy. Sherman and Kharoufeh [18] studied an $M/M/1$ retrial queue with unreliable server. In the queueing theory, vacation queues and retrial queues have been intensive research topics; we can find general models in Artalejo and Gomez-Corral [2]. In 2002, Servi and Finn [17] first

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introduced working vacation policy and studied an $M/M/1$ working Vacation queues. The study of queueing system with working vacations can also provide the theory and analysis method to design the optimal lower speed period. Wu and Takagi [20] extended the $M/M/1/WV$ queue to an $M/M/1/WV$ queue using the matrix-analytic method, Baba [3] considered a $GI/M/1$ queue with working vacations. Krishnamoorthy and Sreenivasan [8] analysed an $M/M/2$ queue with working vacations. Do [6] studied an $M/M/1$ retrial queue with working vacation. Zhang and Xu [21] considered an $M/M/1/WV$ queue with N-policy. Pazhani Bala Murugan and Vijaykrishnaraj [15, 16] studied A bulk arrival retrial queueing models with exponentially distributed multiple working vacation.

2. Model Description

We consider a single arrival queueing system where the primary customers arrive according to a Poisson process with arrival rate $\lambda(\geq 0)$. We assume that there is no waiting space and therefore if an arriving customer (external or repeated) finds the server occupied, he leaves the service area and joins a pool of blocked customers called orbit. We will assume that only the customer at the head of the orbit is allowed to reach the server at a service completion instant. The retrial time follows a general distribution, with distribution function $B(x)$. Let $b(x)$ and $B^{*}(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $B(x)$ respectively for regular service period and let $a(x)$, $A^{*}(\theta)$ denote the probability density function and Laplace Stieltjes Transform of $A(x)$ respectively for working vacation period. Just after the completion of a service, if any customer is in orbit the next customer to gain service is determined by a competition between the primary customer and the orbit customer.

The service discipline is FCFS. Each arriving customer undergoes the first essential service which has general distribution with distribution function $S_{b_1}(x)$, the probability density function $S_{b_1}(x)$ and the Laplace-Stieltjes transform (LST) $S_{b_1}^{*}(\theta)$ where S_{b_1} is the service time of the first essential service.

After completion of first essential service the customer may opt for the second optional service with probability p or the customer may leave the system without taking the second optional service with probability $q(p + q = 1)$. The second optional service follows the general distribution with the distribution function $S_{b_2}(x)$, the probability density function $S_{b_2}(x)$ and the LST $S^*_{b_2(\theta)}$ where S_{b_2} is the service time of the second optional service.

During the working vacation period the server also provides two types of services. The first essential service time S_{v_1} of a typical customer follows a general distribution with the distribution function $S_{v_1}(x)[S_{v_1}(x)$, the probability density function and $S^*_{v_1}(\theta)$, the LST] and the second optional service time S_{v_2} also follows a general distribution with the distribution function $S_{v_2}(x)[s_{v_2}(x)$, the probability density An $M/G/1$ Retrial Queue with Second Optional Service and Multiple Working Vacation function and $S^*_{v_2}(\theta)$, the LST].

Further, it is noted that the service interrupted at the end of a vacation is lost and it is restarted with different distribution at the beginning of the following service period. Inter arrival times, service times and working vacation times are mutually independent of each other.

Let us define the following random variables.

$N(t)$- the orbit size at time t

$A^0(t)$- the remaining retrial time in working vacation period

$B^0(t)$- the remaining retrial time in regular service period

$S^0_{v_i}(t)$- the remaining service time in working vacation period, for $i = 1, 2$

$S^0_{b_i}(t)$- the remaining service time in regular service period, for $i = 1, 2$

$Y(t) = 0$ if the server is on working vacation period at time t but not occupied, 1 if the server is in regular service period at time t but not occupied, 2 if the server is busy for giving first essential service in working vacation period at time t, 3 if the server is busy for giving second optional service in working vacation period at time t, 4 if the server is busy for giving first essential service in regular service period at time t, 5 if the server is busy for giving second optional service in regular service period at time t.

so that the supplementary variables $A^0(t), B^0(t), S_{v_1}^0(t), S_{v_2}^0(t), S_{b_1}^0(t)$ and $S_{b_2}^0(t)$ are introduced in order to obtain the bivariate Markov Process $\{N(t), \partial(t);\ t \geq 0\}$, where $\partial(t) = \{A^0(t)\ if\ Y(t) = 0,\ B^0(t)$ if $Y(t) = 1;\ S_{v_1}^0(t)$ if $Y(t) = 2,\ S_{v_2}^0(t)\ if\ Y(t) = 3;\ S_{b_2}^0(t)\ if\ Y(t) 4;\ S_{b_2}^0(t)\ if\ Y(t) = 5\}$. We define the following limiting probabilities:

$$Q_{0,0} = \lim_{t\to\infty} \Pr\{N(t) = 0,\ Y(t) = 0\}$$

$$Q_{0,n} = \lim_{t\to\infty} \Pr\{N(t) = n,\ Y(t) = 0,\ x < A^0(t) \leq x + dx\};\ n \geq 1$$

$$P_{0,n} = \lim_{t\to\infty} \Pr\{N(t) = n,\ Y(t) = 1,\ x < B^0(t) \leq x + dx\};\ n \geq 1$$

$$Q_{1,n} = \lim_{t\to\infty} \Pr\{N(t) = n,\ Y(t) = 2,\ x < S_{v_1}^0(t) \leq x + dx\};\ n \geq 0$$

$$Q_{2,n} = \lim_{t\to\infty} \Pr\{N(t) = n,\ Y(t) = 3,\ x < S_{v_2}^0(t) \leq x + dx\};\ n \geq 0$$

$$P_{1,n} = \lim_{t\to\infty} \Pr\{N(t) = n,\ Y(t) = 4,\ x < S_{b_1}^0(t) \leq x + dx\};\ n \geq 0$$

$$P_{2,n} = \lim_{t\to\infty} \Pr\{N(t) = n,\ Y(t) = 5,\ x < S_{b_2}^0(t) \leq x + dx\};\ n \geq 0.$$

We define the Laplace Stieltjes Transform and the probability generating functions as follows, for $i = 1, 2,$ $S_{b_i}^*(\theta) = \int_0^\infty e^{-\theta x} S_{b_i}(x)dx;$ $S_{v_i}^*(\theta) = \int_0^\infty e^{-\theta x} S_{v_i}(x)dx;\ A^*(\theta) = \int_0^\infty e^{-\theta x} a(x)dx;$

$$B^*(\theta) = \int_0^\infty e^{-\theta x} b(x) dx; Q_{0,n}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{0,n}(x) dx; Q_{0,n}^*(0) = \int_0^\infty Q_{0,n}(x) dx;$$

$$Q_{i,n}^*(\theta) = \int_0^\infty e^{-\theta x} Q_{i,n}(x) dx; Q_{i,n}^*(0) = \int_0^\infty Q_{i,n}(x) dx; P_{0,n}^*(\theta) = \int_0^\infty e^{-\theta x} P_{0,n}(x) dx;$$

$$P_{0,n}^*(0) = \int_0^\infty P_{0,n}(x) dx; Q_0^*(z, \theta) = \sum_{n=1}^\infty Q_{0,n}^*(\theta) z^n; Q_{0,n}^*(z, 0) = \sum_{n=1}^\infty Q_{0,n}^*(0) z^n;$$

$$Q_0(z, 0) = \sum_{n=1}^\infty Q_{0,n}(0) z^n; \; Q_i^*(z, \theta) = \sum_{n=1}^\infty Q_{i,n}^*(\theta) z^n; \; Q_i^*(z, 0)$$

$$= \sum_{n=1}^\infty Q_{i,n}^*(0) z^n; \; Q_1(z, 0) = \sum_{n=1}^\infty Q_{1,n}(0) z^n; \; P_0^*(z, \theta)$$

$$= \sum_{n=1}^\infty P_{0,n}^*(\theta) z^n, \; P_0^*(z, 0) = \sum_{n=1}^\infty P_{0,n}^*(0) z^n; \; P_0(z, 0)$$

$$= \sum_{n=1}^\infty P_{0,n}(0) z^n; \; P_i^*(z, \theta) = \sum_{n=1}^\infty P_{i,n}^*(\theta) z^n; \; P_i^*(z, 0)$$

$$= \sum_{n=1}^\infty P_{i,n}^*(0) z^n; \; P_i(z, 0) = \sum_{n=1}^\infty P_{i,n}^*(0) z^n.$$

3. The Orbit Size Distribution

By assuming that the system is in steady state condition, the differential difference equations governing the systems are as follows:

$$\lambda Q_{0,0} = q P_{1,0}(0) + P_{2,0}(0) + q Q_{1,0}(0) + Q_{2,0}(0) \tag{1}$$

$$-\frac{d}{dx} Q_{0,n}(x) = -(\lambda + \eta) Q_{0,n}(x) + q Q_{1,n}(0) a(x)$$

$$+ Q_{2,n}(0) a(x); \; n \geq 1 \tag{2}$$

$$-\frac{d}{dx} Q_{1,0}(x) = -(\lambda + \eta) Q_{1,0}(x) + Q_{0,1}(0) s_{v_1}(x) + \lambda Q_{0,0} s_{v_1}(x) \tag{3}$$

$$-\frac{d}{dx}Q_{1,n}(x) = -(\lambda+\eta)Q_{1,n}(x) + \lambda Q_{1,n-1}(x) + Q_{0,n+1}(0)s_{v_1}(x)$$

$$+\lambda s_{v_1}(x)\int_0^\infty Q_{0,n}(x)dx;\ n \geq 1 \tag{4}$$

$$-\frac{d}{dx}Q_{2,0}(x) = -(\lambda+\eta)Q_{2,0}(0) + pQ_{1,0}(0)s_{v_2}(x) \tag{5}$$

$$-\frac{d}{dx}Q_{2,n}(x) = -(\lambda+\eta)Q_{2,n}(x) + pQ_{1,n}(0)s_{v_2}(x) + \lambda Q_{2,n-1}(x);\ n \geq 1 \tag{6}$$

$$-\frac{d}{dx}P_{0,n}(x) = -\lambda P_{0,n}(x) + qP_{1,n}(0)b(x) + P_{2,n}(0)b(x) + \eta b(x)$$

$$\int_0^\infty Q_{0,n}(x)dx;\ n \geq 1 \tag{7}$$

$$-\frac{d}{dx}P_{1,0}(x) = -\lambda P_{1,0}(x) + P_{0,1}(0)s_{b_1}(x) + \eta s_{b_1}(x)\int_0^\infty Q_{1,0}(x)dx \tag{8}$$

$$-\frac{d}{dx}P_{1,n}(x) = -\lambda P_{1,n}(x) + P_{0,n+1}(0)s_{b_1}(x) + \lambda P_{1,n-1}(x) + \eta s_{b_1}(x)\int_0^\infty Q_{1,n}(x)dx$$

$$+\lambda s_{b_1}(x)\int_0^\infty P_{0,n}(x)dx;\ n \geq 1 \tag{9}$$

$$-\frac{d}{dx}P_{2,0}(x) = -\lambda P_{2,0}(x) + pP_{1,0}(0)s_{b_2}(x) + \eta s_{b_2}(x)\int_0^\infty Q_{2,0}(x)dx \tag{10}$$

$$-\frac{d}{dx}P_{2,n}(x) = -\lambda P_{2,n}(x) + pP_{1,n}(0)s_{b_2}(x) + \lambda P_{2,n-1}(x) + \eta s_{b_2}(x)$$

$$\int_0^\infty Q_{2,n}(x)dx. \tag{11}$$

Taking LST on both sides of the equation from (2) to (11) we get,

$$\theta Q_{0,n}^*(\theta) - Q_{0,n}(0) = (\lambda+\eta)Q_{0,n}^*(\theta) - qQ_{1,n}(0)A^*(\theta) - Q_{2,n}(0)A^*(\theta) \tag{12}$$

$$\theta Q_{1,0}^*(\theta) - Q_{1,0}(0) = (\lambda+\eta)Q_{1,0}^*(\theta) - Q_{0,1}(0)S_{v_1}^* - \lambda Q_{0,0}S_{v_1}^*(\theta) \tag{13}$$

$$\theta Q_{1,n}^{*}(\theta) - Q_{1,n}(0) = (\lambda + \eta) Q_{1,n}^{*}(\theta) - \lambda Q_{1,n-1}^{*}(\theta) - Q_{0,n+1}(0) S_{v_1}^{*}(\theta)$$

$$- \lambda S_{v_1}^{*}(\theta) - Q_{0,n}^{*}(0) \quad (14)$$

$$\theta Q_{2,0}^{*}(\theta) - Q_{2,0}(0) = (\lambda + \eta) Q_{2,0}^{*}(\theta) - p Q_{1,0}(0) S_{v_2}^{*}(\theta) \quad (15)$$

$$\theta Q_{2,n}^{*}(\theta) - Q_{2,n}(0) = (\lambda + \eta) Q_{2,n0}^{*}(\theta) - p Q_{1,n}(0) S_{v_2}^{*}(\theta) - \lambda Q_{2,n-1}^{*}(\theta) \quad (16)$$

$$\theta P_{0,n}^{*}(\theta) - P_{0,n}(0) = \lambda P_{0,n}^{*}(\theta) - q P_{1,n}(0) B^{*}(\theta) - P_{2,n}(0)$$

$$B^{*}(\theta) - \eta B^{*}(\theta) Q_{0,n}^{*}(0) \quad (17)$$

$$\theta P_{1,0}^{*}(\theta) - P_{1,0}(0) = \lambda P_{1,0}^{*}(\theta) - P_{0,1}(0) s_{b_1}^{*}(\theta) - \eta s_{b_1}^{*}(\theta) Q_{1,0}^{*}(0) \quad (18)$$

$$\theta P_{1,n}^{*}(\theta) - P_{1,n}(0) = \lambda P_{1,n}^{*}(\theta) - P_{0,n+1}(0) s_{b_1}^{*}(\theta) - \lambda P_{1,n-1}^{*}(\theta)$$

$$- \eta S_{b_1}^{*}(\theta) Q_{1,n}^{*}(0) - \lambda S_{b_1}^{*}(\theta) P_{0,n}^{*}(0) \quad (19)$$

$$\theta P_{2,n}^{*}(\theta) - P_{2,0}(0) = \lambda P_{2,0}^{*}(\theta) - p P_{1,0}(0) s_{b_2}^{*}(\theta) - \eta s_{b_2}^{*}(\theta) Q_{2,0}^{*}(0) \quad (20)$$

$$\theta P_{2,n}^{*}(\theta) - P_{2,n}(0) = \lambda P_{2,n}^{*}(\theta) - p P_{1,n}(0) s_{b_2}^{*}(\theta) - \lambda P_{2,n-1}^{*}(\theta) - \eta s_{b_2}^{*}(\theta) Q_{2,n}^{*}(0). \quad (21)$$

Multiplying (12) with z^n and summed over n from 1 to ∞, we get

$$[\theta - (\lambda + \eta)] Q_0^{*}(z, \theta) = Q_0(z, 0) - A^{*}(\theta) q Q_1(z, 0)$$

$$+ A^{*}(\theta) q Q_{1,0}(0) - A^{*}(\theta) Q_2(z, 0) + A^{*}(\theta) Q_{2,0}(0). \quad (22)$$

Multiplying (14) with z^n and summed over n from 1 to ∞ and added up with (13) gives

$$[\theta - (\lambda - \lambda z + \eta)] Q_1^{*}(z, \theta) = Q_1(z, 0) - \frac{S_{v_1}^{*}(\theta)}{z} Q_0(z, 0)$$

$$- \frac{\lambda z}{z} S_{v_1}^{*}(\theta) - \frac{\lambda z}{z} S_{v_1}^{*}(\theta) Q_{0,0}. \quad (23)$$

Inserting $\theta = \lambda + \eta$ in (22), we get

$$Q_0(z,\,0) = A^*(\lambda+\eta)[qQ_1(z,\,0) - qQ_{1,0}(0) + Q_2(z,\,0) - Q_{2,0}(0)]. \tag{24}$$

Substituting $\theta = 0$ in (22) and using (24), we get

$$Q_0^*(z,\,0) = \frac{1 - A^*(\lambda+\eta)}{\lambda+\eta}[qQ_1(z,\,0) - qQ_{1,0}(0) + Q_2(z,\,0) - Q_{2,0}(0)]. \tag{25}$$

Inserting $\theta = \lambda - \lambda z + \eta$ in (23), we get

$$Q_1(z,\,0) = S_{v_1}^*(\lambda - \lambda z + \eta)[\frac{Q_0(z,0)}{z} + \frac{\lambda z}{x}Q_0^*(z,\,0) + \frac{\lambda z}{z}Q_{0,0}]. \tag{26}$$

Multiplying (16) with z^n and summed over n from 1 to ∞ and added up with (15) gives

$$[\theta - (\lambda - \lambda z + \eta)]Q_2^*(z,\,\theta) = Q_2(z,\,0) - pQ_1(z,\,0)S_{v_2}^*(\theta). \tag{27}$$

Inserting $\theta = \lambda - \lambda z + \eta$ in (27), we get

$$Q_2(z,\,0) = S_{v_2}^*(\lambda - \lambda z + \eta)pQ_1(z,\,0). \tag{28}$$

Substituting (24), (25) and (28) in (26), we get

$$Q_0(z,\,0) =$$

$$\frac{S_{v_1}^*(\lambda-\lambda z+\eta)[\lambda(\lambda+\eta)zQ_{0,0} - [qQ_{1,0}(0)+Q_{2,0}(0)][\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]]}{(\lambda+\eta)z - S_{v_1}^*(\lambda-\lambda z+\eta)[q+ps_{v_2}^*(\lambda-\lambda z+\eta)][\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]}. \tag{29}$$

Substituting (28) and (29) in (24), we get

$$Q_0(z,0) =$$

$$\frac{\begin{matrix}(\lambda+\eta)A^*(\lambda+\eta)[S_{v_1}^*(\lambda-\lambda z+\eta)][q+pS_{v_2}^*(\lambda-\lambda z+\eta)]\lambda zQ_{0,0} \\ -z[qQ_{1,0}(0)+Q_{2,0}(0)]]\end{matrix}}{(\lambda+\eta)z - S_{v_1}^*(\lambda-\lambda z+\eta)[q+ps_{v_2}^*(\lambda-\lambda z+\eta)][\lambda z+(\lambda-\lambda z+\eta)A^*(\lambda+\eta)]}. \tag{30}$$

Let $f(z) = (\lambda+\eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + pS_{v_1}^*(\lambda - \lambda z + \eta)][\lambda z + (\lambda - \lambda z + \eta)$ $A^*(\lambda+\eta)]$ at $z = 0$, $f(0) = -S_{v_1}^*(\lambda+\eta)[q + pS_{v_2}^*(\lambda+\eta)]$ $[\lambda + \eta A^*(\lambda+\eta)] < 0$ and at $z = 1$, $f(1) = (\lambda+\eta) - S_{v_1}^*(\eta)[q + pS_{v_2}^*(\eta)]$ $[\lambda + \eta A^*(\lambda+\eta)] > 0$.

This implies that there exists a real root $z_1 \in (0, 1)$ for the equation $f(z) = 0$. Hence at $z = z_1$ the equation (30) becomes

$$qQ_{1,0}(0) + Q_{2,0}(0) = \lambda U(z_1) Q_{0,0} \tag{31}$$

where

$$U(z_1) = S_{v_1}^*(\lambda - \lambda z_1 + \eta)[q + pS_{v_2}^*(\lambda - \lambda z_1 + \eta)]. \tag{32}$$

Using (31) in (30), we get

$$q_0(z, 0) = \frac{z\lambda(\lambda + \eta)A^*(\lambda + \eta)[zS_{v_1}^*(\lambda - \lambda z + \eta)[q + pS_{v_2}^*(\lambda - \lambda z + \eta)] - U(z_1)]Q_{0,0}}{(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + ps_{v_2}^*(\lambda - \lambda z + \eta)][\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]}. \tag{33}$$

Using (31) in (29), we get

$$Q_1(z, 0) = \frac{S_{v_1}^*(\lambda - \lambda z + \eta)[\lambda(\lambda + \eta)z - \lambda U(z_1)[\lambda z + (\lambda - \lambda z + \eta) A^*(\lambda + \eta)]]Q_{0,0}}{(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + ps_{v_2}^*(\lambda - \lambda z + \eta)][\lambda z + (\lambda - \lambda z + \eta) A^*(\lambda + \eta)]}. \tag{34}$$

Using (33) in (28), we get

$$Q_2(z, 0) = \frac{pS_{v_1}^*(\lambda - \lambda z + \eta)S_{v_2}^*(\lambda - \lambda z + \eta)[\lambda(\lambda + \eta)z - \lambda U(z_1)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]Q_{0,0}}{(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + ps_{v_2}^*(\lambda - \lambda z + \eta)][\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]}. \tag{35}$$

Substituting (31), (33) and (34) in (25), we get

$$Q_0^*(z, 0) = \frac{z\lambda[1 - A^*(\lambda + \eta)][S_{v_1}^*(\lambda - \lambda z + \eta)[q + pS_{v_2}^*(\lambda - \lambda z + \eta)] - U(z_1)]Q_{0,0}}{(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + ps_{v_2}^*(\lambda - \lambda z + \eta)][\lambda z + (\lambda - \lambda z + \eta) A^*(\lambda + \eta)]}. \tag{36}$$

Inserting $\theta = 0$ in (23) and Substituting (32), (33) and (35), we get

$$Q_1^*(z,\ 0) = \frac{\begin{array}{l}\lambda[1 - S_{v_1}^*(\lambda - \lambda z + \eta)][(\lambda + \eta)z - U(z_1)\\ [\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]Q_{0,0}\end{array}}{\begin{array}{l}[\lambda - \lambda z + \eta][(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + ps_{v_2}^*(\lambda - \lambda z + \eta)]\\ [\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]\end{array}}. \quad (37)$$

Inserting $\theta = 0$ in (27) and using (33), we get

$$Q_2^*(z,\ 0) = \frac{\begin{array}{l}\lambda p[S_{v_1}^*(\lambda - \lambda z + \eta)][1 - S_{v_2}^*(\lambda - \lambda z + \eta)][(\lambda + \eta)z - U(z_1)\\ [\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]Q_{0,0}\end{array}}{\begin{array}{l}[\lambda - \lambda z + \eta][(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + ps_{v_2}^*(\lambda - \lambda z + \eta)]\\ [\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]\end{array}}. \quad (38)$$

Multiplying (17) with z^n and summed over n from 1 to ∞, we get

$$(\theta - \lambda)P_0^*(z,\ \theta) = P_0(z,\ 0) - qB^*(\theta)P_1(z,\ 0)$$

$$+ qB^*(\theta)P_{1,0}(0) - B^*(\theta)P_2(z,\ 0) + B^*(\theta)P_{2,0}(0) - \eta B^*(\theta)Q_0^*(z,\ 0). \quad (39)$$

Inserting $\theta = \lambda$ in (38), we get

$$P_0(z, 0) = B^*(\lambda)[qP_1(z, 0) + P_2(z, 0)] - B^*(\lambda)[\lambda Q_{0,0} - \lambda U(z_1)Q_{0,0}]$$

$$+ \eta B^*(\lambda)Q_0^*(z, 0). \quad (40)$$

Inserting $\theta = 0$ in (38) and using (39), we get

$$P_0^*(z,\ 0) = \left[\frac{1 - B^*(\lambda)}{\lambda}\right][qP_1(z,\ 0) + P_2(z,\ 0) - \lambda Q_{0,0} + \lambda U(z_1)$$

$$Q_{0,0} + \eta Q_0^*(z,\ 0)]. \quad (41)$$

Multiplying (19) with z^n and summed over n from 1 to ∞ and added with (18), gives

$$[\theta - (\lambda - \lambda z)]P_1^*(z,\ 0) = P_1(z,\ 0) - \frac{s_{b_1}^*}{z}P_0(z,\ 0) - \eta S_{b_1}^*(\theta)Q_1^*(z,\ 0)$$

$$- \lambda S_{b_1}^*(\theta)P_0^*(z,\ 0). \quad (42)$$

Inserting $\theta = \lambda - \lambda z$ in (41), we get

$$P_1(z, 0) = S_{b_1}^*(\lambda - \lambda z)\left[\frac{P_0(z, 0)}{z} + \eta Q_1^*(z, 0) + \lambda P_0^*(z, 0)\right]. \tag{43}$$

Multiplying (21) with z^n and summed over n from 1 to ∞ and added with (20) gives

$$[\theta - (\lambda - \lambda z)]P_2^*(z, \theta) = P_2(z, 0) - pS_{b_2}^*(\theta)P_1(z, 0) - \eta S_{b_2}^*(\theta)Q_2^*(z, 0). \tag{44}$$

Inserting $\theta = (\lambda - \lambda z)$ in (44), we get

$$P_2(z, 0) = S_{b_2}^*(\lambda - \lambda z)[pP_1(z, 0) + \eta Q_2^*(z, 0)]. \tag{45}$$

Using (39), (40) and (36) in (42), we get

$$P_1(z, 0) = \frac{S_{b_1}^*(\lambda - \lambda z)[B^*(\lambda) + (1 - B^*(\lambda)z][\eta S_{b_2}^*(\lambda - \lambda z) Q_2^*(z, 0) + \eta Q_0^*(z, 0) - \lambda[1 - U(z_1)]Q_{0,0}]}{z - S_{b_1}^*(\lambda - \lambda z)[q + pS_{b_2}^*(\lambda - \lambda z)][B^*(\lambda) + (1 - B^*(\lambda))z]}$$

$$+ \frac{\eta z S_{b_1}^*(\lambda - \lambda z)Q_1^*(z, 0)}{z - S_{b_1}^*(\lambda - \lambda z)[q + pS_{b_2}^*(\lambda - \lambda z)][B^*(\lambda) + (1 - B^*(\lambda))z]}. \tag{46}$$

Using (45) in (44), we get

$$P_2(z, 0) = \frac{S_{b_2}^*(\lambda - \lambda z)\eta[z - qS_{b_1}^*(\lambda - \lambda z)[B^*(\lambda) + (1 - B^*(\lambda))z]]Q_2^*(z, 0)}{z - S_{b_1}^*(\lambda - \lambda z)[q + pS_{b_2}^*(\lambda - \lambda z)][B^*(\lambda) + (1 - B^*(\lambda))z]}$$

$$+ \frac{[pS_{b_1}^*(\lambda - \lambda z)]\eta z Q_1^*(z, 0) + [B^*(\lambda) + (1 - B^*(\lambda))z][\eta Q_0^*(z, 0) - \lambda[1 - U(z_1)]Q_{0,0}]}{z - S_{b_1}^*(\lambda - \lambda z)[q + pS_{b_2}^*(\lambda - \lambda z)][B^*(\lambda) + (1 - B^*(\lambda))z]}. \tag{47}$$

Using (35), (45) and (46) in (40), we get

$$P_0^*(z,\ 0) = \frac{N_0(z)}{D_0(z)} Q_{0,0} \tag{48}$$

where

$$N_0(z) = \eta z[1 - B^*(\lambda)] S_{b_1}^*(\lambda - \lambda z)[q + pS_{b_2}^*(\lambda - \lambda z)][1 - S_{v_1}^*(\lambda - \lambda z + \eta)]$$

$$[(\lambda + \eta)z - U(z_1) \times [\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

$$\eta z(1 - \lambda z)S_{v_1}^*(\lambda - \lambda z + \eta)[1 - S_{v_2}^*(\lambda - \lambda z + \eta)]$$

$$\times [(\lambda + \eta)z - U(z_1)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

$$+ \eta z[1 - B^*(\lambda)][1 - A^*(\lambda + \eta)][\lambda - \lambda z + \eta]$$

$$\times [zS_{v_1}^*(\lambda - \lambda z + \eta)[q + S_{v_2}^*(\lambda - \lambda z + \eta) - U(z_1)z]$$

$$-z[1 - B^*(\lambda)][1 - U(z_1)](\lambda - \lambda z + \eta)$$

$$\times [(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + pS_{v_2}^*(\lambda - \lambda z + \eta)]$$

$$[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]$$

$$D_0(z) = [z - S_{b_1}^*(\lambda - \lambda z)[q + pS_{b_2}^*(\lambda - \lambda z)][B^*(\lambda) + (1 - B^*(\lambda))z]][\lambda - \lambda z + \eta]$$

$$\times [(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + pS_{v_2}^*(\lambda - \lambda z + \eta)]$$

$$[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda\eta)]].$$

Inserting $\theta = 0$ and using (36), (39), (45) and (47) in (41), we get

$$P_1^*(z,\ 0) = \frac{N_1(z)}{D_1(z)} Q_{0,0} \tag{49}$$

where

$$N_1(z) = \lambda\eta pS_{b_2}^*(\lambda - \lambda z)[B^*(\lambda) + (1 - B^*(\lambda))z]S_{v_1}^*(\lambda - \lambda z + \eta)[1 - S_{v_1}^*(\lambda - \lambda z + \eta)]$$

$$\times [(\lambda + \eta)z - U(z_1)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

$$\lambda\eta z[1 - S_{v_1}^*(\lambda - \lambda z + \eta)] \times [(\lambda + \eta)z - U(z_1)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

$$+ \lambda\eta[1 - A^*(\lambda + \eta)][\lambda - \lambda z + \eta] \times [B^*(\lambda) + (1 - B^*(\lambda))z]$$

$$[vS_{v_1}^*(\lambda - \lambda z + \eta)[q + pS_{v_2}^*(\lambda - \lambda z + \eta)]$$

$$-U(z_1)z] - \lambda[1 - U(z_1)] \times [\lambda - \lambda z + \eta]$$

$$[B^*(\lambda) + (1 - B^*(\lambda))z][(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + pS_{v_2}^*(\lambda - \lambda z + \eta)]$$

$$\times [\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]$$

$$D_1(z) = [\lambda - \lambda z][z - S_{b_1}^*(\lambda - \lambda z)[q + pS_{b_2}^*(\lambda - \lambda z)]][B^*(\lambda) + (1 - B^*(\lambda))z[\lambda - \lambda z + \eta].$$

$$\times [(\lambda + \eta)z - S_{v_1}^*(\lambda - \lambda z + \eta)[q + pS_{v_2}^*(\lambda - \lambda z + \eta)]$$

$$[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]].$$

Inserting $\theta = 0$ and using (37), (45) and (46) in (43), we get

$$P_2^*(z, 0) = \frac{N_2(z)}{D_2(z)} Q_{0,0} \tag{50}$$

Where

$$N_2(z) = \eta[z - qS_{b_1}^*(\lambda - \lambda z)[B^*(\lambda) + (1 - B^*(\lambda))z]pS_{v_1}^*(\lambda - \lambda z + \eta)$$

$$[1 - S_{v_1}^*(\lambda - \lambda z + \eta)] \times [\lambda(\lambda + \eta)z - \lambda U(z_1)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

$$+ \eta pzS_{b_1}^*(\lambda - \lambda z) \times [1 - S_{b_1}^*(\lambda - \lambda z + \eta)]$$

$$[\lambda(\lambda + \eta)z - \lambda U(z_1)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

$$+ \eta pS_{b_1}^*(\lambda - \lambda z)[B^*(\lambda) + (1 - B^*(\lambda)z]\lambda[1 - A^*(\lambda + \eta)[\lambda - \lambda z + \eta]$$

$$[zS_{v_1}^*(\lambda - \lambda z + \eta) \times [q + pS_{v_2}^*(\lambda - \lambda z + \eta)] - U(z_1)z]$$

$$-\lambda p[1 - U(z_1)[S_{b_1}^*(\lambda - \lambda z)[B^*(\lambda) + [1 - B^*(\lambda)z]$$

$$\times [\lambda - \lambda z + \eta] [(\lambda + \eta) z - S_{v_1}^* (\lambda - \lambda z + \eta) [q + pS_{v_2}^* (\lambda - \lambda z + \eta)]$$

$$[\lambda z + (\lambda - \lambda z + \eta) \times A^* (\lambda + \eta)]]$$

$$D_2(z) = D_1(z).$$

We define

$$P_V(z) = Q_2^*(z, 0) + Q_1^*(z, 0) + Q_0^*(z, 0) + Q_{0,0} = \frac{N_V(z)}{D_V(z)} Q_{0,0} \tag{51}$$

as the probability generating function for the number of customers in the orbit when the server is on working vacation period, where

$$N_v(z) = pS_{v_1}^* (\lambda - \lambda z + \eta) [1 - S_{v_2}^* (\lambda - \lambda z + \eta)] [\lambda(\lambda + \eta) z - \lambda U(z_1) [\lambda z$$

$$+ (\lambda - \lambda z + \eta) A^* (\lambda + \eta)]] + [1 - S_{v_1}^* (\lambda - \lambda z + \eta)]$$

$$[\lambda(\lambda + \eta) z - \lambda U(z_1) [\lambda z + (\lambda - \lambda z + \eta) A^* (\lambda + \eta)]] + \lambda [1 - A^* (\lambda + \eta)]$$
$$\times [\lambda - \lambda z + \eta] [z S_{v_1}^* (\lambda - \lambda z + \eta) [q + pS_{v_2}^* (\lambda - \lambda z + \eta)] - U(z_1) z]$$

$$+ [\lambda - \lambda z + \eta] [(\lambda + \eta) z - S_{v_1}^* (\lambda - \lambda z + \eta) [q + pS_{v_1}^* (\lambda - \lambda z + \eta)]$$

$$[\lambda z + (\lambda - \lambda z + \eta) A^* (\lambda + \eta)]$$

$$D_V(z) = [\lambda - \lambda z + \eta] [(\lambda + \eta) z - S_{v_1}^* (\lambda - \lambda z + \eta) [q + pS_{v_2}^* (\lambda - \lambda z + \eta)]$$

$$[\lambda z + (\lambda - \lambda z + \eta) A^* (\lambda + \eta)]].$$

We define

$$P_B(z) = P_0^*(z, 0) + P_1^*(z, 0) + P_2^*(z, 0) = \frac{N_B(z)}{D_B(z)} Q_{0,0} \tag{52}$$

as the probability generating function for the number of customers in the orbit when the server is on not working vacation (normal busy) period, where

$$N_B(z) = [\lambda - \lambda z] N_0(z) + [1 - S_{b_1}^* (\lambda - \lambda z)] N_1(z) + [1 - S_{b_2}^* (\lambda - \lambda z)] N_2(z)$$

$$D_B(z) = [\lambda - \lambda z][z - S_{b_1}^* (\lambda - \lambda z)[q + pS_{b_2}^* (\lambda - \lambda z)][B^*(\lambda) + [1 - B^*(\lambda)) z]] D_V(z).$$

We define

$$P(z) = P_V(z) + P_B(z) \tag{53}$$

as the probability generating function for the number of customers in the orbit irrespective of the server state. where $P_V(z)$ and $P_B(z)$ are given in equation (51) and (52). We shall now use the normalizing condition $P(1) = 1$ to determine the unknown $Q_{0,0}$ which appears in (53). Substituting $z = 1$ in (53) and using L'Hospitals rule, we obtain

$$Q_{0,0} = \frac{N_r}{\frac{\lambda - \lambda U(z_1) + \eta}{\eta}\left[\frac{\lambda + \eta - U(z_1)[\lambda + \eta A^*(\lambda + \eta)]}{\lambda + \eta - S^*_{v_1}(\eta)[q + pS^*_{v_2}(\eta)][\lambda + \eta A^*(\lambda + \eta)]}\right][D_r]}. \tag{54}$$

Where

$$N_r = 1 - \lambda E[S_{b_1}] - p\lambda E[S_{b_2}] - [1 - B^*(\lambda)]$$

$$D_r = [\lambda E[S_{b_1}][q + pE[S_{b_2}]]S^*_{v_1}(\eta)[q + pS^*_{v_2}(\eta)] + 1 - B^*(\lambda)].$$

4. Particular Cases

Case (i). If no customer receives the second optional service then by setting $p = 0$ and $S_{b_1} = S_b$, $S_{v_1} = S_v$ in (53), we get

$$P(z) = P_V(z) + P_B(z) \tag{55}$$

$$P_V(z) = \frac{N_V(z)}{D_V} Q_{0,0} \text{ and } P_B(z) = \frac{N_B(z)}{D_B(z)} Q_{0,0}$$

where

$$N_V(z) = \lambda[1 - A^*(\lambda + \eta)][\lambda - \lambda z + \eta][zS^*_v(\lambda - \lambda z + \eta) - U(z_1)z]$$

$$+ \lambda[1 - S^*_v(\lambda - \lambda z + \eta)]$$

$$\times [(\lambda + \eta)z - U(z_1)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

$$+ [\lambda - \lambda z + \eta][(\lambda + \eta)z - S^*_v(\lambda - \lambda z + \eta) \times [\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]$$

$$D_V(z) = [\lambda - \lambda z + \eta][(\lambda + \eta)z - S_v^*(\lambda - \lambda x + \eta)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]$$

$$N_B(z) = \lambda\eta z[1 - S_v^*(\lambda - \lambda z + \eta)][(\lambda + \eta) - U(z_1)[\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

$$[1 - S_b^*(\lambda - \lambda z) \times [B^*(\lambda) + (1 - B^*(\lambda)z]] + \lambda\eta[1 - A^*(\lambda + \eta)][\lambda - \lambda z + \eta]$$

$$[zS_v^*(\lambda - \lambda z + \eta) - U(z_1)z]$$

$$\times [z + [1 - z - S_v^*(\lambda - \lambda z)][B^*(\lambda) + (1 - B^*(\lambda)z - \lambda[1 - U(z_1)]$$

$$[\lambda - \lambda z + \eta]$$

$$\times [(\lambda + \eta)z - S_v^*(\lambda - \lambda z)[\lambda z + (\lambda - \lambda z + \eta)]]$$

$$[z + [1 - z - S_b^*(\lambda - \lambda z)][B^*(\lambda) + (1 - B^*(\lambda)z]]$$

$$D_B(z) = [\lambda - \lambda z][\lambda - \lambda z + \eta][z - S_b^*(\lambda - \lambda z)[B^*(\lambda) + (1 - B^*(\lambda)z]][(\lambda + \eta)z$$

$$- S_v^*(\lambda - \lambda z + \eta) \times [\lambda z + (\lambda - \lambda z + \eta)A^*(\lambda + \eta)]]$$

and

$$Q_{0,0} = \frac{1 - \lambda E[S_b] - (1 - B^*(\lambda))}{\left[\frac{\lambda - \lambda U(z)_1 + \eta}{\eta}\right] - \left[\frac{\lambda + \eta - U(z_1)[\lambda + \eta A^*(\lambda + \eta)]}{\lambda + \eta - S_v^*(\eta)[\lambda + \eta A^*(\lambda + \eta)}\right]}.$$

$$[\lambda E(S_b S_v^*(\eta) + (1 - B^*(\lambda))]$$

Equation (55) is well known generating function of the orbit size distribution of a single arrival retrial queue with exponentially distributed multiple working vacation.

Case (ii):

If no customer receives the Second optional service and no retrial then on setting $A^*(\lambda + \eta) = 1 = B^*(\lambda)$, $p = 0$, $S_{v_1} = S_v$ and $S_{b_1} = S_b$ in (52), we get

$$P(z) = P_V(z) + P_B(z) \tag{56}$$

$$P_V(z) = \frac{N_V(z)}{D_V(z)} Q_{0,0} \text{ and } P_B = \frac{N_B(z)}{D_B(z)} Q_{0,0}.$$

Where

$$N_V(z) = \lambda[1 - S_v^*(\lambda - \lambda + \eta)][z - U(z_1)z] + [\lambda - \lambda z + \eta][z - S_v^*(\lambda - \lambda z + \eta)]$$

$$D_V(z) = [\lambda - \lambda z + \eta][z - S_v^*(\lambda - \lambda z + \eta)]$$

$$N_B(z) = \lambda\eta z[1 - S_v^*(\lambda - \lambda z + \eta)][z - U(z_1)z][1 - S_b^*(\lambda - \lambda z)]$$

$$D_B(z) = [\lambda - \lambda z][z - S_b^*(\lambda - \lambda z)][\lambda - \lambda z + \eta][z - S_v^*(\lambda - \lambda z + \eta)]$$

$$Q_{0,0} = \frac{1 - \lambda E(S_b)}{\left[\frac{\lambda - \lambda U(z_1) + \eta}{\eta} - \frac{1 - U(z_1)}{1 - S_v^*(\eta)}[\lambda E(S_b) S_v^*(\eta)]\right]}.$$

Equation (56) is well known generating function of the queue size distribution of an $M/G/1$ queue with multiple working vacation.

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