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PRIME LABELING OF THE MIDDLE GRAPH OF CERTAIN GRAPHS

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Abstract

A graph $\mathbb{G}(\mathbb{V}, \mathbb{E})$ is observed to concede prime labeling when the vertices of the graph are labeled with unique integral values from $[1, |\mathbb{V}|]$ in a way that for each edge uv, the end vertices u and v are designated labels that share no common positive factors except 1. In this research article we investigate that the middle graph of path, cycle, tadpole graph and crown graph concede prime labeling.

1. Introduction

Herein, this research article the graphs examined are non-oriented with no multiple edges and loops. We symbolize the vertex set by $\mathbb{V}(\mathbb{G})$ with cardinality $|\mathbb{V}(\mathbb{G})|$ and the edge set by $\mathbb{E}(\mathbb{G})$ with cardinality $|\mathbb{E}(\mathbb{G})|$. Graph labeling is one of the important branches of graph theory in which the vertices or edges or both are assigned integral values with some conditions. There are various applications of graph labeling such as astronomy, management of database, theory of coding, communication network addressing, x-ray crystallography, circuit design and much more. The notion of prime labeling was introduced by R. Entringer. The concept of middle

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graph of a graph $\mathbb{G}$ was introduced by T. Hamada and I. Yoshimura in [2]. A graph $\mathbb{G}(\mathbb{V}, \mathbb{E})$ is observed to concede prime labeling when the vertices of the graph are labeled with unique integral values from $[1, |\mathbb{V}|]$ in a way that for each edge uv, the end vertices u and v are designated labels that share no common positive factors except 1. A prime graph is the one that receives a prime labeling [4]. In this research article we investigate that the Middle graph of path, cycle, tadpole and crown graphs concede prime labeling.

2. Preliminaries

Definition 2.1 [6]**.** The graph constructed by sub-dividing every edge of a graph $\mathbb{G}$ by placing into every edge of $\mathbb{G}$ a new vertex and connecting those pairs of newly placed vertices which lie on adjacent edges of the graph $\mathbb{G}$ by introducing new edges is known as the middle graph $\mathcal{M}(\mathbb{G})$ of a graph $\mathbb{G}$.

Definition 2.2 [5]**.** The path graph P_n is constructed by forming a route from one vertex to another by virtue of placing edges linking successive vertices of the path.

Definition 2.3 [5]**.** The cycle graph C_n is constructed by forming a route from one vertex to another by virtue of placing edges linking successive vertices where the last edge connects the last vertex with the starting vertex.

Definition 2.4 [1]**.** The tadpole graph, $T_{m,n}$ is constructed by connecting cycle C_m with path P_n by introducing an edge between any vertex of C_m to the first vertex of P_n whenever $m > 2$ and $n \geq 2$.

Definition 2.5 [3]**.** The crown graph is constructed by attaching absolutely one pendent edge at every vertex of the cycle C_n whenever $n > 2$.

3. Prime Labeling of Middle Graph of Path

Theorem 3.1. *The middle graph of path graph is a prime graph.*

Proof. Let P_n be the Path graph.

The vertex set $\mathbb{V}(P_n) = \{u_1, u_2, \ldots, u_n\}$. In general, $\mathbb{V}(P_n) = \{u_i / 1 \leq i \leq n\}$ and $|\mathbb{V}(P_n)| = n$. The Edge set $\mathbb{E}(P_n) = \{e_1, e_2, \ldots, e_{n-1}\}$.

In general, $\mathbb{E}(P_n) = \{e_i = u_i u_{i+1} / 1 \leq i < n\}$ and $|\mathbb{E}(P_n)| = n - 1$.

Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices affixed with respect to the edges $e_1, e_2, \ldots, e_{n-1}$ of P_n to obtain the middle graph $\mathcal{M}(P_n)$ of the path graph P_n.

Then $\mathbb{V}(\mathcal{M}(P_n)) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}\}$ and $|\mathbb{V}(\mathcal{M}(P_n))| = 2n - 1$

$\mathbb{E}(\mathcal{M}(P_n)) = \{v_i v_{i+1} / 1 \leq i < n-1\} \cup \{u_i v_i / 1 \leq i < n\} \cup \{v_i u_{i+1} / 1 \leq i < n\}$ and $|\mathbb{E}(\mathcal{M}(P_n))| = 3n - 4$.

Let us define a labeling $f : \mathbb{V}(\mathcal{M}(P_n)) \to \{1, 2, 3, \ldots, 2n - 1\}$ by

$f(u_1) = n$

$f(v_i) = 2i - 1, 1 \leq i < n$

$f(u_i) = 2i - 2, 2 \leq i \leq 4n$

Then for the edges,

$v_i v_{i+1}$, $\gcd(f(v_i), f(v_{i+1})) = 1, 1 \leq i < n$

$u_i v_{i+1}$, $\gcd(f(u_i), f(v_i)) = 1$

$v_i u_{i+1}$, $\gcd(f(v_i), f(u_{i+1})) = 1$

Therefore $\mathcal{M}(P_n)$ is a prime graph.

The middle graph of path and its prime labeling are as in the following figures 3.1.1, 3.1.2 and 3.1.3 respectively.

Illustration 3.1.

Figure 3.1.1. $\mathcal{M}(P_n)$.

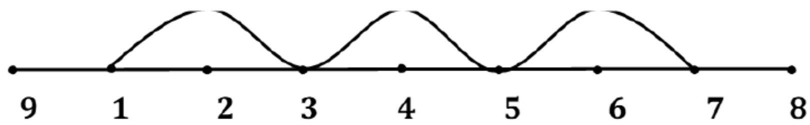

Figure 3.1.2. Prime labeling of $\mathcal{M}(P_5)$.

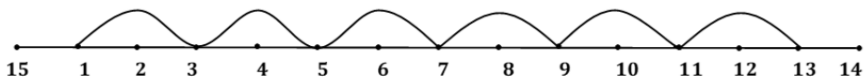

Figure 3.1.3. Prime labeling $\mathcal{M}(P_8)$.

4. Prime Labeling of Middle Graph of Cycle Graph

Theorem 4.1. *The middle graph of cycle graph is a prime graph.*

Proof. Let C_n be the Cycle graph.

The vertex set $\mathbb{V}(C_n) = \{u_1, u_2, \ldots, u_n\}$. In general, $\mathbb{V}(C_n) = \{u_i / 1 \le i \le n\}$ and $|\mathbb{V}(C_n)| = n$

The Edge set $\mathbb{E}(C_n) = \{e_1, e_2, \ldots, e_n\}$.

In general, $\mathbb{E}(C_n) = \{e_i = u_i u_{i+1} / 1 \le i < n\} \cup \{e_n = u_n u_1\}$ and $|\mathbb{E}(C_n)| = n$.

Let $v_1, v_2, \ldots, v_n$ be the vertices added corresponding to the edges $e_1, e_2, \ldots, e_n$ of C_n to obtain the middle graph $\mathcal{M}(C_n)$ of the cycle graph C_n.

Then $\mathbb{V}(\mathcal{M}(C_n)) = \{u_1, u_2, \ldots u_n, v_1, v_2, \ldots v_n\}$ and $|\mathbb{V}(\mathcal{M}(P_n))| = 2n$ $\mathbb{E}(\mathcal{M}(C_n)) = \{v_i v_{i+1} / 1 \le i < n\} \cup \{v_n v_1\} \cup \{u_i v_i / 1 \le i \le n\} \cup \{v_i u_{i+1} / 1 \le i < n\}$ $\cup \{v_n u_1\}$ and $|\mathbb{E}(\mathcal{M}(C_n))| = 3n$.

Let us define a labeling $f : \mathbb{V}(\mathcal{M}(C_n)) \to \{1, 2, 3, \ldots, 2n\}$

$f(v_n) = 1$

$f(v_i) = 2i, 1 \le i < n$

$f(u_i) = 2i + 1 \le i \le n$

Then for the edges,

$v_i v_{i+1}$, $\gcd(f(v_i), f(v_{i+1})) = 1$, $1 \le i < n$

$v_i v_i$, $\gcd(f(u_i), f(v_i)) = 1$, $1 \le i \le n$

$v_i u_{i+1}$, $\gcd(f(v_i), f(u_{i+1})) = 1$, $1 \le i < n$

$v_n v_1$, $\gcd(f(v_n), f(v_1)) = 1$

$v_n u_1$, $\gcd(f(v_n), f(u_1)) = 1$

Therefore $\mathcal{M}(C_n)$ is a prime graph.

The middle graph of cycle graph and its prime labeling are as in the following figures 4.1.1, 4.1.2 and 4.1.3 respectively.

Illustration 4.1:

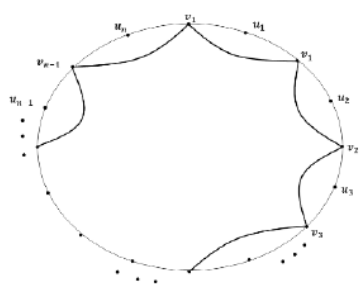

Figure 4.1.1. Middle graph $\mathcal{M}(C_n)$ of cycle graph C_n.

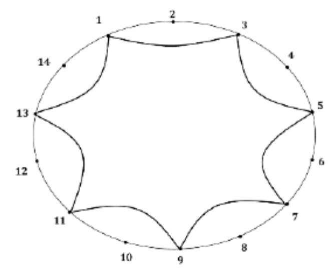

Figure 4.1.2. Prime labeling of Middle graph $\mathcal{M}(C_7)$ of cycle graph C_7.

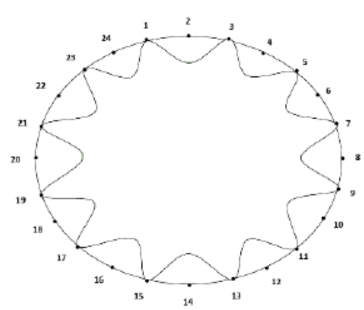

Figure 4.1.3 Prime labeling of Middle graph $\mathcal{M}(C_{12})$ of cycle graph C_{12}.

5. Prime Labeling of Middle Graph of Tadpole Graph

Theorem 5.1. *The middle graph of tadpole graph is a prime graph.*

Proof. Let $T_{m,n}$ be the tadpole graph.

The vertex set $\mathbb{V}(T_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots v_n\}$. In general, $\mathbb{V}(T_{m,n}) = \{u_i, v_j / 1 \leq i \leq m, 1 \leq j \leq n\}$. and $|\mathbb{V}(T_{m,n})| = m + n$.

The Edge set $\mathbb{E}(T_{m,n}) = \{e_1, e_2, \ldots, e_m, w_1, w_2, \ldots w_{n-1}, w_n\}$.

In general, $\mathbb{E}(T_{m,n}) = \{e_i = u_i u_{i+1} / 1 \leq i \leq m-1\} \cup \{e_m = u_m u_1\} \cup \{w_1 = u_1 v_1\} \cup \{w_j = v_j v_{j+1} / 1 \leq j \leq n-1\}$ and $|\mathbb{E}(T_{m,n})| = m + n$.

Let $x_1, x_2, \ldots, x_m$ be the vertices added corresponding to the edges $e_1, e_2, \ldots, e_m$ of cycle C_m and $y_1, y_2, \ldots, y_n$ be the vertices added corresponding to the edges $w_1, w_2, \ldots, w_n$ of path P_n respectively to obtain the middle graph $\mathcal{M}(T_{m,n})$ of the tadpole graph $T_{m,n}$. Then,

$\mathbb{V}(\mathcal{M}(T_{m,n})) = \{u_1, u_2, \ldots, u_m, x_1, x_2, \ldots, x_m, v_1, v_2, \ldots, v_n, y_1, y_2, \ldots, y_n\}$ and $|\mathbb{V}(\mathcal{M}(T_{m,n}))| = 2(m+n) \mathbb{E}(\mathcal{M}(T_{m,n})) = \{u_i x_i / 1 \leq i \leq m\} \cup \{x_i u_{i+1} / 1 \leq i < m\} \cup \{x_m u_1\} \cup \{x_i x_{i+1} / 1 \leq i < m\} \cup \{u_1 y_1\} \cup \{y_i v_i / 1 \leq i \leq n\} \cup \{v_i y_{i+1} / 1 \leq i < n\} \cup \{y_i y_{i+1} / 1 \leq i < n\}$

Let us define a labeling $f : \mathbb{V}(\mathcal{M}(T_{m,n})) \to \{1, 2, 3, \ldots, 2(m+n)\}$

$f(y_1) = 1$

$f(u_i) = 2i, 1 \leq i \leq m$

$f(x_i) = 2i + 1, 1 \leq i \leq m$

$f(v_i) = 2m + 2i, 1 \leq i \leq n$

$f(y_i) = 2m + 2i - 1, 2 \leq i \leq n$

Then for the edges,

$u_i x_i$, $\gcd(f(u_i), f(x_i)) = 1$, $1 \le i \le m$

$x_i u_{i+1}$, $\gcd(f(x_i), f(u_{i+1})) = 1$, $1 \le i \le m$

$x_i x_{i+1}$, $\gcd(f(x_i), f(x_{i+1})) = 1$, $1 \le i < m$

$x_m u_1$, $\gcd(f(x_m), f(u_1)) = 1$

$u_1 y_1$, $\gcd(f(u_1), f(y_1)) = 1$

$y_i v_i$, $\gcd(f(y_i), f(v_i)) = 1$, $1 \le i \le n$

$v_i y_{i+1}$, $\gcd(f(v_i), f(y_{i+1})) = 1$, $1 \le i < n$

$y_i y_{i+1}$, $\gcd(f(y_i), f(y_{i+1})) = 1$, $1 \le i < n$

Therefore $\mathcal{M}(T_{m,n})$ is a prime graph.

The middle graph of tadpole graph and its prime labeling are as in the following figures 5.1.1, and 5.1.2 respectively.

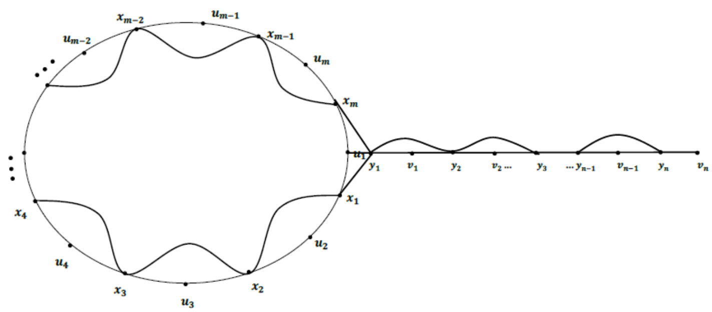

Figure 5.1.1. Middle graph of tadpole graph.

Illustration 5.1.

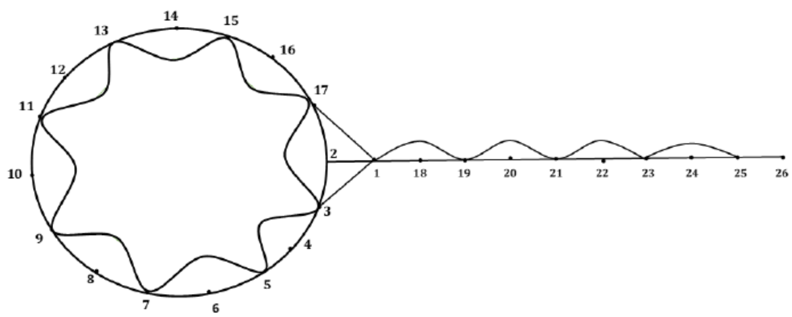

Figure 5.1.2. Prime labeling of middle graph of tadpole graph $\mathcal{M}(T_{8,5})$.

6. Prime Labeling of Middle Graph of Crown Graph

Theorem 6.1. *The middle graph of crown graph is a prime graph.*

Proof. Let $\mathbb{G} = C_n \odot K_1$ be the Crown graph.

The vertex set $\mathbb{V}(\mathbb{G}) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots v_n\}$. In general, $\mathbb{V}(\mathbb{G}) = \{u_i, v_i / 1 \le i \le n\}$. and $|\mathbb{V}(\mathbb{G})| = 2n$.

The Edge set $\mathbb{E}(\mathbb{G}) = \{e_1, e_2, \ldots e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$.

In general, $\mathbb{E}(\mathbb{G}) = \{e_i = u_i u_{i+1} / 1 \le i < n\} \cup \{e_n = u_n u_1\} \cup \{e_{n+i} = u_i v_i / 1 \le i \le n\}$ and $|\mathbb{E}(\mathbb{G})| = 2n$.

Let $x_1, x_2, \ldots, x_n$ be the vertices added corresponding to the edges $e_1, e_2, \ldots, e_n$ of C_n and $y_1, y_2, \ldots, y_n$ be the vertices added corresponding to the edges $e_{n+1}, e_{n+2}, \ldots, e_{2n}$ of P_n respectively to obtain the middle graph $\mathcal{M}(\mathbb{G})$ of the crown graph $\mathbb{G}$.

Then $\mathbb{V}(\mathcal{M}(\mathbb{G})) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. In general, $\mathbb{V}(\mathcal{M}(\mathbb{G})) = \{u_i, v_i, x_i, y_i / 1 \le i \le n\}$ and $|\mathbb{V}(\mathcal{M}(\mathbb{G}))| = 4n$.

$\mathbb{E}(\mathcal{M}(\mathbb{G})) = \{u_i x_i / 1 \le i \le n\} \cup \{x_i u_{i+1} / 1 \le i < n\} \cup \{x_n u_1\} \cup \{x_i x_{i+1} / 1 \le i < n\}$ $\cup \{x_i y_{i+1} / 1 \le i < n\} \cup \{x_n y_1\} \cup \{y_i x_i / 1 \le i \le n\} \cup \{u_i y_i / 1 \le i \le n\} \cup \{y_i v_i / 1 \le i \le n\}$ and $|\mathbb{E}(\mathcal{M}(\mathbb{G}))| = 7n$.

Let us define a labeling $f : \mathbb{V}(\mathcal{M}(\mathbb{G})) \to \{1, 2, 3, \ldots, 4n\}$

$f(x_i) = 4i - 3, 1 \leq i \leq n$

$f(y_i) = 4i - 1, 1 \leq i \leq n$

$$f(u_i) = \begin{cases} 12j + 2, \ i \equiv 1(\operatorname{mod} 3), \ i = 3j + 1, \ j \geq 0 \\ 12j + 8, \ i \equiv 2(\operatorname{mod} 3), \ i = 3j + 2, \ j \geq 0 \\ 12j - 2, \ i \equiv 0(\operatorname{mod} 3), \ i = 3j, \ j \geq 0 \end{cases}$$

$$f(v_i) = \begin{cases} 12j + 4, \ i \equiv 1(\operatorname{mod} 3), \ i = 3j + 1, \ j \geq 0 \\ 12j + 6, \ i \equiv 2(\operatorname{mod} 3), \ i = 3j + 2, \ j \geq 0 \\ 12j, \ i \equiv 0(\operatorname{mod} 3), \ i = 3j, \ j \geq 0 \end{cases}$$

Then for the edges,

$u_i x_i$, $\gcd(f(u_i), f(x_i) = 1, 1 \leq i \leq n$

$x_i u_{i+1}$, $\gcd(f(x_i), f(u_{i+1})) = 1, 1 \leq i < n$

$x_n u_1$, $\gcd(f(x_n), f(u_1)) = 1$

$x_i x_{i+1}$, $\gcd(f(x_i), f(x_{i+1})) = 1, 1 \leq i < n$

$x_i y_{i+1}$, $\gcd(f(x_i), f(y_{i+1})) = 1, 1 \leq i < n$

$x_n x_1$, $\gcd(f(x_n), f(x_1)) = 1$

$x_n y_1$, $\gcd(f(x_n), f(y_1)) = 1$

$y_i x_i$, $\gcd(f(y_i), f(x_i)) = 1, 1 \leq i \leq n$

$u_i y_i$, $\gcd(f(u_i), f(y_i)) = 1, 1 \leq i \leq n$

$y_i v_i$, $\gcd(f(y_i), f(v_i)) = 1, 1 \leq i \leq n$

Therefore $\mathcal{M}(C_n \odot K_1)$ is a prime graph.

The middle graph of crown graph and its prime labeling are as in the following figures 6.1.1 and 6.1.2 respectively.

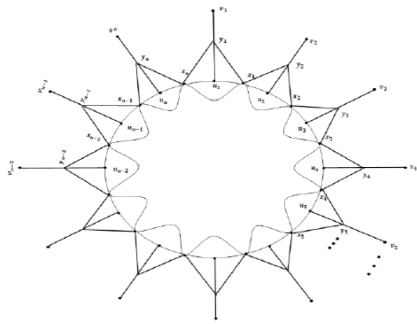

Figure 6.1.1. Middle graph $\mathcal{M}(C_n \odot K_1)$ of crown graph.

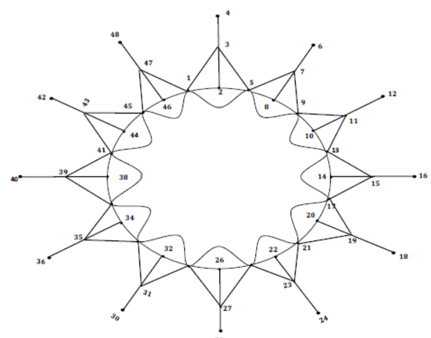

Figure 6.1.2. Prime labeling of middle graph $\mathcal{M}(C_{12} \odot K_1)$ of crown graph.

7. Conclusion

Herein this research article we have established that the Middle graph of path, cycle, tadpole and crown graphs concede prime labeling. To investigate the middle graph of various graphs that admit prime labeling is interesting and engrossing for further study.

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