# COMPLETE AND BOOLEAN LATTICE WHICH ARE DERIVED FROM HYPERLATTICES 

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#### Abstract

Hyperlattices are a suitable generalization of ordinary lattices. In this paper, we introduce two relations $v$ and $\theta$ on a hyperlattice $L$, and we let $v^{*}$ and $\theta^{*}$ be the transitive closure of $\vartheta$ and $\theta$. Then, we investigate the connection between two relations and we show that by the first relation, the quotient of hyperlattice $L$ is a Boolean and residuated lattice and by the second relation and adding distributivity to hyperlattice $L$, the quotient of hyperlattice $L$ is complete.


## 1. Introduction

Algebraic hyperstructures play a prominent role in mathematics with wide ranging applications in many branches such as coding theory, topological spaces, graphs, lattices and the like. One of the structures that are most extensively used and discussed in mathematics and its applications is lattice theory [2, 4]. The concept of a hyperlattice which is based on the hyperoperation was introduced by Konstantinidou and Mittas in [7]. Other contributor to developing of lattice and hyperlattice theory are Serafimidis and Kehagias [12], Varlet [13], Ashrafi [1], Leoreanu-Fotea and Davvaz [9], Leoreanu-Fotea et al. [10] and others.

By the end of 80s the theory of hyperstructures had completed more than half of a century. At that time a lot of theory on hyperstructures had been achieved, for example, the relation $\beta^{*}$ and $\gamma^{*}$ were studied. The main tools in

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the theory of hyperstructures are the fundamental relations. These relations, on the one hand, connect this theory, in some way with the corresponding classical theory and on the other hand, introduce new important classes. Researchers who studied on the fundamental relations are Corsini [3], Freni [5], Vougiouklis [14], and many others. In [11], Rasouli and Davvaz studied lattices which is derived from hyperlattices by means of fundamental relation in hyperlattices. In this paper, we introduce two strongly regular relations on a hyperlattice $L$ and we investigate the structure of the quotient of hyperlattice $L$ and we derive three categories of lattices with these quotient structures.

## 2. Preliminaries

In this section, we provide background information needed in the paper. First, we present some basic definitions and well-known facts about lattices and hyperlattices.

The hyperstructure theory was introduced by Marty [8] in 1934. A function $f$ from $H \times H$ into the set of all nonempty subsets of $H$, is called a binary hyperoperation, and the pair $(H, f)$ is called a hypergroupoid. If $f$ is associative, $H$ is called a semi hypergroup, and it is said to be commutative if $f$ is commutative.

One can see the definition and basic properties of a lattice $L$ in [2]. According to [4], the lattice $L$ is distributive if for all $x, y, z \in L$ one of the following conditions hold:
(1) $x \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z)$;
(2) $x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z)$.

Also, we say $L$ is complete if for every $S \subseteq L, \vee S$ and $\wedge S$ exist and $L$ is Boolean lattice if for each $a \in L$ there exists $a^{\prime} \in L$ such that the following conditions hold:
(1) $(L, \vee, \wedge)$ is a distributive lattice;
(2) $a \vee 0=a$ and $a \wedge 1=a$;
(3) $a \vee a^{\prime}=1$ and $a \wedge a^{\prime}=0$.

Definition 2.1[6]. A residuated lattice is a nonempty set $L$ with four binary operations $\vee, \wedge, \odot, \rightarrow$ and two constants 0,1 such that the following conditions hold:
(1) $(L, \wedge, \vee, 0,1)$ is a bounded lattice;
(2) $(L, \odot, 1)$ is a commutative monoid;
(3) for any $x, y, z \in L, x \odot y \leq z$ if and only if $x \leq y \rightarrow z$.

A generalization of the notion of a lattice is hyperlattice which is defined as follows:

Definition 2.2. Let $L$ be a non-empty set, $\vee: L \times L \rightarrow \wp^{*}(L)$ be a hyperoperation, and $\wedge: L \times L \rightarrow L$ be an operation. Then, $(L, \vee, \wedge)$ is a join hyperlattice if for all $x, y, z \in L$ the following conditions hold:
(1) $x \in x \vee x$ and $x=x \wedge x$;
(2) $x \vee(y \vee z)=(x \vee y) \vee z$ and $x \wedge(y \wedge z)=(x \wedge y) \wedge z$;
(3) $x \vee y=y \vee x$ and $x \wedge y=y \wedge x$;
(4) $x \in x \wedge(x \vee y) \cap x \vee(x \wedge y)$.

Now, we recall the definition of $H_{v}$-lattice which is defined already and we use it in this paper.

Definition 2.3. $(L, \vee, \wedge)$ is an $H_{v}$-lattice if for all $x, y, z \in L$ we have:
(1) $(x \wedge y) \wedge z \cap x \vee(y \vee z) \neq \emptyset$;
(2) $x \vee L=L \vee x=L$;
(3) $x \in x \vee x$ and $x \in x \wedge x$;
(4) $x \vee y=y \vee x$ and $x \wedge y=y \wedge x$;
(5) $x \wedge(y \wedge z) \cap(x \wedge y) \wedge z \neq \emptyset$.

Let $(L, \wedge, \vee)$ be a join hyperlattice. According to [11], we say that $L$ is a strong join hyperlattice if for all $x, u \in L, x \wedge y$ implies that $x=x \wedge y$. We
say that 0 is a zero element of $L$, if for all $x \in L$ we have $0 \leq x$ and 1 is a unit of $L$ if for all $x \in L, x \leq 1$. We say $L$ is bounded if $L$ has 0 and 1 . And $y$ is a complement of $x$ if $1 \in x \vee y$ and $0=x \wedge y$. A complemented hyperlattice is a bounded hyperlattice which every element has a complement. We say that $L$ is distributive if for all $x, y, z \in L, x \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z)$. And $L$ is $s$-distributive if $x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z)$. The map $f: L_{1} \rightarrow L_{2}$ is called a homomorphism if for all $x, y \in L_{1}$ we have $f(x \vee y)=f(x) \vee f(y)$ and $f(x \wedge y)=f(x) \wedge f(y)$. Moreover, $f$ is an isomorphism if it is bijection too.

Definition 2.4. Let $\mathcal{R}$ be an equivalence relation on a nonempty set $L$ and $A, B \subseteq L$, then
(1) $A \overline{\mathcal{R}} B$ means that for all $a \in A$, there exists $b \in B$ such that $a \mathcal{R} b$ and for all $b^{\prime} \in B$, there exists $a^{\prime} \in A$ such that $a^{\prime} \mathcal{R} b^{\prime}$;
(2) $A \overline{\overline{\mathcal{R}}} B$ means that for all $a \in A$, for all $b \in B$, we have $a \mathcal{R} b$.

Definition 2.5[11]. Let $\mathcal{R}$ be an equivalence relation on a hyperlattice $(L, \vee, \wedge)$ and $X, Y \subseteq L$.
(1) $\mathcal{R}$ is called a regular relation respect to $\vee$ (respect to $\wedge$ ) if $x \mathcal{R} y$ implies that $x \vee z \overline{\mathcal{R}} y \vee z(x \wedge z \overline{\mathcal{R}} y \wedge z)$, for all $x, y, z \in L \cdot \mathcal{R}$ is called a regular relation if it is regular respect to $\vee$ and $\wedge$, at the same time.
(2) $\mathcal{R}$ is called a strongly regular relation respect to $\vee$ (respect to $\wedge$ ) if $x \mathcal{R} y$ implies $x \vee z \overline{\overline{\mathcal{R}}} y \vee z(x \wedge z \overline{\overline{\mathcal{R}}} y \wedge z)$, for all $x, y, z \in L \cdot \mathcal{R}$ is called a strongly regular relation if it is strongly regular respect to $\vee$ and $\wedge$, at the same time.

Let $\mathcal{R}$ be a reflexive and symmetric relation on a nonempty set $L$. The transitive closure of $\mathcal{R}$ is denoted by $\mathcal{R}^{*}$ and defined as follows:

$$
x \mathcal{R}^{*} y \Leftrightarrow \exists n \in \mathbb{N}, \exists\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in L^{n}, x_{1}=x \mathcal{R} x_{2} \mathcal{R} \ldots x_{n-1} \mathcal{R} x_{n}=y
$$

The fundamental relations $\beta^{*}$ and $\gamma^{*}$ are defined in hypergroups,
hyperrings, as the smallest equivalence relations so that the quotient would be group and ring. The way to find the fundamental classes is given by analogous theorems to the following.

Theorem 2.6[14]. Let $(H, \circ)$ be a hypergroup and let us denote by $U$ the set of all finite products of elements of $H$. We define the relation $\beta$ in $H$ as follows: $x \beta y$ iff $\{x, y\} \subset u$ where $u \in U$. Then, the fundamental relation $\beta^{*}$ is the smallest equivalence relation on $H$ such that the quotient $H / \beta^{*}$ is a group.

## 3. Boolean and Complete Lattices which is Derived from Hyperlattices

In this section, we introduce two relations $v$ and $\theta$ on a hyperlattice $L$, and we let $v^{*}$ and $\theta^{*}$ be the transitive closure of $v$ and $\theta$. Then, we investigate the connection between two relations and we show that by the first relation, the quotient of hyperlattice $L$ is complete, Boolean and residuated lattice and by the second relation and adding distributivity to hyperlattice $L$, the quotient of hyperlattice $L$ is complete. Then, we define $\theta$-complete in hyperlattices and we prove some theorems and propositions which hold in the special category of hyperlattices such as $H_{v}$-lattices.

According to [3], if $R$ is an equivalence relation on a hypergroup $H$, then $R$ is strongly regular if and only if $(H / R, \otimes)$ is a group. According to [11], in the category of hyperlattices, if $(L, \vee, \wedge)$ is a hyperlattice (superlattice) and $R$ be an equivalence relation on $L, \vee, \wedge$ be hyperoperations of $L / \mathcal{R}$, then, if $R$ is strongly regular relation, $(L / \mathcal{R}, \vee, \wedge)$ is a lattice.

Definition 3.1. If $(L, \vee, \wedge)$ is a hyperlattice, then we set $\theta_{1}=\{(x, x) \mid x \in L\}$, and for every integer $n>1, \theta_{n}$ is the relation defined as follows

Let $\theta=\bigcup_{n \geqslant 1} \theta_{n}$. Clearly, the relation $\theta$ is reflexive and symmetric. Denote by $\theta^{*}$ the transitive closure of $\theta$.

Proposition 3.2. Let $(L, \vee, \wedge)$ be a distributive hyperlattice. Then, $\theta^{*}$ is a strongly regular relation on hyperlattice L. (Notice that if L is a hyperlattice, $\theta^{*}$ is a strongly regular relation with respect to $\vee$ ).

Proof. Let $a \theta^{*} b$. Then, there exist $r \in N,\left(x_{0}, x_{1}, \ldots, x_{r}\right) \in L^{r+1}$ such that $x_{0}=a, x_{1}, \ldots, x_{r}=b$ and $\left(q_{1}, q_{2}, \ldots, q_{r}\right) \in \mathbb{N}^{r}$ such that for all $i \in\{1,2, \ldots, r-1\} \quad$ we have $x_{i} \theta_{q_{i}} x_{i+1}$. Let $z \in L$ and $u_{1} \in x_{i} \wedge z$, $v \in x_{i+1} \wedge z$. We check that $u_{1} \theta^{*} u_{2}$. From $x_{i} \theta_{q_{i}} x_{i+1}$ it follows that there exist $\left(z_{i 1}, z_{i 2}, \ldots, z_{i k_{i}}\right) \in L^{q_{i}},\left(k_{1}, k_{2}, \ldots, k_{q_{i}}\right) \in \mathbb{N}^{q_{i}}$ such that $\left\{x_{i}, x_{i+1}\right\}$ $\subseteq \vee_{i=1}^{q_{i}}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right) . \quad$ So, we have $\quad\left\{x_{i} \wedge z, x_{i+1} \wedge z\right\} \subseteq \vee_{i=1}^{q_{i}\left(\wedge \wedge_{j=1}^{k_{i}} z_{i j}\right) \wedge z .}$ Therefore, by distributivity of hyperlattice $L$, we have $\left(x_{i} \wedge z\right) \theta_{q_{i+1}}\left(x_{i+1} \wedge z\right)$ and we have $u \theta^{*} v$. Similarly, we can easily show that $\theta^{*}$ is strongly regular with respect to $\vee$ and proof is completed.

The relation $\theta$ which is defined above is not transitive. Now, we investigate under what conditions $\theta$ is transitive.

Definition 3.3. Let $(L, \vee, \wedge)$ be a hyperlattice and $M \subseteq L$ be a nonempty subset of $L$. We say $M$ is a $\theta$-part of $L$, if for every $n \in \mathbb{N}, i=1,2, \ldots, n, \forall k_{i} \in \mathbb{N}, \forall\left(z_{i 1}, z_{i 2}, \ldots, z_{i k_{i}}\right) \in L^{k_{i}}$, we have

$$
\left.\underset{i=1}{\stackrel{n}{\vee}\left(\hat{j}_{i}\right.} z_{i j}\right) \cap M \neq \emptyset \Rightarrow \underset{i=1}{\vee}\left(\hat{j}_{j=1}^{k_{i}} z_{i j}\right) \subseteq M .
$$

Notice that for every $x \in L$, we define $P(x)=\{y \in L: x \theta y\}$.
Theorem 3.4. Let $(L, \vee, \wedge)$ be a hyperlattice. Then, the following conditions are equivalent:
(1) $\theta$ is transitive;
(2) for every $x \in L, \theta^{*}(x)=P(x)$;
(3) for every $x \in L, P(x)$ is a $\theta$-part of $L$.

Proof. (1) $\Rightarrow$ (2): If $y \in P(x)$, then $y \theta x$. Thus, $y \in \theta^{*}(x)$. Also, if $y \in \theta^{*}(x)$, then by transitivity of $\theta$, we have $y \in \theta(x)$. Thus, $y \in P(x)$ and therefore, $P(x)=\theta^{*}(x)$.
(2) $\Rightarrow$ (3): Let $P(x) \cap \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right) \neq \emptyset$ and $z \in \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right) \cap P(x)$. Then, for any $y \in \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)$, we have $\{y, z\} \subseteq \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)$. Thus, $y \theta_{n} z \in P(x)$. Therefore, $y \in \theta^{*}(x)=P(x)$ and $\vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right) \subseteq P(x)$. Thus, $P(x)$ is a $\theta$-part of $L$ and the proof is completed.
(3) $\Rightarrow$ (1): Let $x \theta y \theta z$. We show that $x \theta z$. Thus, there exist $n, n^{\prime} \in \mathbb{N}$ such that $x \theta_{n} y$ and $y \theta_{n^{\prime}} z$. Therefore, there exist $\left(k_{1}, k_{2}, \ldots, k_{n}\right) \in \mathbb{N}$, $\left(z_{i 1}, z_{i 2}, \ldots, z_{i n}\right) \in L^{n}, \quad$ such that $\{x, y\} \subseteq \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right) \quad$ and similarly there exist $\left(k_{1}^{\prime}, k_{2}^{\prime}, \ldots, k_{n}^{\prime}\right) \in \mathbb{N},\left(z_{i 1}^{\prime}, z_{i 2}^{\prime}, \ldots, z_{i n}^{\prime}\right) \in L^{n} \quad$ such that $\{y, z\} \subseteq \vee_{i=1}^{n^{\prime}}\left(\wedge_{j=1}^{k_{i}^{\prime}} z_{i j}^{\prime}\right)$. Thus, $x \in P(x) \cap \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)$. By the hypothesis, $\vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right) \subseteq P(x)$. Moreover, $y \in P(x) \cap \vee_{i=1}^{n^{\prime}}\left(\wedge_{j=1}^{k_{i}^{\prime}} z_{i j}^{\prime}\right)$ and by hypothesis $\vee_{i=1}^{n^{\prime}}\left(\wedge_{j=1}^{k_{i}^{\prime}} z_{i j}\right) \subseteq P(x)$. Thus, $z \in P(x)$ and $x \theta z$.

Theorem 3.5. Let $L$ be a distributive hyperlattice. Then, $L / \theta^{*}$ is a complete lattice.

Proof. Since $\theta^{*}$ is strongly regular, thus $L / \theta^{*}$ is a lattice. By the theorem of [4] it suffices to show that $L / \theta^{*}$ has a least element and for every $S \subseteq L / \theta^{*}, \vee S$ exists. If $L / \theta^{*}$ has a least element, the first condition is ensumered. Otherwise, the lack of a bottom element can be easily remedied by adding one. For every $S \subseteq L / \theta^{*}$, since $\theta^{*}$ is an equivalence relation and
by properties of $\theta^{*}$, we have $L=\bigcup_{a_{i} \in L} \theta^{*}\left(a_{i}\right)$, and if $a / \theta^{*} \neq b / \theta^{*}$, then $a / \theta^{*} \cap b / \theta^{*}=\emptyset$. Thus, for every $S \subseteq L / \theta^{*}$, we have $S=\bigcup_{a_{i} \in L} \theta^{*}\left(a_{i}\right)$, such that $\theta^{*}\left(a_{i}\right) \cap \theta^{*}\left(a_{j}\right)=\emptyset \quad$ for every $\quad \theta^{*}\left(a_{i}\right) \in S$. Therefore, $\vee S=\left\{y \in L / \theta^{*}, \forall x \in S, x \leq y\right\}$. Since for every $x \in S$, we have $x \in \theta^{*}\left(a_{i}\right)$, for one element $a_{i} \in L$, thus we consider the union of blocks which is contained $S$ by the equivalence relation $\theta^{*}$. This is the least upper bound of $S$ and $L / \theta^{*}$ is a complete lattice.

Theorem 3.6. Let $L$ be a finite distributive hyperlattice. Then, the relation $\theta^{*}$ is the smallest strongly regular relation on the hyperlattice $L$ such that $L / \theta^{*}$ is a complete lattice.

Proof. Let $v$ be a strongly regular relation on hyperlattice $L$ such that $L / v$ is a complete lattice. Then, consider the canonical map $\varphi: L \rightarrow L / v$. Suppose that $(x, y) \in \theta$. Then, there exist $\left(k_{1}, k_{2}, \ldots, k_{n}\right) \in \mathbb{N}^{n}$ and $\left(z_{i 1}, z_{i 2}, \ldots, z_{i n}\right) \in L^{n}$ such that $\{x, y\} \subseteq \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)$. Thus, $\{\varphi(x), \varphi(y)\}$ $\subseteq \varphi\left(\vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)\right)$. Since the cardinal of $\varphi \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)$ is equal to 1 , we have $\varphi(x)=\varphi(y)$. Therefore, $(x, y) \in v$ and $\theta \subseteq v, \theta^{*} \subseteq v$. Thus, $\theta^{*}$ is the smallest strongly regular relation on finite hyperlattice $L$ such that $L / \theta^{*}$ is a complete lattice.

Now, we consider the relation $v$ which is introduced in [11], and we investigate the quotient of an arbitrary hyperlattice $L$ with this relation is what type of lattices. Then, we obtain the connection between this relation and the relation $\theta$ which is defined before.

Definition 3.7[11]. Let $(L, \vee, \wedge)$ be a hyperlattice. Then, we set $\mathrm{v}_{1}=\{(x, x) \mid x \in L\}$ and for every integer $n>1, v_{n}$ is defined as follows:

$$
x v_{n} y \Leftrightarrow \exists\left(z_{1}, z_{2}, \ldots, z_{n}\right) \in L^{n}, \exists z \in \xi\left(\left\{z_{1}, z_{2}, \ldots, z_{n}\right\}\right),\{x, y\} \subseteq z
$$

where $\xi\left(\left\{z_{1}, z_{2}, \ldots, z_{n}\right\}\right)$ is the set of all finite combinations of $z_{1}, z_{2}, \ldots, z_{n}$
respect to $\vee$ and $\wedge$. We set $v=\bigcup_{n \geq 1} v_{n}$ and $v^{*}$ is the transitive closure of $v$. By theorem of [11] $v^{*}$ is a strongly regular relation on a hyperlattice $L$, and therefore $L / \theta^{*}$ is a lattice.

Theorem 3.8. Let $(L, \vee, \wedge)$ be a hyperlattice and $v^{*}$ be an equivalence relation which is defined above. Then, $L / v^{*}$ is a Boolean lattice.

Proof. Let $L$ be a hyperlattice and $a, b, c \in L$ be arbitrary elements. Then, we have $\{a \wedge(b \vee c),(a \wedge b) \vee(a \wedge c)\} \subseteq \xi\{a, b, c\}$. Therefore, $L / v^{*}$ is a distributive lattice. Now, if $L / v^{*}$ is bounded, the second condition of Boolean lattice is ensumered, otherwise by adding elements such as $v^{*}(0), v^{*}(1)$ to $L / v^{*}$, we have a bounded lattice $L / v^{*}$. Let $v^{*}(a) \in L / v^{*}$. Then, we have $\{a \vee a, 1\} \subseteq \xi\{a, 1\} \quad$ and $\quad\{a \wedge a, 0\} \subseteq \xi\{a, 0\}$. Therefore, $(a \vee a) v^{*} 1$ and $(a \wedge a) v^{*} 0$. Thus, $v^{*}(a) \vee v^{*}(a)=v^{*}(1)$ and $v^{*}(a) \wedge v^{*}(a)=v^{*}(0)$. Hence, the complement of every element in $L / v^{*}$ exists and $L / v^{*}$ is a Boolean lattice.

Theorem 3.9. Let $L$ be a hyperlattice and $v^{*}$ be an equivalence relation on $L$ which is defined above. Then, $L / v^{*}$ is a complete lattice.

Proof. The proof is similar to the proof of Theorem 3.6.
Corollary 3.10. Let L be a distributive hyperlattice. Then, $\theta^{*}=v^{*}$.
Proof. Clearly by definitions $\theta^{*} \subseteq v^{*}$. It suffices to prove $v^{*} \subseteq \theta^{*}$. By the corollary in [11], $v^{*}$ is the smallest equivalence relation such that the quotient $L / v^{*}$ is a lattice. So, by previous theorem, we have $v^{*} \subseteq \theta^{*}$ and proof is complete.

Notice that by [6] a residuated lattice is defined. Now, we show that by the relation $v^{*}$ which is defined above, we can make this category of lattices from an arbitrary hyperlattice $L$.

Theorem 3.11. Let $L$ be a hyperlattice and $v^{*}$ be the relation which is defined above. Then, $L / v^{*}$ is a residuated lattice.

Proof. By the proof of Theorem 3.8, the first condition of residuated lattice is ensumered. Now, we define in $L / v^{*}$ two binary operation $\odot$ and $\rightarrow$ as follows: for any $v^{*}(a), v^{*}(b) \in L / v^{*}, v^{*}(a) \rightarrow v^{*}(b)=\left\{v^{*}(c) ; v^{*}(a) \wedge v^{*}(b)\right.$ $\left.\ll v^{*}(c)\right\}$ and $v^{*}(a) \odot v^{*}(b)=\left\{v^{*}(c) ; v^{*}(a) \vee v^{*}(b) \ll v^{*}(c)\right\}$. For two arbitrary sets $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$. Now, by these definitions for any $v^{*}(a), v^{*}(b) \in L / v^{*}$ we have $v^{*}(a) \odot v^{*}(b)$ $=v^{*}(a) \odot v^{*}(b)$ and we can easily prove binary operation $\odot$ is associative. Now, for arbitrary element $x \in L, v^{*}(x) \odot v^{*}(1)=\left\{v^{*}(d) ; v^{*}(x) \vee v^{*}(1) \ll v^{*}(d)\right\}$. Since $\quad x \in v^{*}(x \vee 1) \quad$ and $\quad 1 \in v^{*}(x), x \leq 1$, thus $\quad v^{*}(x) \in v^{*}(x) \odot v^{*}(1)$. Therefore, $\left(L / v^{*}, \odot\right)$ is a commutative monoid. Let $v^{*}(b), v^{*}(b), v^{*}(c) \in L / v^{*}$. We have $v^{*}(1) \in v^{*}(b) \rightarrow v^{*}(c)$. Therefore, $v^{*}(1) \leq v^{*}(b) \rightarrow v^{*}(c)$. Thus, third condition of residuated lattice in $L / v^{*}$ holds and $L / v^{*}$ is a residuated lattice. $\square$

Notice that in all hyperlattices the following proposition is not true. So, in the category of $H_{v}$-lattices, the following proposition is true.

Proposition 3.12. Let $L$ be an $H_{v}$-lattice. Then, we have $\theta_{n} \subseteq \theta_{n+1}$ (for $n \geq 1$ ).

Proof. If $x \theta_{n} y$ then there exist $\left(z_{i 1}, z_{i 2}, \ldots, z_{i n}\right)$ and $\left(k_{1}, k_{2}, \ldots, k_{n}\right) \in \mathbb{N}^{n}$ such that $\{x, y\} \subseteq \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)$. Since $z_{n k_{n}} \in L$ and $L$ is a $H_{v}$-lattice, by reproduction axiom $z_{n k_{n}} a \vee b$, where $a, b \in L$. Thus, we set for $i=1,2, \ldots$, $n-1, j=1,2, \ldots, k_{i}, z_{i j}^{\prime}=z_{i j} \quad$ and $\quad k_{i}^{\prime}=k_{i}, k_{n+1}^{\prime}=k_{n+1}, z_{n j}^{\prime}=z_{n j}, z_{n+1, j}^{\prime}=z_{n j}$. Thus, $\{x, y\} \subseteq \vee_{i=1}^{n+1}\left(\wedge_{j=1}^{k_{i}^{\prime}} z_{i j}^{\prime}\right)$ and therefore $x \theta_{n} y$.

Corollary 3.13. Let $L$ be an $H_{v}$-lattice. Then, for all $n \geqslant 1, \theta=\theta_{n}$.
Proof. It suffices to prove that $\theta \subseteq \theta_{n}$. Let $x \theta y$. Thus, $\exists m \in \mathbb{N}$ such that $x \theta_{m} y$. If $m \leq n$, then by the previous proposition we have $\theta_{m} \subseteq \theta_{n}$. If $m>n$ then it can easily proved that there exist $s \in \vee_{i=1}^{m}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)$ such that $\{x, y\} \subseteq \vee_{i=1}^{n-1}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right) \vee s$. Thus, $x \theta_{n} y$.

Definition 3.14. A hyperlattice $L$ is said to be $\theta_{n}$-complete if $\forall\left(k_{1}, k_{2}, \ldots, k_{n}\right) \in \mathbb{N}^{n}, i=1,2, \ldots, n, \forall\left(x_{i 1}, x_{i 2}, \ldots, x_{i k_{i}}\right) \in L^{K}$, we have

$$
\theta\left(\underset{i=1}{\vee}\left(\hat{k}_{j=1}^{k_{i}} z_{i j}\right)\right)=\stackrel{n}{\vee}\left(\hat{i=1}_{k_{i}}^{\wedge_{j=1}} z_{i j}\right) .
$$

Theorem 3.15. Let $L$ be a $\theta_{n}$-complete $H_{v}$-lattice. Then, for all $\left(k_{1}, k_{2}, \ldots, k_{n}\right) \in \mathbb{N}^{n},\left(z_{i 1}, \ldots, z_{i k_{i}}\right) \in L^{k_{i}}, \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)$ is a $\theta$-part of $L$.

Proof. Consider $n \in \mathbb{N},\left(k_{1}^{\prime}, \ldots, k_{m}^{\prime}\right) \in \mathbb{N}^{m}$, and $\left(y_{i 1}, \ldots, y_{i k^{\prime}}\right) \in L^{k_{i}^{\prime}}$ such that the condition $\vee_{i=1}^{m}\left(\wedge_{j=1}^{k_{i}^{\prime}} y_{i j}\right) \cap \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right) \neq \emptyset$ holds. Thus, there exists $x \in \vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}^{\prime}} y_{i j}\right) \cap \vee_{i=1}^{n}\left(\wedge_{j=1}^{n} z_{i j}\right)$. Let $y \in \vee_{i=1}^{m}\left(\wedge_{j=1}^{k_{i}^{\prime}} y_{i j}\right)$. Then, we have $x \theta_{m} y$. Thus, by Corollary $3.13 x \theta y$. Therefore, $y \in \theta(x) \subseteq \theta\left(\vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}} z_{i j}\right)\right)$ $=\vee_{i=1}^{n} \wedge_{j=1}^{k_{i}} z_{i j}$ and $\vee_{i=1}^{n}\left(\wedge_{j=1}^{k_{i}^{\prime}} y_{i j}\right) \subseteq \vee_{i=1}^{n}\left(\wedge_{j=1}^{n} z_{i j}\right)$, the proof is completed.

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