Advances and Applications in Mathematical Sciences
Volume 21, Issue 7, May 2022, Pages 3595-3615

SPHERICAL FUZZY PREFERENCE RELATION AND ITS APPLICATION TO GROUP DECISION MAKING WITH CONSENSUS ANALYSIS

V. DHANALAKSHMI

Department of Mathematics
Stella Maris College
Chennai, Tamil Nadu, India

Abstract

In a group decision-making process, the consensus within the experts is required. Consensus analysis is constructed using similarity functions which measures the closeness of the experts preferences. Spherical fuzzy sets are more generalized form of fuzzy sets that can handle many situations with its membership, non-membership and neutral membership grades. Also the requirement for spherical fuzzy sets is that the sum of squares of the above mentioned membership grades to be less than or equal to 1. This widens the range of the three grades for the experts in making decision valuations. In this paper, distance and similarity measures on spherical fuzzy sets and the spherical fuzzy preference relation are discussed. Also the aggregation of several spherical fuzzy preference relations are discussed. An algorithm is proposed to solve the group decision-making problem after verifying the consensus of the experts. Finally the algorithm is demonstrated with an illustration.

1. Introduction

Decision making is an immense part of human culture and applied to fields like financial matters, engineering, and management. With the improvement of science and technology, the uncertainty likewise plays a predominant factor during the decision making process. Further, gathering exact information during the decision making process is challenging. A large portion of the data gathered from the different sources are either uncertain or imprecise, thus leading to an incorrect result. Hence, it is important to identify the proper method to handle these imprecise and uncertain

2020 Mathematics Subject Classification: 03E72, 90B50.
Keywords: Spherical Fuzzy Sets, Fuzzy Preference Relation, Group Decision Making, Consensus Analysis.
Received July 27, 2021; Accepted January 1, 2022

information.

Fuzzy set theory was introduced by Zadeh [25] for representing and manipulating data that was not precise and designed to mathematically represent uncertainty and vagueness. For the universal set X, a fuzzy subset A is usually denoted by

$$A = \{(x, \mu_A(x)) \mid x \in X, \mu_A(x) \in [0, 1]\}$$

where $\mu_A(x)$ is the degree of membership of the element x into the fuzzy set A. The set of all fuzzy sets on a set X is denoted by $\mathcal{F}(X)$. If for a fuzzy set $A \in \mathcal{F}(X)$, $\mu_A(x) = 0, \forall x \in X$, then A is an empty set. If the set $\{x \in X \mid \mu_A(x) > 0\}$ is finite, then the fuzzy set A is called a discrete fuzzy set. In the case when $\mu_A(x) \in \{0, 1\}, \forall x \in X$ then the fuzzy set A is reduced to a crisp set and $\mu_A = \chi_A$. But in the fuzzy sets, the non-membership value of an element could not be derived from its membership value. Hence to overcome this, Atanassov developed intuitionistic fuzzy sets [2] by introducing membership degree and non-membership degree for every element of the universal set, which adds upto 1. Later, Atanassov and Gargov [11] developed the interval-valued intuitionistic fuzzy sets as an extension of intuitionistic fuzzy sets. But in some practical instances, there arises situations in which the sum of the two membership grades exceed 1. Yager developed the Pythagorean fuzzy sets [18] to overcome this situation with the condition that the sum of squares of the two membership grades is upto 1. Still these concepts did not capture the neutrality in the situation, hence Cuong and Kreinovich [5] developed picture fuzzy set by utilizing three grades, namely membership, non-membership and neutral membership degree whose sum does not exceed 1. Picture fuzzy sets was more generalized and was widely applied in various fields [6, 3]. Still there are situations where the sum of the three considered membership grades exceeds 1, which cannot be handled by these fuzzy sets. Kutlu Gundogdu and C. Kahraman [8] introduced the idea of spherical fuzzy sets, where the sum of squares of the three membership grades does not exceed 1. Spherical fuzzy sets is more realistic and able to handle many different situations over the other fuzzy sets. Since its introduction, spherical fuzzy sets gained attention of researchers from different fields and is applied for various problems. Some of the applications are: multi-criteria decision making problems [1, 26, 4], medical diagnosis problem [15], pattern recognition [21, 22, 23, 14], clustering [20, 17], selection problems [12, 9, 7, 13, 16],

The objective of this paper is to utilize the concept of spherical fuzzy sets to a decision-making problem in which the consensus analysis is rendered using a similarity measure which in turn is based on the distance measure between the spherical fuzzy sets. Section 2 discusses the definition, arithmetic operations and score function on spherical fuzzy sets. In section 3, distance measure and similarity measure and their properties are developed to carry out consensus analysis. Section 4 discusses the spherical fuzzy preference relations on the cartesian product of universal set $U \times U$ and its aggregation. In section 5, an algorithm is presented to solve a decision making problem, where the evaluations are represented in spherical fuzzy preference relations and illustrates the same with an example. Finally the conclusion is presented in section 6.

2. Preliminaries

For the sake of completeness we recall the required definitions and results.

Definition 2.1 [8]. Spherical fuzzy sets $\widetilde{A}_S$ of the universe of discourse U is given by

$$\widetilde{A}_S = \{(x, \langle \mu_{\widetilde{A}_S}(x), \nu_{\widetilde{A}_S}(x), \eta_{\widetilde{A}_S}(x) \rangle)\} \tag{2.1}$$

where

$$\mu_{\widetilde{A}_S} : U \to [0, 1], \nu_{\widetilde{A}_S} : U \to [0, 1], \eta_{\widetilde{A}_S} : U \to [0, 1] \tag{2.2}$$

and

$$0 \le \mu_{\widetilde{A}_S}(x)^2 + \nu_{\widetilde{A}_S}(x)^2 + \eta_{\widetilde{A}_S}(x)^2 \le 1, \forall\; x \in U \tag{2.3}$$

for each x, the numbers $\mu_{\widetilde{A}_S}(x), \nu_{\widetilde{A}_S}(x)$ and $\eta_{\widetilde{A}_S}(x)$ are the degree of membership, non-membership and neutral(hesitancy)-membership of x to $\widetilde{A}_S$, respectively.

Remark 2.1.1. On the surface of the unit sphere 2.3 becomes

$$\mu_{\tilde{A}_S}(x)^2 + \nu_{\tilde{A}_S}(x)^2 + \eta_{\tilde{A}_S}(x)^2 = 1, \forall\ x \in U$$

2. The quantity $\rho_{\tilde{A}_S}(x) = \sqrt{1 - (\mu_{\tilde{A}_S}(x)^2 + \nu_{\tilde{A}_S}(x)^2 + \eta_{\tilde{A}_S}(x)^2)}$ is considered as the degree of refusal membership.

Definition 2.2. The complement of a spherical fuzzy set

$\tilde{A}_S = \{(x, \langle \mu_{\tilde{A}_S}(x), \nu_{\tilde{A}_S}(x), \eta_{\tilde{A}_S}(x) \rangle)\}$ is defined as

$\tilde{A}_S^c = \{(x, \langle \nu_{\tilde{A}_S}(x), \mu_{\tilde{A}_S}(x), \eta_{\tilde{A}_S}(x) \rangle)\}$

Definition 2.3 [8]**.** Let $\tilde{A}_S$ and $\tilde{B}_S$ be two spherical fuzzy sets and λ be a positive real number, the arithmetic operations are defined as follows:

$$\tilde{A}_S \oplus \tilde{B}_S = \{(x, \langle \sqrt{\mu^2_{\tilde{A}_S}(x) + \mu^2_{\tilde{B}_S}(x) - \mu^2_{\tilde{A}_S}(x)\mu^2_{\tilde{B}_S}(x)}, \nu_{\tilde{A}_S}(x)\nu_{\tilde{B}_S}(x),$$

$$\sqrt{\left[1 - \mu^2_{\tilde{B}_S}(x)\right]\eta^2_{\tilde{A}_S}(x) + \left[1 - \mu^2_{\tilde{A}_S}(x)\right]\eta^2_{\tilde{B}_S}(x) - \eta^2_{\tilde{A}_S}(x)\eta^2_{\tilde{B}_S}(x)} \rangle)\}$$

$$\tilde{A}_S \otimes \tilde{B}_S = \{(x, \langle \mu_{\tilde{A}_S}(x)\mu_{\tilde{B}_S}(x), \sqrt{\nu^2_{\tilde{A}_S}(x) + \nu^2_{\tilde{B}_S}(x) - \nu^2_{\tilde{A}_S}(x)\nu^2_{\tilde{B}_S}(x)},$$

$$\sqrt{\left[1 - \nu^2_{\tilde{B}_S}(x)\right]\eta^2_{\tilde{A}_S}(x) + \left[1 - \nu^2_{\tilde{A}_S}(x)\right]\eta^2_{\tilde{B}_S}(x) - \eta^2_{\tilde{A}_S}(x)\eta^2_{\tilde{B}_S}(x)} \rangle)\}$$

$$\lambda\tilde{A}_S = \{(x, \langle \sqrt{1 - (1 - \eta^2_{\tilde{A}_S}(x)(x))^{\lambda}}, \nu^{\lambda}_{\tilde{A}_S}(x), \sqrt{\left[1 - \mu^2_{\tilde{A}_S}(x)\right] - \left[1 - \mu^2_{\tilde{A}_S}(x) - \eta^2_{\tilde{A}_S}(x)\right]^{\lambda}} \rangle)\}$$

$$\tilde{A}_S^{\lambda} = \{(x, \langle \mu^{\lambda}_{\tilde{A}_S}(x), \sqrt{1 - (1 - \nu^2_{\tilde{A}_S}(x))^{\lambda}}, \sqrt{\left[1 - \nu^2_{\tilde{A}_S}(x)\right]^{\lambda} - \left[1 - \nu^2_{\tilde{A}_S}(x) - \eta^2_{\tilde{A}_S}(x)\right]^{\lambda}} \rangle)\}$$

Theorem 2.1 [10]**.** *The following properties hold good for any two spherical fuzzy sets* $\tilde{A}_S$ *and* $\tilde{B}_S$ *and* $\lambda, \lambda_1, \lambda_2$ *being positive real numbers*:

(1) $\tilde{A}_S \oplus \tilde{B}_S = \tilde{B}_S \oplus \tilde{A}_S$

(2) $\tilde{A}_S \otimes \tilde{B}_S = \tilde{B}_S \otimes \tilde{A}_S$

(3) $\lambda(\widetilde{A}_S \oplus \widetilde{B}_S) = \lambda\widetilde{A}_S \oplus \lambda\widetilde{B}_S$

(4) $\lambda_1\widetilde{A}_S \oplus \lambda_2\widetilde{A}_S = (\lambda_1 + \lambda_2)\widetilde{A}_S$

(5) $(\widetilde{A}_S \otimes \widetilde{B}_S)^{\lambda} = \widetilde{A}_S^{\lambda} \otimes \widetilde{B}_S^{\lambda}$

(6) $\widetilde{A}_S^{\lambda_1} \otimes \widetilde{A}_S^{\lambda_2} = \widetilde{A}_S^{\lambda_1 + \lambda_2}$

Definition 2.4. Score function on spherical fuzzy sets is defined as

$$Score(\widetilde{A}_S) = (\mu_{\widetilde{A}_S}(x) - \eta_{\widetilde{A}_S}(x))^2 - (\nu_{\widetilde{A}_S}(x) - \eta_{\widetilde{A}_S}(x))^2 \tag{2.4}$$

The above defined score functions is used to sort the spherical fuzzy sets as $\widetilde{A}_S \prec \widetilde{B}_S$ if and only if

(i) $Score(\widetilde{A}_S) < Score(\widetilde{B}_S)$ or

(ii) $Score(\widetilde{A}_S) = Score(\widetilde{B}_S)$ and $\rho_{\widetilde{A}_S}(x) > \rho_{\widetilde{B}_S}(x)$

3. Distance and Similarity Measure between two Spherical Fuzzy Sets

Let $U = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourse, $\widetilde{A}_S$ and $\widetilde{B}_S$ be two spherical fuzzy sets in U, then the distance between $\widetilde{A}_S$ and $\widetilde{B}_S$ is defined as:

Distance between the two spherical fuzzy elements $\widetilde{A}_S(x)$ and $\widetilde{B}_S(x)$ can be defined considering the two cases, where both the points lie on the unit sphere [24] and otherwise

$$\begin{cases} \dfrac{2}{\pi}\arccos\begin{bmatrix} \mu_{\tilde{A}_S}(x)\mu_{\tilde{B}_S}(x)+\nu_{\tilde{A}_S}(x)\nu_{\tilde{B}_S}(x) \\ +\eta_{\tilde{A}_S}(x)\eta_{\tilde{B}_S}(x) \end{bmatrix} & \text{if } \begin{matrix}\rho_{\tilde{A}_S}(x)=0\\ \rho_{\tilde{B}_S}(x)=0\end{matrix} \\ \dfrac{2}{\pi}\arccos\left[1+\dfrac{1}{2}\begin{bmatrix} (\mu_{\tilde{A}_S}(x)-\mu_{\tilde{B}_S}(x)) \\ +(\nu_{\tilde{A}_S}(x)-\nu_{\tilde{B}_S}(x)) \\ +(\eta_{\tilde{A}_S}(x)-\eta_{\tilde{B}_S}(x)) \\ +(\rho_{\tilde{A}_S}(x)-\rho_{\tilde{B}_S}(x)) \end{bmatrix}\right], & \text{if } \begin{matrix}\rho_{\tilde{A}_S}\neq 0\\ \text{or}\\ \rho_{\tilde{B}_S}\neq 0\end{matrix} \end{cases}$$

Both the cases simplifies to the following expression and thus the distance between the two spherical fuzzy elements $\tilde{A}_S(x)$ and $\tilde{B}_S(x)$ is defined as

$$dis(\tilde{A}_S(x), \tilde{B}_S(x)) = \frac{2}{\pi}\arccos\begin{bmatrix} \mu_{\tilde{A}_S}(x)\mu_{\tilde{B}_S}(x)+\nu_{\tilde{A}_S}(x)\nu_{\tilde{B}_S}(x) \\ \eta_{\tilde{A}_S}(x)\eta_{\tilde{B}_S}(x)+\rho_{\tilde{A}_S}(x)\rho_{\tilde{B}_S}(x) \end{bmatrix} \tag{3.1}$$

Thus, the distance measure between the two spherical fuzzy sets $\tilde{A}_S$ and $\tilde{B}_S$ is

$$d(\tilde{A}_S, \tilde{B}_S) = \sum_{x\in U} dis(\tilde{A}_S(x), \tilde{B}_S(x))$$

Let the spherical fuzzy elements $\tilde{A}_S(x)$ and $\tilde{B}_S(x)$ be denoted by a_x and b_x respectively as $a_x = \langle \mu_{a_x}, \nu_{a_x}, \eta_{a_x}\rangle$ and $b_x = \langle \mu_{b_x}, \nu_{b_x}, \eta_{b_x}\rangle$ with the refusal degrees ρ_{a_x} and ρ_{b_x}.

Definition 3.1. For the given spherical fuzzy elements a_x and b_x and b_x^c, the complement of b_x, define

$$s(a_x, b_x) = \begin{cases} 0.5 & \text{if } a_x = b_x = b_x^c \\ \dfrac{dis(a_x, b_x^c)}{dis(a_x, b_x)+dis(a_x, b_x^c)} & \text{otherwise.} \end{cases} \tag{3.2}$$

the degree of similarity between the spherical fuzzy elements a_x and b_x.

Theorem 3.1. *The similarity measure satisfies the following properties*:

1. $0 \leq s(a_x, b_x) \leq 1$

2. $s(a_x, b_x) = s(b_x, a_x) = s(a_x^c, b_x^c)$

3. $s(a_x, b_x^c) = s(a_x^c, b_x)$

4. $s(a_x, b_x) = 1 \Leftrightarrow a_x = b_x$

5. $s(a_x, b_x) > 0.5 \Leftrightarrow dis(a_x, b_x) < dis(a_x, b_x^c)$ *i.e., a_x is more similar to b_x than b_x^c*

6. $s(a_x, b_x) = 0.5 \Leftrightarrow dis(a_x, b_x) = dis(a_x, b_x^c)$ *i.e., a_x is to the same extent similar to b_x and b_x^c*

7. $s(a_x, b_x) < 0.5 \Leftrightarrow dis(a_x, b_x) < dis(a_x, b_x^c)$ *i.e., a_x is more similar to b_x^c than b_x*

8. $s(a_x, b_x) = 0 \Leftrightarrow a_x = b_x^c$ *i.e., a_x and b_x are completely dissimilar.*

Proof. 1. According to the arc-cosine value $dis(a_x, b_x)$ lie between 0 and 1 and hence

$$0 \leq s(a_x, b_x) \leq 1$$

2. The case when $a_x = b_x = b_x^c$ is obvious. Now consider the case $a_x \neq b_x \neq b_x^c$

$$\begin{aligned} dis(b_x, a_x) &= \frac{2}{\pi} \arccos [\mu_{b_x} \mu_{a_x} + \nu_{b_x} \nu_{a_x} + \eta_{b_x} \eta_{a_x} + \rho_{b_x} \rho_{a_x}] \\ &= \frac{2}{\pi} \arccos [\mu_{a_x} \mu_{b_x} + \nu_{a_x} \nu_{b_x} + \eta_{a_x} \eta_{b_x} + \rho_{a_x} \rho_{b_x}] \\ &= dis(a_x, b_x). \end{aligned}$$

Also,

$$dis(b_x, a_x^c) = \frac{2}{\pi} \arccos [\mu_{b_x} \nu_{a_x} + \nu_{b_x} \mu_{a_x} + \eta_{b_x} \eta_{a_x} + \rho_{b_x} \rho_{a_x}]$$

$$= \frac{2}{\pi} \arccos [\nu_{b_x} \mu_{a_x} + \mu_{b_x} \nu_{a_x} + \eta_{b_x} \eta_{a_x} + \rho_{b_x} \rho_{a_x}]$$

$$= \frac{2}{\pi} \arccos [\mu_{a_x} \nu_{b_x} + \nu_{a_x} \mu_{b_x} + \eta_{a_x} \eta_{b_x} + \rho_{a_x} \rho_{b_x}]$$

$$= dis(a_x, b_x^c).$$

Thus,

$$s(a_x, b_x) = \frac{dis(a_x, b_x^c)}{dis(a_x, b_x) + dis(a_x, b_x^c)}$$

$$= \frac{dis(b_x, a_x^c)}{dis(b_x, a_x) + dis(b_x, a_x^c)}$$

$$= s(b_x, a_x)$$

and

$$s(a_x^c, b_x^c) = \frac{dis(a_x, (b_x^c)^c)}{dis(a_x^c, (b_x)^c) + dis(a_x^c, (b_x^c)^c)}$$

$$= \frac{dis(a_x^c, b_x)}{dis(a_x^c, b_x) + dis(a_x^c, b_x)}$$

$$= \frac{dis(b_x, a_x^c)}{dis(b_x, a_x^c) + dis(b_x, a_x^c)}$$

$$= s(b_x, a_x)$$

3.

$$s(a_x, b_x^c) = \frac{dis(a_x, (b_x^c)^c)}{dis(a_x, b_x^c) + dis(a_x, (b_x^c)^c)}$$

$$= \frac{dis(a_x, b_x)}{dis(a_x, b_x^c) + dis(a_x, b_x)}$$

$$= \frac{dis(b_x, a_x)}{dis(b_x, a_x^c) + dis(b_x, a_x)}$$

$$= \frac{dis(b_x, (a_x^c)^c)}{dis(b_x, a_x^c) + dis(b_x, (a_x^c)^c)}$$

$$= s(b_x, a_x^c) = s(a_x^c, b_x)$$

4.

$$s(a_x, b_x) = 1 \Leftrightarrow \frac{dis(a_x, b_x^c)}{dis(a_x, b_x) + dis(a_x, b_x^c)} = 1$$

$$\Leftrightarrow dis(a_x, b_x^c) = dis(a_x, b_x) + dis(a_x, b_x^c)$$

$$\Leftrightarrow dis(a_x, b_x^c) = dis(a_x, b_x) + dis(a_x, b_x^c)$$

$$\Leftrightarrow dis(a_x, b_x) = 0$$

$$\Leftrightarrow \frac{2}{\pi} \arccos [\mu_{a_x}\mu_{b_x} + \nu_{a_x}\nu_{b_x} + \eta_{a_x}\eta_{b_x} + \rho_{a_x}\rho_{b_x}] = 0$$

$$\Leftrightarrow \mu_{a_x}\mu_{b_x} + \nu_{a_x}\nu_{b_x} + \eta_{a_x}\eta_{b_x} + \rho_{a_x}\rho_{b_x} = 1$$

$$\Leftrightarrow a_x = b_x \text{ as } \langle \mu_{a_x}, \nu_{a_x}, \eta_{a_x}, \rho_{a_x} \rangle$$

$$\text{and } \langle \mu_{b_x}, \nu_{b_x}, \eta_{b_x}, \rho_{b_x} \rangle \text{ are unit vectors}$$

5.

$$s(a_x, b_x) > 0.5 \Leftrightarrow \frac{dis(a_x, b_x^c)}{dis(a_x, b_x) + dis(a_x, b_x^c)} > 0.5$$

$$\Leftrightarrow 2dis(a_x, b_x^c) > dis(a_x, b_x) + dis(a_x, b_x^c)$$

$$\Leftrightarrow dis(a_x, b_x^c) > dis(a_x, b_x)$$

i.e., a_x is more similar to b_x than b_x^c

6. $s(a_x, b_x) = 0.5 \Leftrightarrow dis(a_x, b_x) = dis(a_x, b_x^c)$. This is obvious from the

definition of similarity measure.

7.

$$s(a_x, b_x) < 0.5 \Leftrightarrow \frac{dis(a_x, b_x^c)}{dis(a_x, b_x) + dis(a_x, b_x^c)} < 0.5$$

$$\Leftrightarrow 2dis(a_x, b_x^c) < dis(a_x, b_x) + dis(a_x, b_x^c)$$

$$\Leftrightarrow dis(a_x, b_x^c) < dis(a_x, b_x)$$

i.e., a_x is more similar to b_x^c than b_x.

8.

$$s(a_x, b_x) = 0 \Leftrightarrow dis(a_x, b_x^c) = 0 \Leftrightarrow a_x = b_x^c$$

i.e., a_x and b_x are complement of each other. □

Based on 3.1, we define the degree of similarity between two spherical fuzzy sets:

Definition 3.2. Let $\widetilde{A}_S$ and $\widetilde{B}_S$ be two spherical fuzzy sets in U, and $\widetilde{B}_S^c$ be the complement of $\widetilde{B}_S$, then

$$\widetilde{s}(\widetilde{A}_S, \widetilde{B}_S) = \frac{1}{n}\sum_{i=1}^{n} s(\widetilde{A}_S(x_i), \widetilde{B}_S(x_i)) \tag{3.3}$$

is called the degree of similarity between $\widetilde{A}_S$ and $\widetilde{B}_S$

From 3.1, we have,

(1) $0 \leq \widetilde{s}(\widetilde{A}_S, \widetilde{B}_S) \leq 1$

(2) $\widetilde{s}(\widetilde{A}_S, \widetilde{B}_S) = \widetilde{s}(\widetilde{B}_S, \widetilde{A}_S) = \widetilde{s}(\widetilde{B}_S^c, \widetilde{A}_S^c)$

(3) $\widetilde{s}(\widetilde{B}_S^c, \widetilde{A}_S^c) = \widetilde{s}(\widetilde{A}_S^c, \widetilde{B}_S^c)$

4. Spherical Fuzzy Preference Relation

Definition 4.1 [19]**.** A fuzzy preference relation B on the set $A = \{A_1, \ldots, A_n\}$ is represented by a matrix $B = [b_{ij}]_{n \times n}$, where b_{ik} is the intensity of preference of A_i over A_k, and satisfies:

$$b_{ik} + b_{ki} = 1, \, b_{ij} \in [0, 1], \, \forall \; A_i, \, A_k \in A$$

Definition 4.2. Spherical fuzzy preference relation on the set $U = \{x_1, \ldots, x_n\}$ represented by a matrix $\widetilde{R}_S = [\widetilde{r}_{ij}]_{n \times n}$, where $\widetilde{r}_{ij} = \langle (x_i, x_j), \mu_{\widetilde{R}_S}(x_i, x_j), \nu_{\widetilde{R}_S}(x_i, x_j), \eta_{\widetilde{R}_S}(x_i, x_j) \rangle, \forall i, \, j = 1, 2, \ldots, n.$ In short, let us represent $\widetilde{r}_{ij} = (\mu_{ij}, \nu_{ij}, \eta_{ij}), \forall i, \, j = 1, 2, \ldots, n,$ where μ_{ij} denotes the degree to which the object x_i is preferred to the object x_j, ν_{ij} denotes the degree to which the object x_i is not preferred to the object x_j, η_{ij} denotes the degree of neutrality in preference and $\rho_{ij} = \sqrt{1 - (\mu_{ij}^2 + \nu_{ij}^2 + \eta_{ij}^2)}$ is the refusal degree of preference, with the following conditions: $\mu_{ij} = \nu_{ji}, \, \mu_{ii} = \nu_{ii} = \eta_{ii} = 0.5, \, \forall \; i, \, j = 1, 2, \ldots, n.$

Remark 4.1. $\rho_{ii} = 0.5.$

Definition 4.3. Let $\widetilde{R}_S = [\widetilde{r}_{ij}]_{n \times n}$ be a spherical preference relation, then it is called a consistent spherical preference relation, if it satisfies $r_{ij} = r_{ik} r_{kj}, \, \forall \, i, \, j, \, k = 1, 2, \ldots, n.$

In a consistent spherical preference relation, the alternative x_i is preferred to x_j with a spherical fuzzy value r_{ij} equal to the product of the preferences when using an intermediate alternative x_k.

Lemma 4.1. $\widetilde{R}_S = \widetilde{R}_S^T.$

Proof. The proof follows from 4.2. □

Using the 2.1 and mathematical induction, we have

Theorem 4.1. *Let* $\widetilde{R}_S^{(k)} = [\widetilde{r}_{ij}^{(k)}]_{n \times n}, \, k = 1, 2, \ldots, m$ *be* m *spherical fuzzy* *preference relations on* $U = \{x_1, x_2, \ldots, x_n\}$, *where* $\widetilde{r}_{ij}^{(k)} = (\mu_{ij}^{(k)}, \nu_{ij}^{(k)}, \eta_{ij}^{(k)})$, *and let* $w = (w_1, w_2, \ldots, w_m)^T$ *be the weight vector of the spherical preference relations* $\widetilde{R}_S^{(1)}, \widetilde{R}_S^{(2)}, \ldots, \widetilde{R}_S^{(m)}, w_k > 0, k = 1, 2, \ldots, m, \sum_{k=1}^{m} w_k = 1,$ *then* $\widetilde{R}_S = [\widetilde{r}_{ij}]_{n\times n}$ *is also a spherical fuzzy preference relation, where* $\widetilde{r}_{ij}$ *is the aggregated value of* $\widetilde{r}_{ij}^{(1)}, \widetilde{r}_{ij}^{(2)}, \ldots, \widetilde{r}_{ij}^{(m)}$ *obtained by using the spherical fuzzy weighted arithmetic averaging operator*:

$$\widetilde{r}_{ij} = \sum_{k=1}^{m} w_k \widetilde{r}_{ij}^{(k)}, \quad i, j = 1, 2, \ldots, n \tag{4.1}$$

or by using the spherical fuzzy weighted geometric averaging operator:

$$\widetilde{r}_{ij} = \prod_{k=1}^{m} (\widetilde{r}_{ij}^{(k)})^{w_k}, \quad i, j = 1, 2, \ldots, n \tag{4.2}$$

In particular, if $w = (1/m, 1/m, \ldots, 1/m)^T$, then 4.1 and 4.2 are, respectively, reduced to the spherical fuzzy arithmetic averaging operator:

$$\widetilde{r}_{ij} = \frac{1}{m} \sum_{k=1}^{m} \widetilde{r}_{ij}^{(k)}, \quad i, j = 1, 2, \ldots, n \tag{4.3}$$

and the intuitionistic fuzzy geometric averaging operator

$$\widetilde{r}_{ij} = \left(\prod_{k=1}^{m} \widetilde{r}_{ij}^{(k)} \right)^{\frac{1}{m}}, i, j = 1, 2, \ldots, n \tag{4.4}$$

Definition 4.4. Let $\widetilde{R}_S^{(k)} = [\widetilde{r}_{ij}^{(k)}]_{n\times n}, k = 1, 2, \ldots, m$ be m spherical fuzzy preference relations on $U = \{x_1, x_2, \ldots, x_n\}$, and $\widetilde{R}_S = [\widetilde{r}_{ij}]_{n\times n}$ be their aggregated spherical fuzzy preference relations, then

$$\widetilde{s}(\widetilde{R}_S^{(k)}, \widetilde{R}_S) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} s(r_{ij}^{(k)}, r_{ij}) \tag{4.5}$$

is called the degree of similarity between $\widetilde{R}^{(k)}$ and $\widetilde{R}$, where $s(r_{ij}^{(k)}, r_{ij})$ is the similarity between the spherical fuzzy elements calculated as in 3.1.

Definition 4.5. Let $\widetilde{R}_S^{(k)} = [\widetilde{r}_{ij}^{(k)}]_{n \times n}$, $k = 1, 2, \ldots, m$ be m spherical fuzzy preference relations on $U = \{x_1, x_2, \ldots, x_n\}$, and $\widetilde{R}_S = [\widetilde{r}_{ij}]_{n \times n}$ be their aggregates spherical fuzzy preference relations, then $\widetilde{R}_S^{(k)}$ and $\widetilde{R}_S$ are of acceptable similarity, if $\widetilde{s}(\widetilde{R}_S^{(k)}, \widetilde{R}_S) > \alpha$, where α is the threshold of acceptable similarity.

5. Group Decision Making with Consensus Analysis based on Spherical Fuzzy Preference Relations

In this section, let us consider an algorithm to solve a group decision-making problem with a check on the concurrence level in the experts' evaluations.

Let $U = \{x_1, x_2, \ldots, x_n\}$ be the set of alternatives evaluated by m decision makers. Let $(w_1, w_2, \ldots, w_m)^T$ be the weight vector of the experts with $w_i > 0$ and $\sum_i w_i = 1$. An expert must usually provide preference information over alternatives during the decision-making process and the information provided by the k^{th} expert is stored in the matrix $\widetilde{R}_S^{(k)}$.

5.1 Algorithm.

Step 1. Obtain the aggregated spherical fuzzy preference relation $\widetilde{R}_S$ from the given m spherical fuzzy preference relations using 4.1 or 4.2.

Step 2. Obtain the degree of similarity between $\widetilde{R}_S^{(k)}$ and $\widetilde{R}_S$ using the 4.4.

Step 3. Fix a threshold level α for acceptance of similarity. If the calculated degree of similarity of expert e_l (say) is less than the fixed level, then return the matrix to the expert e_l and suggest him/her to revaluate the preference relation.

Step 4. When each individual spherical fuzzy preference relation and the aggregated spherical fuzzy preference relation are of acceptable similarity, calculate the new aggregated spherical fuzzy preference relation $\widetilde{RN}_S$.

Step 5. Using spherical fuzzy arithmetic / geometric averaging operators, aggregate the row entries of $\widetilde{RN}_S$.

Step 6. Rank the alternatives using the score function and refusal degree as in 2.4.

5.2 Illustration. There are four alternatives $U = \{x_1, \ldots, x_4\}$ and three experts $E = \{e_1, e_2, e_3\}$ for a group decision making problem (whose weight vector is $w = (0.5\ 0.3\ 0.2)^T$. The experts compare the four alternatives and form the following spherical fuzzy preference relations:

$$\widetilde{R}_S^{(1)} = \begin{bmatrix} (0.5, 0.5, 0.5) & (0.2, 0.5, 0.4) & (0.5, 0.4, 0.3) & (0.7, 0.2, 0.3) \\ (0.5, 0.2, 0.4) & (0.5, 0.5, 0.5) & (0.3, 0.4, 0.5) & (0.5, 0.5, 0.2) \\ (0.4, 0.5, 0.3) & (0.4, 0.3, 0.5) & (0.5, 0.5, 0.5) & (0.8, 0.2, 0.1) \\ (0.2, 0.7, 0.3) & (0.5, 0.5, 0.2) & (0.2, 0.8, 0.1) & (0.5, 0.5, 0.5) \end{bmatrix}$$

$$\widetilde{R}_S^{(2)} = \begin{bmatrix} (0.5, 0.5, 0.5) & (0.3, 0.4, 0.4) & (0.4, 0.5, 0.2) & (0.7, 0.2, 0.3) \\ (0.4, 0.3, 0.4) & (0.5, 0.5, 0.5) & (0.4, 0.4, 0.5) & (0.5, 0.5, 0.2) \\ (0.5, 0.4, 0.2) & (0.4, 0.4, 0.5) & (0.5, 0.5, 0.5) & (0.7, 0.2, 0.3) \\ (0.2, 0.7, 0.3) & (0.5, 0.5, 0.2) & (0.2, 0.7, 0.3) & (0.5, 0.5, 0.5) \end{bmatrix}$$

$$\widetilde{R}_S^{(3)} = \begin{bmatrix} (0.5, 0.5, 0.5) & (0.8, 0.2, 0.2) & (0.3, 0.4, 0.3) & (0.6, 0.4, 0.3) \\ (0.2, 0.8, 0.2) & (0.5, 0.5, 0.5) & (0.5, 0.3, 0.3) & (0.5, 0.5, 0.2) \\ (0.4, 0.3, 0.3) & (0.3, 0.5, 0.3) & (0.5, 0.5, 0.5) & (0.3, 0.7, 0.2) \\ (0.4, 0.6, 0.3) & (0.5, 0.5, 0.2) & (0.7, 0.3, 0.2) & (0.5, 0.5, 0.5) \end{bmatrix}$$

5.2.1 Using Arithmetic Aggregation. The aggregated spherical preference relation with respect to the weighted arithmetic averaging operator is

$$\widetilde{R}_S = \begin{bmatrix} (0.500, 0.500, 0.500) & (0.473, 0.389, 0.351) \\ (0.430, 0.298, 0.377) & (0.500, 0.500, 0.500) \\ (0.434, 0.422, 0.272) & (0.383, 0.362, 0.378) \\ (0.256, 0.679, 0.300) & (0.500, 0.500, 0.200) \end{bmatrix}$$

$$\begin{matrix} (0.440,\ 0.428,\ 0.276) & (0.683,\ 0.230,\ 0.301) \\ (0.381,\ 0.378,\ 0.465) & (0.500,\ 0.500,\ 0.200) \\ (0.500,\ 0.500,\ 0.500) & (0.720,\ 0.257,\ 0.194) \\ (0.393,\ 0.632,\ 0.169) & (0.500,\ 0.500,\ 0.500) \end{matrix}\right]$$

The degree of similarity matrix are as follows:

$$[s_{ij}^{(1)}]_{n\times n} = \begin{pmatrix} 0.500 & 0.399 & 0.542 & 0.951 \\ 0.712 & 0.500 & 0.491 & 0.500 \\ 0.465 & 0.535 & 0.500 & 0.840 \\ 0.917 & 0.500 & 0.686 & 0.500 \end{pmatrix}$$

$$[s_{ij}^{(2)}]_{n\times n} = \begin{pmatrix} 0.500 & 0.422 & 0.474 & 0.951 \\ 0.815 & 0.500 & 0.500 & 0.500 \\ 0.527 & 0.500 & 0.500 & 0.853 \\ 0.917 & 0.500 & 0.706 & 0.500 \end{pmatrix}$$

$$[s_{ij}^{(3)}]_{n\times n} = \begin{pmatrix} 0.500 & 0.552 & 0.488 & 0.713 \\ 0.434 & 0.500 & 0.503 & 0.500 \\ 0.514 & 0.462 & 0.500 & 0.072 \\ 0.731 & 0.500 & 0.204 & 0.500 \end{pmatrix}$$

Now the degree of similarity of the preference relations of each of the experts with that of the aggregated preference relation are

$$\widetilde{s}(\widetilde{R}_S^{(1)}, \widetilde{R}_S) = 0.596$$

$$\widetilde{s}(\widetilde{R}_S^{(2)}, \widetilde{R}_S) = 0.604$$

$$\widetilde{s}(\widetilde{R}_S^{(3)}, \widetilde{R}_S) = 0.479$$

Considering $\alpha = 0.5$ to be the threshold level of acceptance of similarity, we find that the similarity index of expert 3 is less than the threshold level. Hence the spherical fuzzy preference relation $\widetilde{R}_S^{(3)}$ is returned to expert 3 along with the aggregated preference relation $\widetilde{R}_S$ with a special mention to reconsider valuation of the entries $r_{3,4}^{(3)}, r_{4,3}^{(3)}$.

Suppose the revaluated spherical fuzzy preference relation of expert 3 is as follows:

$$\widetilde{R}_S^{(3)} = \begin{bmatrix} (0.5,\ 0.5,\ 0.5) & (0.8,\ 0.2,\ 0.2) & (0.3,\ 0.4,\ 0.3) & (0.6,\ 0.4,\ 0.3) \\ (0.2,\ 0.8,\ 0.2) & (0.5,\ 0.5,\ 0.5) & (0.5,\ 0.3,\ 0.3) & (0.5,\ 0.5,\ 0.2) \\ (0.4,\ 0.3,\ 0.3) & (0.3,\ 0.5, 0.3) & (0.5,\ 0.5,\ 0.5) & (0.6,\ 0.5,\ 0.2) \\ (0.4,\ 0.6,\ 0.3) & (0.5,\ 0.5,\ 0.2) & (0.5,\ 0.6,\ 0.2) & (0.5,\ 0.5,\ 0.5) \end{bmatrix} \tag{5.1}$$

then the aggregated preference relation with the modified entries is

$$\widetilde{RN}_S = \begin{bmatrix} (0.500, 0.500, 0.500) & (0.473, 0.389, 0.351) \\ (0.430, 0.298, 0.377) & (0.500, 0.500, 0.500) \\ (0.434, 0.422, 0.272) & (0.383, 0.362, 0.378) \\ (0.256, 0.679, 0.300) & (0.500, 0.500, 0.200) \end{bmatrix}$$

$$\begin{matrix} (0.440,\ 0.428,\ 0.276) & (0.683,\ 0.230,\ 0.301) \\ (0.381,\ 0.378,\ 0.465) & (0.500,\ 0.500,\ 0.200) \\ (0.500,\ 0.500,\ 0.500) & (0.743,\ 0.240,\ 0.191) \\ (0.291,\ 0.726,\ 0.202) & (0.500,\ 0.500,\ 0.500) \end{matrix}\Bigg]$$

and

$$\widetilde{s}(\widetilde{R}_S^{(1)},\ \widetilde{R}_S) = 0.607$$

$$\widetilde{s}(\widetilde{R}_S^{(2)},\ \widetilde{R}_S) = 0.612$$

$$\widetilde{s}(\widetilde{R}_S^{(3)},\ \widetilde{R}_S) = 0.538$$

hence, each individual spherical fuzzy preference relation are of acceptable similarity with the aggregated spherical fuzzy preference relation.

Following step 5 of the algorithm in 5.1, the aggregated entry for each alternative with respect to the arithmetic averaging operator is

$$\widetilde{RV}_S = \begin{bmatrix} (0.541, 0.372, 0.371) \\ (0.457, 0.410, 0.408) \\ (0.553, 0.368, 0.349) \\ (0.409, 0.592, 0.342) \end{bmatrix}$$

The score function of each of the alternatives is

$$score(\widetilde{RV}_S) = \begin{bmatrix} 0.028899 \\ 0.002397 \\ 0.041255 \\ -0.058011 \end{bmatrix}$$

Observe that the alternative x_3 is preferred over other alternatives.

5.2.2 Using Geometric Aggregation. The aggregated spherical preference relation with respect to the weighted geometric averaging operator is

$$\widetilde{R}_S = \begin{bmatrix} (0.500,\ 0.500,\ 0.500) & (0.298,\ 0.430,\ 0.377) & (0.422,\ 0.434,\ 0.272) & (0.679,\ 0.256,\ 0.300) \\ (0.389,\ 0.473,\ 0.351) & (0.500,\ 0.500,\ 0.500) & (0.362,\ 0.383,\ 0.473) & (0.500,\ 0.500,\ 0.200) \\ (0.428,\ 0.440,\ 0.276) & (0.378,\ 0.381,\ 0.379) & (0.500,\ 0.500,\ 0.500) & (0.632,\ 0.393,\ 0.205) \\ (0.230,\ 0.683,\ 0.301) & (0.500,\ 0.500,\ 0.200) & (0.257,\ 0.720,\ 0.191) & (0.500,\ 0.500,\ 0.500) \end{bmatrix}$$

The degree of similarity matrix are as follows:

$$[s_{ij}^{(1)}]_{n\times n} = \begin{pmatrix} 0.500 & 0.712 & 0.465 & 0.917 \\ 0.399 & 0.500 & 0.577 & 0.500 \\ 0.542 & 0.495 & 0.500 & 0.682 \\ 0.951 & 0.500 & 0.840 & 0.500 \end{pmatrix}$$

$$[s_{ij}^{(2)}]_{n\times n} = \begin{pmatrix} 0.500 & 0.815 & 0.527 & 0.917 \\ 0.422 & 0.500 & 0.500 & 0.500 \\ 0.474 & 0.500 & 0.500 & 0.706 \\ 0.951 & 0.500 & 0.853 & 0.500 \end{pmatrix}$$

$$[s_{ij}^{(3)}]_{n\times n} = \begin{pmatrix} 0.500 & 0.434 & 0.514 & 0.731 \\ 0.552 & 0.500 & 0.480 & 0.500 \\ 0.488 & 0.506 & 0.500 & 0.204 \\ 0.713 & 0.500 & 0.072 & 0.500 \end{pmatrix}$$

Now the degree of similarity of the preference relations of each of the experts with that of the aggregated preference relation are

$$\widetilde{s}(\widetilde{R}_S^{(1)}, \widetilde{R}_S) = 0.599$$

$$\widetilde{s}(\widetilde{R}_S^{(2)}, \widetilde{R}_S) = 0.604$$

$$\widetilde{s}(\widetilde{R}_S^{(3)}, \widetilde{R}_S) = 0.481$$

Here again, the similarity index of expert 3 is below the threshold level and thus considering the revaluation as in 5.1, the aggregated preference relation with the modified entries is

$$\widetilde{RV}_S = \begin{bmatrix} (0.500, 0.500, 0.500) & (0.298, 0.430, 0.377) \\ (0.389, 0.473, 0.351) & (0.500, 0.500, 0.500) \\ (0.428, 0.440, 0.276) & (0.378, 0.381, 0.379) \\ (0.230, 0.983, 0.301) & (0.500, 0.500, 0.200) \end{bmatrix}$$

$$\begin{bmatrix} (0.422, 0.434, 0.272) & (0.679, 0.256, 0.300) \\ (0.362, 0.383, 0.473) & (0.500, 0.500, 0.200) \\ (0.500, 0.500, 0.500) & (0.726, 0.294, 0.202) \\ (0.240, 0.743, 0.191) & (0.500, 0.500, 0.500) \end{bmatrix}$$

and

$$\widetilde{s}(\widetilde{R}_S^{(1)}, \widetilde{R}_S) = 0.610$$

$$\widetilde{s}(\widetilde{R}_S^{(2)}, \widetilde{R}_S) = 0.612$$

$$\widetilde{s}(\widetilde{R}_S^{(3)}, \widetilde{R}_S) = 0.539$$

hence, each individual spherical fuzzy preference relation are of acceptable similarity with the aggregated spherical fuzzy preference relation.

Following step 5 of the algorithm in 5.1, the aggregated entry for each alternative with respect to the geometric averaging operator is

$$\widetilde{RV}_S = \begin{bmatrix} (0.455, 0.418, 0.385) \\ (0.433, 0.468, 0.404) \\ (0.492, 0.414, 0.373) \\ (0.343, 0.628, 0.318) \end{bmatrix}$$

The score function of each of the alternatives is

$$score(\widetilde{RV}_S) = \begin{bmatrix} 0.003811 \\ -0.003255 \\ 0.012480 \\ -0.095475 \end{bmatrix}$$

Observe that the alternative x_3 is preferred over other alternatives.

6. Conclusion

Group decision-making is an integral part of every organization and is successful when fostering consensus. While making a decision on selection, the preference of an alternative over the other is considered rather than the direct evaluations. The present work utilizes the concept of spherical fuzzy sets for preference relations as this gives the expert more space for expressing his/her preference or non-preference or neutrality of one alternative over the other. For this the distance measure, similarity measure, score function and aggregation are defined and an algorithm is proposed based on these operators to solve the group decision-making with consensus analysis.

References

[1] Shahzaib Ashraf, Saleem Abdullah, Tahir Mahmood, Fazal Ghani and Tariq Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision making problems, Journal of Intelligent and Fuzzy Systems 36(3) (2019), 2829-2844.

[2] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87-96.

[3] T. M. Athira, S. J. John and H. Garg, Entropy and distance measures of Pythagorean fuzzy soft sets and their applications, Journal of Intelligent and Fuzzy Systems (2019), 1-14.

[4] Abit Balin, A novel fuzzy multi-criteria decision-making methodology based upon the spherical fuzzy sets with a real case study, Iranian Journal of Fuzzy Systems 17(4) (2020), 167-177.

[5] B. C. Cuong and V. Kreinovich, Picture fuzzy sets-a new concept for computational intelligence problems, Proceedings of 2013 Third World Congress on Information and Communication Technologies (WICT 2013), IEEE, (2013), 1-6.

[6] H. Garg, Some picture fuzzy aggregation operators and their applications to multicriteria decision-making, Arabian Journal for Science and Engineering 42(12) (2017), 5275-5290.

[7] Fatma Kutlu Gundogdu and Cengiz Kahraman, Spherical fuzzy analytic hierarchy process (ahp) and its application to industrial robot selection, In International Conference on Intelligent and Fuzzy Systems (2019), 988-996.

[8] F. Kutlu Gundogdu and C. Kahraman, A novel fuzzy topsis method using emerging interval-valued spherical fuzzy sets, Eng. Appl. Artif. Intell. 85 (2019), 307-323.

[9] Cengiz Kahraman, Fatma Kutlu Gundogdu, Sezi C, evik Onar and Basar Oztaysi, Hospital location selection using spherical fuzzy topsis, In EUSFLAT Conf., 2019.

[10] Cengiz Kahraman and Fatma Kutlu Gundogdu, Decision Making with Spherical Fuzzy Sets: Theory and Applications - Studies in Fuzziness and Soft Computing 392, (2021).

[11] K. T. Atanassov and G. Gargov, Interval valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989), 343-349.

[12] Fatma Kutlu Gundogdu and Cengiz Kahraman, A novel vikor method using spherical fuzzy sets and its application to warehouse site selection, Journal of Intelligent and Fuzzy Systems 37(1) (2019), 1197-1211.

[13] Peide Liu, Baoying Zhu and Peng Wang, A multi-attribute decision-making approach based on spherical fuzzy sets for yunnan baiyaos r and d project selection problem, International Journal of Fuzzy Systems 21(7) (2019), 2168-2191.

[14] Tahir Mahmood, Muhammad Ilyas, Zeeshan Ali and Abdu Gumaei, Spherical fuzzy sets-based cosine similarity and information measures for pattern recognition and medical diagnosis, IEEE Access 9 (2021), 25835-25842.

[15] Tahir Mahmood, Kifayat Ullah, Qaisar Khan and Naeem Jan. An approach toward decisionmaking and medical diagnosis problems using the concept of spherical fuzzy sets, Neural Computing and Applications 31(11) (2019), 7041-7053.

[16] Manoj Mathew, Ripon K Chakrabortty and Michael J. Ryan, A novel approach integrating ahp and topsis under spherical fuzzy sets for advanced manufacturing system selection, Engineering Applications of Artificial Intelligence 96 (2020), 103988.

[17] Serif Ozlu and Faruk Karaaslan, Correlation coefficient of t-spherical type-2 hesitant fuzzy sets and their applications in clustering analysis, Journal of Ambient Intelligence and Humanized Computing (2021), 1-29.

[18] R. R. Yager, Pythagorean fuzzy subsets, IFSA World Congress and NAFIPS Annual Meeting IEEE, (2013), 57-61.

[19] T. Tanino, Fuzzy preference orderings in group decision making, Fuzzy Sets and Systems 12 (1984), 117-131.

[20] Kifayat Ullah, Harish Garg, Tahir Mahmood, Naeem Jan and Zeeshan Ali, Correlation coefficients for t-spherical fuzzy sets and their applications in clustering and multi-attribute decision making, Soft Computing 24(3) (2020), 1647-1659.

[21] Kifayat Ullah, Tahir Mahmood and Naeem Jan, Similarity measures for t-spherical fuzzy sets with applications in pattern recognition, Symmetry 10(6) (2018), 193.

[22] Mei-Qin Wu, Ting-You Chen and Jian-Ping Fan, Divergence measure of t-spherical fuzzy sets and its applications in pattern recognition, IEEE Access 8 (2019), 10208-10221.

[23] Mei-Qin Wu, Ting-You Chen and Jian-Ping Fan, Similarity measures of t-spherical fuzzy sets based on the cosine function and their applications in pattern recognition, IEEE Access 8 (2020), 98181-98192.

[24] Y. Yang and F. Chiclana, Intuitionistic fuzzy sets: spherical representation and distances, Int. J. Intell. Syst. 24(4) (2009), 399-420.

[25] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965), 338-353.

[26] Shouzhen Zeng, Harish Garg, Muhammad Munir, Tahir Mahmood and Azmat Hussain, A multiattribute decision making process with immediate probabilistic interactive averaging aggregation operators of t-spherical fuzzy sets and its application in the selection of solar cells, Energies 12(23) (2019), 4436.