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DISTANCES BASED SIMILARITY MEASURES BETWEEN TRAPEZOIDAL INTUITIONISTIC FUZZY NUMBERS USING SCORE FUNCTIONS AND ITS APPLICATION TO PATTERN RECOGNITION PROBLEMS

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Abstract

In this article, three new distances based similarity measures between any two trapezoidal intuitionistic fuzzy numbers (TraIFNs) using some score functions discussed in Lakshmana Gomathi Nayagam, V., S. Jeevaraj and P. Dhanasekaran, A Linear Ordering on the class of Trapezoidal Intuitionistic Fuzzy numbers, Expert Systems with Applications, (2016) DOI: 10.1016/j.eswa.2016.05.003 [9] are defined and some of its important properties are proved and validated by numerical examples. Distances based similarity measures between TraIFNs were introduced by many researchers using different methods with real world applications such as TOPSIS method, pattern recognition problems and multi-criteria decision making (MCDM) method and so on. In some situations, many of the existing distances based similarity measures between TraIFNs are given unreasonable results due to problem dependent. The proposed method is satisfied human intuition and is applied to pattern recognition problems to show the effectiveness of the proposed method using TraIFNs environment. Finally, we obtain general

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conclusions and future scope of the proposed method.

1. Introduction

Trapezoidal intuitionistic fuzzy number (TraIFN) has been studied by Nehi and Maleki [12] which is a generalization of fuzzy sets [19], interval-valued fuzzy sets [1] and interval-valued intuitionistic fuzzy sets [2] and is applied in many real world applications such as multi-criteria decision making (MCDM) method, pattern recognition problems, technique for order of preference by similarity to ideal solution (TOPSIS) method, image processing, artificial intelligence and so on. In literature, distances based similarity measures between trapezoidal intuitionistic fuzzy numbers (TraIFNs) play an important role in pattern recognition problems. Similarity measures using different concepts have been introduced by various researchers in the literature [3, 7, 11, 15, 18].

In [3], the similarity measure of vague set was introduced and applied on behavior analysis problems. In [6], modified Chen's similarity measures defined and also applied on the same behavior analysis problems. Dengfeng and Chuntian similarity measure has modified a new similarity measure by Mitchell [11] and its application to pattern recognition problems. In 2007, Xu has proposed another distance based similarity measure and applied on the MCDM problem in intuitionistic fuzzy environment. In [16], based on the Hamming, Euclidean and Hausdorff distances, distance and similarity measures of IVIFSs developed and applied on aggregation operators. In [17], a vector representation with the membership and non-membership degrees studied to form a cosine similarity measure and used for pattern recognition and medical diagnosis. In [18], distance based similarity measure between trapezoidal intuitionistic fuzzy numbers introduced and applied on multi-criteria decision making problems. In [14], based on membership, non-membership and hesitation degrees, a new similarity and weighted similarity measures proposed and application to pattern recognition problem.

In [4], a novel similarity measure based on transformation techniques between intuitionistic fuzzy sets developed and applied to pattern recognition problems. Xu (2017) has improved the measure of distance between IFSs using Minkowski distance and applied to pattern recognition problems and medical diagnosis. In [13], a new H-max distance measures introduced and applied to medical diagnosis problems. In [7], based on two isosceles triangles in a square area of two IFSs, a novel similarity measure for IFSs proposed and application to pattern recognition problems. Lakshmana et al., [10] introduced a new distance based similarity measure on incomplete imprecise interval information and is applied to technique for order of preference by similarity to ideal solution (TOPSIS) method and pattern recognition problems. The proposed distances based similarity measures using the total order relation introduced by Lakshmana et al., [9] can overcome some drawbacks of the some existing methods [3, 17, 18].

This paper is organized as follows. In Sec. 2, we recall some basic definitions and results on trapezoidal intuitionistic fuzzy numbers and similarity measure. Sec. 3 is devoted for defining three new distances based similarity measures using some score functions on the class of TraIFNs discussed in [9] and some of its important properties are studied. In Sec. 4, the proposed distances based similarity measures are compared with some existing methods and are applied to pattern recognition problems for its applicability and effectiveness. Finally, conclusions and prospective scopes are given in Sec. 5.

2. Preliminaries

In this section, we review some basic definitions related to trapezoidal intuitionsitic fuzzy numbers and similarity measures.

Definition 2.1 ([1]). Let X be a nonempty set. An intuitionistic fuzzy set (IFS) $\tilde{\tilde{A}}$ in X is defined by $\tilde{\tilde{A}} = \langle(x, \mu_{\tilde{\tilde{A}}}(x), \nu_{\tilde{\tilde{A}}}(x)) : x \in X\rangle$, where $\mu_{\tilde{\tilde{A}}}$ and $\nu_{\tilde{\tilde{A}}}$ are membership and non-membership functions with the condition $0 \leq \mu_{\tilde{\tilde{A}}}(x) + \nu_{\tilde{\tilde{A}}}(x) \leq 1, \forall x \in X$.

Here, $\mu_{\tilde{\tilde{A}}}(x), \nu_{\tilde{\tilde{A}}}(x) \in [0, 1]$ denote the degree of membership and non-membership of x to lie in $\tilde{\tilde{A}}$ respectively. For each intuitionistic fuzzy subset $\tilde{\tilde{A}}$ in X, the intuitionistic fuzzy index is $\pi_{\tilde{\tilde{A}}}(x) = 1 - \mu_{\tilde{\tilde{A}}}(x) - \nu_{\tilde{\tilde{A}}}(x)$ of x to lie in $\tilde{\tilde{A}}$.

Definition 2.2 ([12]). A trapezoidal intuitionistic fuzzy number (Grzegorzewski, [5]) $\widetilde{A}$ with parameters $\widetilde{a}_{1\nu} \leq \widetilde{a}_{1\mu}, \widetilde{a}_{2\nu} \leq \widetilde{a}_{2\mu} \leq \widetilde{a}_{3\mu} \leq \widetilde{a}_{3\nu}$ and $\widetilde{a}_{4\mu} \leq \widetilde{a}_{4\nu}$, denoted as $\widetilde{A} = \langle(\widetilde{a}_{1\mu}, \widetilde{a}_{2\mu}, \widetilde{a}_{3\mu}, \widetilde{a}_{4\mu}), (\widetilde{a}_{1\nu}, \widetilde{a}_{2\nu}, \widetilde{a}_{3\nu}, \widetilde{a}_{4\nu})\rangle$ in the set of real numbers R whose membership function is defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - \widetilde{a}_{1\mu}}{\widetilde{a}_{2\mu} - \widetilde{a}_{1\mu}} & \text{if } \widetilde{a}_{1\mu} \leq x \leq \widetilde{a}_{2\mu} \\ 1 & \text{if } \widetilde{a}_{2\mu} \leq x \leq \widetilde{a}_{3\mu} \\ \dfrac{\widetilde{a}_{4\mu} - x}{\widetilde{a}_{4\mu} - \widetilde{a}_{3\mu}} & \text{if } \widetilde{a}_{3\mu} \leq x \leq \widetilde{a}_{4\mu} \\ 0 & \text{Otherwise} \end{cases}$$

and non-membership function is defined as

$$\nu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x - \widetilde{a}_{2\nu}}{\widetilde{a}_{1\nu} - \widetilde{a}_{2\nu}} & \text{if } \widetilde{a}_{1\nu} \leq x \leq \widetilde{a}_{2\nu} \\ 0 & \text{if } \widetilde{a}_{2\nu} \leq x \leq \widetilde{a}_{3\nu} \\ \dfrac{x - \widetilde{a}_{3\nu}}{\widetilde{a}_{4\nu} - \widetilde{a}_{3\nu}} & \text{if } \widetilde{a}_{3\nu} \leq x \leq \widetilde{a}_{4\nu} \\ 1 & \text{Otherwise.} \end{cases}$$

The triangular intuitionistic fuzzy numbers (TriIFNs) is considered as special cases of the trapezoidal intuitionistic fuzzy numbers (TraIFNs) when $\widetilde{a}_{2\mu} = \widetilde{a}_{3\mu}$ (and $\widetilde{a}_{2\nu} = \widetilde{a}_{3\nu}$).

The following operations for trapezoidal intuitionistic fuzzy numbers have been given in Nehi and Maleki [12].

Let $\widetilde{A} = \langle(\widetilde{a}_{1\mu}, \widetilde{a}_{2\mu}, \widetilde{a}_{3\mu}, \widetilde{a}_{4\mu}), (\widetilde{a}_{1\nu}, \widetilde{a}_{2\nu}, \widetilde{a}_{3\nu}, \widetilde{a}_{4\nu})\rangle$ and $\widetilde{B} = \langle(\widetilde{b}_{1\mu}, \widetilde{b}_{2\mu}, \widetilde{b}_{3\mu}, \widetilde{b}_{4\mu}), (\widetilde{b}_{1\nu}, \widetilde{b}_{2\nu}, \widetilde{b}_{3\nu}, \widetilde{b}_{4\nu})\rangle$ be two trapezoidal intuitionistic fuzzy numbers and r be a positive scalar. Then, $\widetilde{A} + \widetilde{B}$ $= \langle(\widetilde{a}_{1\mu} + \widetilde{b}_{1\mu}, \widetilde{a}_{2\mu} + \widetilde{b}_{2\mu}, \widetilde{a}_{3\mu} + \widetilde{b}_{3\mu}, \widetilde{a}_{4\mu} + \widetilde{b}_{4\mu}), (\widetilde{a}_{1\nu} + \widetilde{b}_{1\nu}, \widetilde{a}_{2\nu} + \widetilde{b}_{2\nu}, \widetilde{a}_{3\nu} + \widetilde{b}_{3\nu}, \widetilde{a}_{4\nu} + \widetilde{b}_{4\nu})\rangle$, $r\widetilde{A} = \langle(r\widetilde{a}_{1\mu}, r\widetilde{a}_{2\mu}, r\widetilde{a}_{3\mu}, r\widetilde{a}_{4\mu}), (r\widetilde{a}_{1\nu}, r\widetilde{a}_{2\nu}, r\widetilde{a}_{3\nu}, r\widetilde{a}_{4\nu})\rangle$.

The concept of distance measure between two TraIFS(X) is given as follows.

Definition 2.3 ([16]). A real-valued function $D : IFS(X) \times IFS(X)$

$\rightarrow [0, 1]$ is called a distance measure on $IFS(X)$, if D satisfies the following postulates:

(D1) $0 \leq D(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \leq 1$, for all $\tilde{\tilde{A}}, \tilde{\tilde{B}} \in IFS(X)$,

(D2) $D(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 0 \Leftrightarrow \tilde{\tilde{A}} = \tilde{\tilde{B}}$;

(D3) $D(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = D(\tilde{\tilde{B}}, \tilde{\tilde{A}})$;

(D4) If $\tilde{\tilde{A}} \subseteq \tilde{\tilde{B}} \subseteq \tilde{\tilde{C}}$, then $D(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \leq D(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ and $D(\tilde{\tilde{B}}, \tilde{\tilde{C}}) \leq D(\tilde{\tilde{A}}, \tilde{\tilde{C}})$, $\forall \tilde{\tilde{C}} \in IFS(X)$.

The property (D4) is equivalently written as (D5): If $\tilde{\tilde{A}} \subseteq \tilde{\tilde{B}} \subseteq \tilde{\tilde{C}}$, then $D(\tilde{\tilde{A}}, \tilde{\tilde{C}}) \geq \max(D(\tilde{\tilde{A}}, \tilde{\tilde{B}}), D(\tilde{\tilde{B}}, \tilde{\tilde{C}}))$.

The similarity measure between two $IFS(X)$ can be described in the following definition.

Definition 2.4 ([16]). A real-valued function $S : IFS(X) \times IFS(X) \rightarrow [0, 1]$ is called a similarity measure on $IFS(X)$, if S satisfies the following postulates:

(S1) $0 \leq S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \leq 1$, for all $\tilde{\tilde{A}}, \tilde{\tilde{B}} \in IFS(X)$,

(S2) $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = 1 \Leftrightarrow \tilde{\tilde{A}} = \tilde{\tilde{B}}$;

(S3) $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) = S(\tilde{\tilde{B}}, \tilde{\tilde{A}})$;

(S4) If $\tilde{\tilde{A}} \subseteq \tilde{\tilde{B}} \subseteq \tilde{\tilde{C}}$, then $S(\tilde{\tilde{A}}, \tilde{\tilde{B}}) \geq S(\tilde{\tilde{A}}, \tilde{\tilde{C}})$ and $S(\tilde{\tilde{B}}, \tilde{\tilde{C}}) \geq S(\tilde{\tilde{A}}, \tilde{\tilde{C}})$, $\forall \tilde{\tilde{C}} \in IFS(X)$.

The property (S4) is equivalently written as (S5): If $\tilde{\tilde{A}} \subseteq \tilde{\tilde{B}} \subseteq \tilde{\tilde{C}}$, then $S(\tilde{\tilde{A}}, \tilde{\tilde{C}}) \leq \min(S(\tilde{\tilde{A}}, \tilde{\tilde{B}}), S(\tilde{\tilde{B}}, \tilde{\tilde{C}}))$.

Definition 2.5. If $\widetilde{A}$ and $\widetilde{B}$ are two TraIFNs of the set X, then

- $\widetilde{A} \leq \widetilde{B}$ if and only if $\widetilde{a}_{1\mu} \leq \widetilde{b}_{1\mu}$, $\widetilde{a}_{2\mu} \leq \widetilde{b}_{2\mu}$, $\widetilde{a}_{3\mu} \leq \widetilde{b}_{3\mu}$, $\widetilde{a}_{4\mu} \leq \widetilde{b}_{4\mu}$ and $\widetilde{a}_{1\nu} \geq \widetilde{b}_{1\nu}$, $\widetilde{a}_{2\nu} \geq \widetilde{b}_{2\nu}$, $\widetilde{a}_{3\nu} \geq \widetilde{b}_{3\nu}$, $\widetilde{a}_{4\nu} \geq \widetilde{b}_{4\nu}$;
- $\widetilde{A} = \widetilde{B}$ if and only if $\widetilde{a}_{1\mu} = \widetilde{b}_{1\mu}$, $\widetilde{a}_{2\mu} = \widetilde{b}_{2\mu}$, $\widetilde{a}_{3\mu} = \widetilde{b}_{3\mu}$, $\widetilde{a}_{4\mu} = \widetilde{b}_{4\mu}$ and $\widetilde{a}_{1\nu} = \widetilde{b}_{1\nu}$, $\widetilde{a}_{2\nu} = \widetilde{b}_{2\nu}$, $\widetilde{a}_{3\nu} = \widetilde{b}_{3\nu}$, $\widetilde{a}_{4\nu} = \widetilde{b}_{4\nu}$;
- $\widetilde{A}^c = \langle(\widetilde{a}_{1\nu}, \widetilde{a}_{2\nu}, \widetilde{a}_{3\nu}, \widetilde{a}_{4\nu}), (\widetilde{a}_{1\mu}, \widetilde{a}_{2\mu}, \widetilde{a}_{3\mu}, \widetilde{a}_{4\mu})\rangle$, where $\widetilde{A}^c$ is the complement of $\widetilde{A}$.

Lakshmana et al. [9] gave the following definitions and theorems for different score functions of a trapezoidal intuitionistic fuzzy number (TraIFN).

Definition 2.6 [9]**.** Let $\widetilde{A} \in$TraIFN. Then the membership and non-membership scores of a TraIFN $\widetilde{A}$ is defined as

$$L(\widetilde{A}) = \frac{1}{8}(2(\widetilde{a}_{1\mu} + \widetilde{a}_{2\mu} + \widetilde{a}_{3\mu} + \widetilde{a}_{4\mu}) - 2(\widetilde{a}_{1\nu} + \widetilde{a}_{2\nu} + \widetilde{a}_{3\nu} + \widetilde{a}_{4\nu})$$
$$+ (\widetilde{a}_{1\mu} + \widetilde{a}_{2\mu})(\widetilde{a}_{1\nu} + \widetilde{a}_{2\nu}) + (\widetilde{a}_{3\mu} + \widetilde{a}_{4\mu})(\widetilde{a}_{3\nu} + \widetilde{a}_{4\nu})).$$

and

$$LG(\widetilde{A}) = \frac{1}{8}(-2(\widetilde{a}_{1\mu} + \widetilde{a}_{2\mu} + \widetilde{a}_{3\mu} + \widetilde{a}_{4\mu}) + 2(\widetilde{a}_{1\nu} + \widetilde{a}_{2\nu} + \widetilde{a}_{3\nu} + \widetilde{a}_{4\nu})$$
$$+ (\widetilde{a}_{1\mu} + \widetilde{a}_{2\mu})(\widetilde{a}_{1\nu} + \widetilde{a}_{2\nu}) + (\widetilde{a}_{3\mu} + \widetilde{a}_{4\mu})(\widetilde{a}_{3\nu} + \widetilde{a}_{4\nu})).$$

Theorem 2.1 [9]**.** Let $\widetilde{A}, \widetilde{B} \in TraIFN$. If $\widetilde{A} \leq \widetilde{B}\,(\widetilde{A} \geq \widetilde{B})$ with $\widetilde{a}_{1\mu} \leq \widetilde{b}_{1\mu}$, $\widetilde{a}_{2\mu} \leq \widetilde{b}_{2\mu}$, $\widetilde{a}_{3\mu} \leq \widetilde{b}_{3\mu}$, $\widetilde{a}_{4\mu} \leq \widetilde{b}_{4\mu}$ and $\widetilde{a}_{1\nu} \geq \widetilde{b}_{1\nu}$, $\widetilde{a}_{2\nu} \geq \widetilde{b}_{2\nu}$, $\widetilde{a}_{3\nu} \geq \widetilde{b}_{3\nu}$, $\widetilde{a}_{4\nu} \geq \widetilde{b}_{4\nu}$ ($\widetilde{a}_{1\mu} \geq \widetilde{b}_{1\mu}$, $\widetilde{a}_{2\mu} \geq \widetilde{b}_{2\mu}$, $\widetilde{a}_{3\mu} \geq \widetilde{b}_{3\mu}$, $\widetilde{a}_{4\mu} \geq \widetilde{b}_{4\mu}$ and $\widetilde{a}_{1\nu} \leq \widetilde{b}_{1\nu}$, $\widetilde{a}_{2\nu} \leq \widetilde{b}_{2\nu}$, $\widetilde{a}_{3\nu} \leq \widetilde{b}_{3\nu}$, $\widetilde{a}_{4\nu} \leq \widetilde{b}_{3\nu}$), then $L(\widetilde{A}) \leq L(\widetilde{B})$ and $LG(\widetilde{A}) \geq LG(\widetilde{B})$ $(L(\widetilde{A}) \geq L(\widetilde{B}))$ and $LG(\widetilde{A}) \leq LG(\widetilde{B})$.

Definition 2.7 [9]**.** Let $\widetilde{A} \in TraIFN$. Then the widespread score of a

TraIFN $\widetilde{A}$ is defined as

$$WS(\widetilde{A}) = \frac{1}{8}((\widetilde{a}_{1\mu} - \widetilde{a}_{2\mu} + \widetilde{a}_{3\mu} - \widetilde{a}_{4\mu}) + (\widetilde{a}_{1\nu} - \widetilde{a}_{2\nu} + \widetilde{a}_{3\nu} - \widetilde{a}_{4\nu}) + (\widetilde{a}_{1\mu} + \widetilde{a}_{3\mu})(\widetilde{a}_{1\nu} + \widetilde{a}_{3\nu}) - (\widetilde{a}_{2\mu} + \widetilde{a}_{4\mu})(\widetilde{a}_{2\nu} + \widetilde{a}_{4\nu})).$$

Theorem 2.2 [9]**.** *Let* $\widetilde{A}, \widetilde{B} \in TraIFN$. *If* $\widetilde{A} \leq \widetilde{B}\,(\widetilde{A} \geq \widetilde{B})$ *with* $\widetilde{a}_{1\mu} \geq \widetilde{b}_{1\mu}$, $\widetilde{a}_{2\mu} \leq \widetilde{b}_{2\mu}$, $\widetilde{a}_{3\mu} \geq \widetilde{b}_{3\mu}$, $\widetilde{a}_{4\mu} \leq \widetilde{b}_{4\mu}$ *and* $\widetilde{a}_{1\nu} \geq \widetilde{b}_{1\nu}$, $\widetilde{a}_{2\nu} \leq \widetilde{b}_{2\nu}$, $\widetilde{a}_{3\nu} \geq \widetilde{b}_{3\nu}$, $\widetilde{a}_{4\nu} \leq \widetilde{b}_{4\nu}$ $(\widetilde{a}_{1\mu} \leq \widetilde{b}_{1\mu}$, $\widetilde{a}_{2\mu} \geq \widetilde{b}_{2\mu}$, $\widetilde{a}_{3\mu} \leq \widetilde{b}_{3\mu}$, $\widetilde{a}_{4\mu} \geq \widetilde{b}_{4\mu}$ *and* $\widetilde{a}_{1\nu} \leq \widetilde{b}_{1\nu}$, $\widetilde{a}_{2\nu} \geq \widetilde{b}_{2\nu}$, $\widetilde{a}_{3\nu} \leq \widetilde{b}_{3\nu}$, $\widetilde{a}_{4\nu} \geq \widetilde{b}_{3\nu})$, *then* $WS(\widetilde{A}) \geq WS(\widetilde{B})$ $(WS(\widetilde{A}) \leq WS(\widetilde{B}))$.

Definition 2.8 [9]**.** Let $\widetilde{A} \in TraIFN$. Then the exact score of a $TraIFN$ $\widetilde{A}$ is defined as

$$J_8(\widetilde{A}) = \frac{1}{8}((-\widetilde{a}_{1\mu} + \widetilde{a}_{2\mu} + \widetilde{a}_{3\mu} - \widetilde{a}_{4\mu}) + (-\widetilde{a}_{1\nu} + \widetilde{a}_{2\nu} + \widetilde{a}_{3\nu} - \widetilde{a}_{4\nu}) - (\widetilde{a}_{1\mu} + \widetilde{a}_{4\mu})(\widetilde{a}_{1\nu} + \widetilde{a}_{4\nu}) + (\widetilde{a}_{2\mu} + \widetilde{a}_{3\mu})(\widetilde{a}_{2\nu} + \widetilde{a}_{3\nu})).$$

Theorem 2.3 [9]**.** *Let* $\widetilde{A}, \widetilde{B} \in TraIFN$. *If* $\widetilde{A} \leq \widetilde{B}\,(\widetilde{A} \geq \widetilde{B})$ with $\widetilde{a}_{1\mu} \leq \widetilde{b}_{1\mu}$, $\widetilde{a}_{2\mu} \geq \widetilde{b}_{2\mu}$, $\widetilde{a}_{3\mu} \geq \widetilde{b}_{3\mu}$, $\widetilde{a}_{4\mu} \leq \widetilde{b}_{4\mu}$ and $\widetilde{a}_{1\nu} \leq \widetilde{b}_{1\nu}$, $\widetilde{a}_{2\nu} \geq \widetilde{b}_{2\nu}$, $\widetilde{a}_{3\nu} \geq \widetilde{b}_{3\nu}$, $\widetilde{a}_{4\nu} \leq \widetilde{b}_{4\nu}$ $(\widetilde{a}_{1\mu} \geq \widetilde{b}_{1\mu}$, $\widetilde{a}_{2\mu} \leq \widetilde{b}_{2\mu}$, $\widetilde{a}_{3\mu} \leq \widetilde{b}_{3\mu}$, $\widetilde{a}_{4\mu} \geq \widetilde{b}_{4\mu}$ and $\widetilde{a}_{1\nu} \geq \widetilde{b}_{1\nu}$, $\widetilde{a}_{2\nu} \leq \widetilde{b}_{2\nu}$, $\widetilde{a}_{3\nu} \leq \widetilde{b}_{3\nu}$, $\widetilde{a}_{4\nu} \geq \widetilde{b}_{4\nu})$, then $J_8(\widetilde{B}) \leq J_8(\widetilde{A})$ $(J_8(\widetilde{A}) \geq J_8(\widetilde{B}))$.

3. Distances Based Similarity Measures on TraIFNs

In this section, membership, non-membership, widespread and exact score functions are used to introduce three new distances based similarity measures on the set of TraIFNs and some of its properties are examined with numerical example.

3.1 Membership and non-membership scores based similarity measure on TraIFNs

Definition 3.1.1. Let $\widetilde{A}$ and $\widetilde{B}$ be two TraIFNs. Then the membership and non-membership scores based distance measure between TraIFNs $\widetilde{A}$ and $\widetilde{B}$ are defined as

$$D_1(\widetilde{A}, \widetilde{B}) = \frac{1}{2}\{| L(\widetilde{A}) - L(\widetilde{B}) | + | LG(\widetilde{A}) - LG(\widetilde{B}) |\} \tag{1}$$

where $L(\widetilde{A})$, $L(\widetilde{B})$, $LG(\widetilde{A})$ and $LG(\widetilde{B})$ are score functions [9] of two TraIFNs $\widetilde{A}$ and $\widetilde{B}$ respectively.

The following propositions are obtained from the Definition 3.1.1.

Proposition 3.1.1. Let $\widetilde{A} = ((\widetilde{a}_{1\mu}, \widetilde{a}_{2\mu}, \widetilde{a}_{3\mu}), (\widetilde{a}_{1\nu}, \widetilde{a}_{2\nu}, \widetilde{a}_{3\nu}))$ and $\widetilde{B} = ((\widetilde{b}_{1\mu}, \widetilde{b}_{2\mu}, \widetilde{b}_{3\mu}), (\widetilde{b}_{1\nu}, \widetilde{b}_{2\nu}, \widetilde{b}_{3\nu}))$ be two TriIFNs. Then the membership and non-membership scores based distance measure between TriIFNs $\widetilde{A}$ and $\widetilde{B}$ is defined as $D_1(\widetilde{A}, \widetilde{B}) = \frac{1}{2}\{| L(\widetilde{A}) - L(\widetilde{B}) | + | LG(\widetilde{A}) - LG(\widetilde{B}) |\}$, where $L(\widetilde{A})$, $L(\widetilde{B})$, $LG(\widetilde{A})$ and $LG(\widetilde{B})$ are the membership and non-membership scores of two TriIFNs $\widetilde{A}$ and $\widetilde{B}$ respectively.

Proposition 3.1.2. *Let* $\widetilde{A} = ([\widetilde{a}_{1\mu}, \widetilde{a}_{2\mu}], [\widetilde{a}_{1\nu}, \widetilde{a}_{2\nu}])$ and $\widetilde{B} = ([\widetilde{b}_{1\mu}, \widetilde{b}_{2\mu}], [\widetilde{b}_{1\nu}, \widetilde{b}_{2\nu}])$ *be two interval-valued intuitionistic fuzzy numbers (IVIFNs). Then the membership and non-membership scores based distance measure between IVIFNs* $\widetilde{A}$ *and* $\widetilde{B}$ *is defined as* $D_1(\widetilde{A}, \widetilde{B}) = \frac{1}{2}\{| L(\widetilde{A}) - L(\widetilde{B}) | + | LG(\widetilde{A}) - LG(\widetilde{B}) |\}$, *where* $L(\widetilde{A})$, $L(\widetilde{B})$, $LG(\widetilde{A})$ *and* $LG(\widetilde{B})$ *are the membership and non-membership scores of two* IVIFNs $\widetilde{A}$ *and* $\widetilde{B}$ *respectively*.

Proposition 3.1.3. *Let* $\widetilde{A} = (\widetilde{a}_{1\mu}, \widetilde{a}_{1\nu})$ *and* $\widetilde{B} = (\widetilde{b}_{1\mu}, \widetilde{b}_{1\nu})$ *be two IFSs. Then the membership and non-membership scores based distance measure between IFSs* $\widetilde{A}$ *and* $\widetilde{B}$ *is defined as* $D_1(\widetilde{A}, \widetilde{B}) = \frac{1}{2}\{| L(\widetilde{A}) - L(\widetilde{B}) | + | LG(\widetilde{A}) - LG(\widetilde{B}) |\}$, *where* $L(\widetilde{A})$, $L(\widetilde{B})$, $LG(\widetilde{A})$ *and* $LG(\widetilde{B})$ *are the membership and non-membership scores of two IFSs* $\widetilde{A}$ *and* $\widetilde{B}$ *respectively.*

The following theorem proves that the function D_1 is a distance measure on the class of TraIFNs.

Theorem 3.1.1. *The measure* $D_1(\widetilde{A}, \widetilde{B})$ *is a distance measure between TraIFNs* $\widetilde{A}$ *and* $\widetilde{B}$.

Proof. It is very easy to prove that $D_1(\widetilde{A}, \widetilde{B})$ satisfies (D1)-(D3). To prove the property (D4), suppose that $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$. By hypothesis, $\widetilde{a}_{1\mu} \leq \widetilde{b}_{1\mu} \leq \widetilde{c}_{1\mu}$; $\widetilde{a}_{2\mu} \leq \widetilde{b}_{2\mu} \leq \widetilde{c}_{2\mu}$; $\widetilde{a}_{3\mu} \leq \widetilde{b}_{3\mu} \leq \widetilde{c}_{3\mu}$; $\widetilde{a}_{4\mu} \leq \widetilde{b}_{4\mu} \leq \widetilde{c}_{1\mu}$; $\widetilde{a}_{1\nu} \geq \widetilde{b}_{1\nu} \geq \widetilde{c}_{1\nu}$; $\widetilde{a}_{2\nu} \geq \widetilde{b}_{2\nu} \geq \widetilde{c}_{2\nu}$; $\widetilde{a}_{3\nu} \geq \widetilde{b}_{3\nu} \geq \widetilde{c}_{3\nu}$ and $\widetilde{a}_{4\nu} \geq \widetilde{b}_{4\nu} \geq \widetilde{c}_{1\nu}$. By Theorem 2.1, if $\widetilde{A} \leq \widetilde{B}$, $L(\widetilde{A}) \leq L(\widetilde{B})$ and $LG(\widetilde{A}) \geq LG(\widetilde{B})$ implies $| L(\widetilde{A}) - L(\widetilde{B}) | \geq 0$ and $| LG(\widetilde{A}) - LG(\widetilde{B}) | \geq 0$. Now,

$$D_1(\widetilde{A}, \widetilde{B}) = \frac{1}{2}\{| L(\widetilde{A}) - L(\widetilde{B}) | + | LG(\widetilde{A}) - LG(\widetilde{B}) |\}$$

$$= \frac{1}{2}\{| L(\widetilde{A}) - L(\widetilde{C}) + L(\widetilde{C}) - L(\widetilde{B}) | + | LG(\widetilde{A}) - LG(\widetilde{C}) + LG(\widetilde{C}) - LG(\widetilde{B}) |\}$$

$$\leq \frac{1}{2}\{| L(\widetilde{A}) - L(\widetilde{C}) | + | LG(\widetilde{A}) - LG(\widetilde{C}) |\}$$

$$+ \frac{1}{2}\{| L(\widetilde{C}) - L(\widetilde{B}) | + | LG(\widetilde{C}) - LG(\widetilde{B}) |\}$$

$= D_1(\widetilde{A}, \widetilde{C}) + D_1(\widetilde{C}, \widetilde{B})$. That is, $D_1(\widetilde{A}, \widetilde{B}) \leq D_1(\widetilde{A}, \widetilde{C}) + D_1(\widetilde{C}, \widetilde{B})$. Hence the proof. □

The similarity measure between two TraIFNs is introduced as follows.

Definition 3.1.2. Let $\widetilde{A}$ and $\widetilde{B}$ be two TraIFNs in $X = \{x_1, x_2, \ldots, x_n\}$. Then the distance based similarity measure S_1 using membership and non-membership scores between two TraIFNs $\widetilde{A}$ and $\widetilde{B}$ is defined as $S_1(\widetilde{A}, \widetilde{B}) = 1 - D_1(\widetilde{A}, \widetilde{B})$, where $D_1(\widetilde{A}, \widetilde{B})$ is in Definition 3.1.1.

The meaning of the larger value of $S_1(\widetilde{A}, \widetilde{B})$, the more the similarity between $\widetilde{A}$ and $\widetilde{B}$. The following theorem proves that the function S_1 is a similarity measure using the membership and non-membership scores between TraIFNs.

Theorem 3.1.2. *The measure* $S_1(\widetilde{A}, \widetilde{B})$ *is a similarity measure between TraIFNs* $\widetilde{A}$ *and* $\widetilde{B}$.

Proof. It is very easy to prove that $S_1(\widetilde{A}, \widetilde{B})$ satisfies (S1)-(S3). To prove the property (S4), suppose that $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$. By Theorem 2.1, if $\widetilde{A} \leq \widetilde{B}$, $L(\widetilde{A}) \leq L(\widetilde{B})$ and $LG(\widetilde{A}) \geq LG(\widetilde{B})$ implies $| L(\widetilde{A}) - L(\widetilde{B}) | \geq 0$ and $| LG(\widetilde{A}) - LG(\widetilde{B}) | \geq 0$. Obviously, $D_1(\widetilde{A}, \widetilde{B}) \leq D_1(\widetilde{A}, \widetilde{C}) \Rightarrow 1 - D_1(\widetilde{A}, \widetilde{B}) \geq 1 - D_1(\widetilde{A}, \widetilde{C}) \Rightarrow S_1(\widetilde{A}, \widetilde{B}) \geq S_1(\widetilde{A}, \widetilde{C})$. Similarly, $S_1(\widetilde{B}, \widetilde{C}) \geq S_1(\widetilde{A}, \widetilde{C})$. Hence the proof. □

Example 3.1.1. Let $\widetilde{A} = ((0.16, 0.26, 0.30, 0.60), (0.08, 0.14, 0.52, 0.68))$ and $\widetilde{B} = ((0.23, 0.38, 0.47, 0.65), (0.02, 0.12, 0.49, 0.65))$ be two TraIFNs. Then $S_1(\widetilde{A}, \widetilde{B}) = 0.8625$. In some cases, S_1 is not enough to distinguish any two TraIFNs, which is shown in the next example.

Example 3.1.2. Let $\widetilde{A} = ((0.1, 0.3, 0.4, 0.5), (0.05, 0.2, 0.5, 0.6))$ and $\widetilde{B} = ((0.1, 0.3, 0.4, 0.5), (0.01, 0.24, 0.45, 0.65))$ be two TraIFNs. Obviously, $\widetilde{A}$ and $\widetilde{B}$ are not equal (same) even though $S_1(\widetilde{A}, \widetilde{B}) = 1$. Hence, we need another distance based similarity measure to measure the degree of similarity between any two TraIFNs. In the next section, we will discuss the widespread score based similarity measure between TraIFNs.

3.2 Widespread score based similarity measure on TraIFNs

Definition 3.2.1. Let $\widetilde{A}$ and $\widetilde{B}$ be two TraIFNs. Then the widespread score based distance measure between TraIFNs $\widetilde{A}$ and $\widetilde{B}$ is defined as

$$D_2(\widetilde{A}, \widetilde{B}) = | WS(\widetilde{A}) - WS(\widetilde{B}) | \tag{2}$$

where $WS(\widetilde{A})$ and $WS(\widetilde{B})$ are widespread score functions [9] of two TraIFNs $\widetilde{A}$ and $\widetilde{B}$ respectively.

The following theorem proves that the function D_2 is a distance measure on the widespread class of TraIFNs.

Theorem 3.2.1. *The measure* $D_2(\widetilde{A}, \widetilde{B})$ *is a distance measure between TraIFNs* $\widetilde{A}$ *and* $\widetilde{B}$.

Proof. The proof is similar to the Theorem 3.1.1. □

The similarity measure between two TraIFNs is introduced as follows.

Definition 3.2.2. Let $\widetilde{A}$ and $\widetilde{B}$ be two TraIFNs in $X = \{x_1, x_2, \ldots, x_n\}$. Then the widespread score based similarity measure between two TraIFNs $\widetilde{A}$ and $\widetilde{B}$ is defined as $S_2(\widetilde{A}, \widetilde{B}) = 1 - D_2(\widetilde{A}, \widetilde{B})$, where $D_2(\widetilde{A}, \widetilde{B})$ is in Definition 3.2.1.

The meaning of the larger the value of $S_2(\widetilde{A}, \widetilde{B})$, the more the similarity between $\widetilde{A}$ and $\widetilde{B}$. The following theorem proves that the function S_2 is a similarity measure on the class of TraIFNs.

Theorem 3.2.2. *The measure* $S_2(\widetilde{A}, \widetilde{B})$ *is a similarity measure between TraIFNs* $\widetilde{A}$ *and* $\widetilde{B}$.

Proof. The proof is similar to the Theorem 3.1.2. □

Example 3.2.1. Let $\widetilde{A} = ((0.1, 0.3, 0.4, 0.5), (0.05, 0.2, 0.5, 0.6))$ and $\widetilde{B} = ((0.1, 0.3, 0.4, 0.5), (0.01, 0.24, 0.45, 0.65))$ be two TraIFNs. Here $S_1(\widetilde{A}, \widetilde{B}) = 1$ and $S_2(\widetilde{A}, \widetilde{B}) = 0.9629$. Hence, S_2 gives the degree of similarity between TraIFNs $\widetilde{A}$ and $\widetilde{B}$.

Example 3.2.2. Let $\widetilde{A} = ((0.3, 0.4, 0.6, 0.7), (0.2, 0.4, 0.65, 0.75))$ and $\widetilde{B} = ((0.3, 0.4, 0.55, 0.75), (0.3, 0.3, 0.6, 0.8))$ be two TraIFNs. Here $S_1(\widetilde{A}, \widetilde{B}) = 1$ and $S_2(\widetilde{A}, \widetilde{B}) = 1$. Therefore, the similarity measures S_1 and S_2 are not sufficient to measure the degree of similarity between any two TraIFNs. Hence, we need another distance based similarity measure to measure the degree of similarity between any two TraIFNs. In the next section, we will discuss the exact score based similarity measure between TraIFNs.

3.3 Exact score based similarity measure on TraIFNs

Definition 3.3.1. Let $\widetilde{A}$ and $\widetilde{B}$ be two TraIFNs. Then the exact score based distance measure between TraIFNs $\widetilde{A}$ and $\widetilde{B}$ is defined as

$$D_3(\widetilde{A}, \widetilde{B}) = | J_8(\widetilde{A}) - J_8(\widetilde{B}) | \tag{3}$$

where $J_8(\widetilde{A})$ and $J_8(\widetilde{B})$ are exact score functions [9] of two TraIFNs $\widetilde{A}$ and $\widetilde{B}$ respectively.

The following theorem proves that the function D_3 is a distance measure on the exact class of TraIFNs.

Theorem 3.3.1. *The measure* $D_3(\widetilde{A}, \widetilde{B})$ *is a distance measure between TraIFNs* $\widetilde{A}$ *and* $\widetilde{B}$.

Proof. The proof is similar to the Theorem 3.1.1. □

The exact score based similarity measure between two TraIFNs is introduced as follows.

Definition 3.3.2. Let $\widetilde{A}$ and $\widetilde{B}$ be two TraIFNs in $X = \{x_1, x_2, \ldots, x_n\}$. Then the exact score based similarity measure between two TraIFNs $\widetilde{A}$ and $\widetilde{B}$ is defined as $S_3(\widetilde{A}, \widetilde{B}) = 1 - D_3(\widetilde{A}, \widetilde{B})$, where $D_3(\widetilde{A}, \widetilde{B})$ is in Definition 3.3.1.

The meaning of the larger the value of $S_3(\widetilde{A}, \widetilde{B})$, the more the similarity between $\widetilde{A}$ and $\widetilde{B}$. The following theorem proves that the function S_3 is a exact score based similarity measure on the exact class of TraIFNs.

Theorem 3.3.2. *The measure* $S_3(\widetilde{A}, \widetilde{B})$ *is a similarity measure between TraIFNs* $\widetilde{A}$ *and* $\widetilde{B}$.

Proof. The proof is similar to the Theorem 3.1.2. □

Example 3.3.1. Let $\widetilde{A} = ((0.3, 0.4, 0.6, 0.7), (0.2, 0.4, 0.65, 0.75))$ and $\widetilde{B} = ((0.3, 0.4, 0.55, 0.75), (0.3, 0.3, 0.6, 0.8))$ be two TraIFNs. Here $S_1(\widetilde{A}, \widetilde{B}) = 1$, $S_2(\widetilde{A}, \widetilde{B}) = 1$ and $S_3(\widetilde{A}, \widetilde{B}) = 1.9$. Hence, S_3 gives the degree of similarity between two TraIFNs $\widetilde{A}$ and $\widetilde{B}$.

3.4 Axiomatic relation on distances based similarity measures between TraIFNs

The axiomatic relation on distances based similarity measures between any two TraIFNs is defined as follows.

Definition 3.4.1. Let $\widetilde{A}$ and $\widetilde{B}$ be any two TraIFNs. Then, first apply the similarity measure $S_1(\widetilde{A}, \widetilde{B})$ for two TraIFNs $\widetilde{A}$ and $\widetilde{B}$.

If the degree of similarity of $S_1(\widetilde{A}, \widetilde{B})$ is equal to one, then apply the similarity measure $S_2(\widetilde{A}, \widetilde{B})$ for any two TraIFNs.

If the degree of similarity of $S_1(\widetilde{A}, \widetilde{B})$ and $S_2(\widetilde{A}, \widetilde{B})$ is equal to one, then apply the similarity measure $S_3(\widetilde{A}, \widetilde{B})$ for any two TraIFNs.

If the degree of similarity of $S_1(\widetilde{A}, \widetilde{B})$, $S_2(\widetilde{A}, \widetilde{B})$ and $S_3(\widetilde{A}, \widetilde{B})$ is equal to one, then we can conclude that the two TraIFNs $\widetilde{A}$ and $\widetilde{B}$ are equal (same).

The proposed method is validated by comparing some of existing methods and applied to pattern recognition problems which is discussed in the next section.

4. Applications

In this section, first we see the efficiency of the proposed method over familiar existing methods by using numerical examples. Further the applicability and importance of the proposed method in solving pattern recognition problems are shown by illustrative examples.

Table 1. Existing similarity measures.

Author(s)	Similarity measures
Chen [3]	$S_C(A, B) = 1 - \frac{\sum_{i=1}^{n} \lvert (\mu_A(x_i) - \nu_{\tilde{A}}(x_i)) - (\mu_B(x_i) - \nu_B(x_i)) \rvert}{2n}$
Hong and Kim [6]	$S_H(A, B) = 1 - \frac{\sum_{i=1}^{n} \lvert (\mu_A(x_i) - \nu_B(x_i)) - (\mu_{\tilde{A}}(x_i) - \nu_B(x_i)) \rvert}{2n}$
Ye [17]	$C_{IFS}(A, B) = \frac{1}{n} - \sum_{i=1}^{n} \frac{\mu_A(x_i)\mu_B(x_i) + \nu_{\tilde{A}}(x_i)\nu_B(x_i)}{\sqrt{(\mu_A(x_i))^2 + (\nu_A(x_i))^2} + \sqrt{(\mu_B(x_i))^2 + (\nu_B(x_i))^2}}$
Xu and Chen [16]	$S_{Xu_1}(A, B) = 1 - [\frac{1}{4n} \sum_{j=1}^{n} (\lvert \mu_{A_L}(x_j) - \mu_{B_L}(x_j) \rvert^{\alpha} + \lvert \mu_{A_U}(x_j) - \mu_{B_U}(x_j) \rvert^{\alpha} + \lvert \nu_{A_L}(x_j) - \nu_{B_L}(x_j) \rvert^{\alpha} + \lvert \nu_{A_U}(x_j) - \nu_{B_U}(x_j) \rvert^{\alpha})]^{\frac{1}{\alpha}}, \alpha > 0$
Ye [18]	$S_H(A, B) = 1 - \frac{1}{8} (\sum_{i=1}^{4} \lvert a_{1i} - a_{2i} \rvert + \sum_{j=1}^{4} \lvert b_{1j} - b_{2j} \rvert)$
Ye [18]	$S_E(A, B) = 1 - \sqrt{\frac{1}{8} (\sum_{i=1}^{4} (a_{1i} - a_{2i})^2 + \sum_{j=1}^{4} (b_{1j} - b_{2j})^2)}$
Song et al. [14]	$S_S(\tilde{A}, \tilde{B}) = \frac{1}{2n} \sum_{i=1}^{n} (\sqrt{\mu_A(x_i)\mu_B(x_i)} + 2\sqrt{\nu_A(x_i)\nu_B(x_i)} + \sqrt{(1 - \nu_A(x_i))(1 - \nu_B(x_i))} + \sqrt{\pi_A(x_i)\pi_B(x_i)})$
Chen and Cheng [4]	$S_{CC}(A, B) = 1 - \frac{\lvert 2(\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) \rvert}{3} \times (1 - \frac{\pi_A(x_i) - \pi_B(x_i)}{2})$

	$-\dfrac{\mid 2(\nu_A(x_i)-\nu_B(x_i))-(\mu_A(x_i)-\mu_B(x_i))\mid}{3}$ $\times(\dfrac{\pi_A(x_i)-\pi_B(x_i)}{2})$
Ngan et al. [13]	$d_{Hm}(A, B)=\dfrac{1}{3n}\sum_{i=1}^{n}(\mid \mu_A(x_i)-\mu_B(x_i)\mid$ $+\mid \nu_A(x_i)-\nu_B(x_i)\mid+\mid \max\{\mu_A(x_i), \nu_B(x_i)\}$ $-\max\{\mu_A(x_i), \nu_B(x_i)\}\mid)$
Jiang et al. [7]	$S_{(A, B)}=1-\dfrac{1}{2n}\sum(\mid \dfrac{2(\mu_A(x_i)\pi_B(x_i)-\mu_A(x_i)\pi_B(x_i)) - 4(\mu_A(x_i)-\mu_B(x_i))\mid}{4-\pi_A(x_i)\pi_B(x_i)}\mid$ $+\mid \dfrac{4(\nu_A(x_i)-\nu_B(x_i))+2(\nu_A(x_i)\pi_B(x_i)-\nu_B(x_i)\pi_A(x_i)) + 2(\pi_A(x_i)-\pi_B(x_i))}{4-\pi_A(x_i)\pi_B(x_i)}\mid)$
Lakshmana et al. [9]	$S_{(A, B)}=1-\dfrac{1}{4}(\mid a_1-a_2\mid+\mid b_1-b_2\mid$ $+\mid c_2(1-a_2)-c_1(1-a_1)\mid+\mid d_2(1-b_2)-d_1(1-b_1)\mid)$

4.1 Proposed method compared with some existing methods

In this subsection, the proposed method is compared with some of the existing similarity measures by numerical examples whose definitions are given in Table 1. The advantage of the proposed method over familiar existing methods is illustrated in Table 2.

According to the method of Chen [3] and Ye [17], the results of similarity measure for the sets $\widetilde{A}=((0.1, 0.1, 0.1, 0.1), (0.1, 0.1, 0.1, 0.1))$ and $\widetilde{B}=((0.2, 0.2, 0.2, 0.2), (0.2, 0.2, 0.2, 0.2))$ are equal to one but its geometrical figures imply that it is not correct whereas our proposed method gives different value of similarity for those two sets. That is, $S(\widetilde{A}, \widetilde{B})=0.97$.

Table 2. Comparison with some of existing similarity measures.

Author (s)	1	2	3	4
	(0.2, 0.2) (0.3, 0.3)	(0.1, 0.1) (0.2, 0.2)	((0.1, 0.2, 0.3, 0.4) (0.1, 0.1, 0.5, 0.6)) ((0.2, 0.3, 0.4, 0.5) (0.0, 0.1, 0.4, 0.5))	((0.1, 0.2, 0.3, 0.4) (0.05, 0.15, 0.7, 0.8)) ((0.2, 0.3, 0.4, 0.5) (0.1, 0.2, 0.7, 0.8))
Chen [3]	1	1	NA	NA
Hong and Kim [6]	0.9	0.9	NA	NA
Ye [17]	1	1	NA	NA
Xu and Chen1 [16]	0.9	0.9	NA	NA
Ye [18]	0.9	0.9	0.9125	0.9375
Ye [18]	0.9	0.9	0.9065	0.925
Song et al. [14]	0.9865	0.9828	NA	NA
Chen and Cheng [4]	0.9667	0.9667	NA	NA
Ngan et al. [13]	0.0667	0.9333	NA	NA
Jiang et al. [7]	0.8404	0.8295	NA	NA
Lakshmana et al. [9]	0.925	0.915	NA	NA
Proposed method	0.95	0.97	0.825	0.925

4.2 Pattern recognition. TraIFSs and IFSs are mathematical tools to process imprecise information. In this subsection, the proposed similarity measures S_K, $K = 1, 2, 3$ for TraIFNs are applied to pattern recognition problems to demonstrate its effectiveness.

We apply the similarity measures S_K, $K = 1, 2, 3,$ using the Definition 3.4.1, to solve the pattern recognition problems with trapezoidal intuitionistic fuzzy information.

Algorithm.

Step 1. In the pattern recognition problem, suppose that there exist m patterns represented by TraIFSs

$$\widetilde{A}^i = \{\langle x_j, (\widetilde{a}^i_{1\mu}(x_j), \widetilde{a}^i_{2\mu}(x_j), \widetilde{a}^i_{3\mu}(x_j), \widetilde{a}^i_{4\mu}(x_j)), (\widetilde{a}^i_{1\nu}(x_j), \widetilde{a}^i_{2\nu}(x_j), \widetilde{a}^i_{3\nu}(x_j), \widetilde{a}^i_{4\nu}(x_j))\rangle \mid x_j \in X\}$$

for $i = 1, 2, \ldots, m$ in $X = \{x_1, x_2, \ldots, x_n\}$, and suppose that there is a sample to be recognized, which is represented by a TraIFS

$$\widetilde{B} = \{\langle x_j, (\widetilde{b}^i_{1\mu}(x_j), \widetilde{b}^i_{2\mu}(x_j), \widetilde{b}^i_{3\mu}(x_j), \widetilde{b}^i_{4\mu}(x_j)), (\widetilde{b}^i_{1\nu}(x_j), \widetilde{b}^i_{2\nu}(x_j), \widetilde{b}^i_{3\nu}(x_j), \widetilde{b}^i_{4\nu}(x_j))\rangle \mid x_j \in X\}$$

Step 2. Calculate the similarity measure $S_K(A^i, B)$, $K = 1, 2, 3$ between $\widetilde{A}^i$ and $\widetilde{B}$ using Definition 3.4.1.

Step 3. Select the largest one, denoted by $S_K(A^k, B)$, $K = 1, 2, 3$ from $S_K(A^i, B)$, $K = 1, 2, 3$, $i = 1, 2, \ldots, m$. Then $\widetilde{B}$ is more similar (close) to the pattern $\widetilde{A}^k$.

The proposed method for TraIFSs is compared and validated in the following examples.

Example 4.2.1. Let $\widetilde{A} = ((0.3, 0.3, 0.3, 0.3), (0.3, 0.3, 0.9, 0.9))$, $\widetilde{B} = ((0.2, 0.2, 0.2, 0.2), (0.2, 0.2, 0.8, 1.0))$ and $\widetilde{C} = ((0.2, 0.2, 0.2, 0.2), (0.2, 0.2, 1.0, 1.0))$ be three TraIFNs. Then $S_1(\widetilde{A}, \widetilde{B}) = 0.93$ and $S_1(\widetilde{A}, \widetilde{C}) = 0.9$. From this example, we observed that the sample $\widetilde{A}$ is close to pattern $\widetilde{B}$ compared with the pattern $\widetilde{C}$.

But, the results of Hamming and Euclidean distances based similarity measures proposed by Ye [18] are $S_H(\tilde{A}, \tilde{B}) = S_H(\tilde{A}, \tilde{C}) = 0.9$ and $S_E(\tilde{A}, \tilde{B}) = S_E(\tilde{A}, \tilde{C}) = 0.9$ which is illogical to human intuition.

Example 4.2.2. Let $\tilde{A} = ((0.1, 0.2, 0.3, 0.4), (0.1, 0.2, 0.3, 0.4))$, $\tilde{B} = ((0.1, 0.1, 0.3\ 0.3), (0.1, 0.1, 0.3, 0.3))$ and $\tilde{C} = ((0.2, 0.2, 0.4, 0.4), (0.2, 0.2, 0.4, 0.4))$ be three TraIFNs. Then $S_1(\tilde{A}, \tilde{B}) = 0.9775$ and $S_1(\tilde{A}, \tilde{C}) = 0.9725$. Although in the proposed method presented above that sample $\tilde{A}$ is close to pattern $\tilde{B}$, which indicates that sample $\tilde{A}$ should be classified to pattern $\tilde{B}$.

In the result of proposed method by Ye [18], $S_H(\tilde{A}, \tilde{B}) = S_H(\tilde{A}, \tilde{C}) = 0.95$ and $S_E(\tilde{A}, \tilde{B}) = S_E(\tilde{A}, \tilde{C}) = 0.9293$ which is illogical to human intuition.

Example 4.2.3. Let $\tilde{A} = ((0.15, 0.25, 0.45, 0.55), (0.1, 0.2, 0.5, 0.6))$, $\tilde{B} = ((0.15, 0.25, 0.45\ 0.55), (0.11, 0.21, 0.51, 0.61))$ and $\tilde{C} = ((0.15, 0.25, 0.46, 0.56), (0.1, 0.2, 0.51, 0.61))$ be three TraIFNs. Then $S_1(\tilde{A}, \tilde{B}) = 0.99$ and $S_1(\tilde{A}, \tilde{C}) = 0.9947$. Although in the proposed method presented above that sample $\tilde{A}$ is close to pattern $\tilde{C}$, which indicates that sample $\tilde{A}$ should be classified to pattern $\tilde{C}$.

It is observed that $\tilde{B}$ and $\tilde{C}$ are two different TraIFNs. However, we can apply the existing similarity measure proposed by Ye [18], $S_H(\tilde{A}, \tilde{B}) = S_H(\tilde{A}, \tilde{C}) = 0.995$ and $S_E(\tilde{A}, \tilde{B}) = S_E(\tilde{A}, \tilde{C}) = 0.9929$ which is illogical to human intuition.

5. Conclusions

In this paper, three new distances based similarity measures between trapezoidal intuitionistic fuzzy numbers for pattern recognition problems using some score functions discussed in [9] have introduced and studied to measure the closeness between any two TraIFNs. Generally, trapezoidal intuitionistic fuzzy number is very effective to deal with pattern recognition problems. From Table 2, the proposed method is compared with some of the existing methods to show the effectiveness by using numerical examples. The proposed method can overcome the drawbacks of the some existing similarity measures. Finally, the proposed distances based similarity measures of TraIFNs are applied to the pattern recognition problems by numerical examples. For future research, we may extend the same work to decision making problem and image processing domain.

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