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A NOVEL ANALYTICAL APPROACH FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract

In this paper, a unified form of the Homotopy perturbation technique, the Yang integral transform and J. H. He's polynomials is proposed to solve some non-linear partial differential equations of fractional order. The fact that the nonlinear equations can be easily manipulated by J. H. He's polynomials rather than Adomian's polynomials, is referred as a benefit of the proposed technique. Moreover, the proposed technique gives the solutions in expeditions convergent series leading to the closed form solutions without any discretization or limitation on assumptions and so is a refinement of the many existing techniques.

1. Introduction

Fractional calculus which concerns arbitrary order derivatives and integrals plays a prominent role in a variety of science and engineering domains. The linear and nonlinear partial differential equations of fractional order have broad applications in Acoustic, Analytical chemistry, Biology, Signal processing, Fluid mechanics, Electromagnetism and so forth. Most nonlinear differential equations do not attain analytical solutions. In the last

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few decades, several numerical techniques have been devised to solve linear and nonlinear partial differential equations of fractional order such as, Homotopy analysis technique, Laplace decomposed technique, Adomian decomposition technique, Homotopy perturbation sumudu transform technique, Jacobi spectral collocation technique, Homotopy perturbation technique, Fractional complex transform technique, Yang-Laplace transform technique and so forth. Howsoever, due to several shortcomings and computational complexities such as discretization of variables, unnecessary linearization, transformation or the use of restrictive assumptions, these numerical approaches cannot be considered as universal for solving linear and nonlinear partial differential equations of fractional order.

The major goal of this paper is to propose a novel analytical approach termed the Homotopy Perturbation Yang Integral Transform Technique (HPYITT) for solving linear and nonlinear partial differential equations of fractional order without the shortcomings listed above. The proposed technique generates a series solutions that converges quickly to an exact solution with precise computational parameters. Futhermore, the nonlinear equations are enumerated using J. H. He's polynomials in this novel analytical approach.

The present article is organized as follows. In Section 2, some basic definitions, some properties of Yang integral transform and Homotopy perturbation technique are discussed. In section 3, the analysis of the proposed technique is discussed. In section 4, some examples are illustrated to elucidate the applicability and the efficiency of the proposed technique.

2. Preliminaries

Definition 2.1. The fractional derivative with order $\lambda \geq 0,$ of a function $g(\tau) \in L^1([0, \infty))$ in Caputo sense, is defined by

$$ {}^{c}D_{\tau}^{\lambda} g(\tau) = \begin{cases} \dfrac{1}{\lambda(n-\lambda)} \displaystyle\int_0^{\tau} (\tau - \xi)^{n-\lambda-1} g^{(n)}(\xi) d\xi, & n-1 < \lambda < n,\ n \in \mathbb{N} \\ \dfrac{d^n}{d\tau^n} g(\tau) & \lambda = n \in \mathbb{N}. \end{cases} $$

The Caputo time fractional derivative of order λ of a function $g(y, \tau) \in L^1([0, \infty) \times [0, \infty)),$ is defined by

$$ {}^{c}D_{\tau}^{\lambda} g(y,\ \tau) = \begin{cases} \dfrac{1}{\lambda(n-\lambda)} \displaystyle\int_0^{\tau} (\tau-\xi)^{n-\lambda-1} \dfrac{\partial^{(n)} g(y,\ \xi)}{\partial \xi^{n}} d\xi, & n-1 < \lambda < n,\ n \in \mathbb{N} \\ \dfrac{\partial^{(n)} g(y,\ \xi)}{\partial \xi^{n}} & \lambda = n \in \mathbb{N}. \end{cases} $$

Definition 2.2. If $f(\tau)$ is a function defined for $\tau \geq 0,$ then the integral

$$ \int_0^{\infty} e^{-s\tau} f(\tau) d\tau,\ 0 < \tau < \infty,\ s \in \mathbb{C} $$

is called the Laplace transform of $f(\tau),$ assuming that the integral exists and is usually denoted by $L\{f(\tau); s\}$ or $F(s).$

Definition 2.3. Let $A = \{g(\tau) \in L^{1}([0,\ \infty)); \exists N,\ p_1$ and/or $p_2 > 0 \in \mathbb{R}$ such that $|g(\tau)| < Ne^{\frac{|\tau|}{p_i}},$ if $\tau \in (-1)^{i} \times [0,\ \infty),\ i = 1,\ 2,\}$

where N must be a finite constant, while $p_1,\ p_2$ may be finite and need not exist simultaneously.

The Yang integral transform of a function $g(\tau) \in A$ is defined by

$$ Y\{g(\tau); w\} = G(w) = \int_0^{\infty} e^{\frac{-\tau}{w}} g(\tau) d\tau, $$

assuming that the integral exists for some $w \in (-p_1,\ p_2).$

Some important properties of Yang integral transform are listed below.

1. Yang integral transform for some basic functions:

- $Y\{\tau^{n}; w\} = n!\, w^{n+1}$
- $Y\{e^{a\tau}; w\} = \dfrac{w}{1-aw}$
- $Y\{\sin a\tau; w\} = \dfrac{aw^{2}}{1+a^{2}w^{2}}$
- $Y\{\cos a\tau; w\} = \dfrac{w}{1+a^{2}w^{2}}$

2. Linear property: If $Y\{g(\tau)\} = G(w)$ and $Y\{h(\tau)\} = H(w)$, then

$Y\{c_1 g(\tau) + c_2 h(\tau)\} = c_1 G(w) + c_2 H(w)$, where c_1 and c_2 are constants.

3. Laplace-Yang duality property: $G(w) = F\left(\frac{1}{w}\right)$, where $F(w) = L\{f(\tau); w\}$.

Definition 2.4. Let $B = \{g(y, \tau) \in L^1([0, \infty) \times [0, \infty)); \exists N, p_1$ and/or $p_2 > 0 \in \mathbb{R}$ such that $|g(y, \tau)| < Ne^{\frac{|\tau|}{p_i}}$, if $\tau \in (-1)^i \times [0, \infty), i = 1, 2\}$ where $y \in \mathbb{R}$, N must be a finite constant, while p_1, p_2 may be finite and need not exist simultaneously.

The Yang integral transform of a function $g(y, \tau) \in B$ is defined by

$$Y\{g(y, \tau); w\} = G(y, w) = \int_0^\infty e^{\frac{-\tau}{w}} g(y, \tau) d\tau,$$

assuming that the integral exists for some $w \in (-p_1, p_2)$.

Theorem 2.5. *If $G(w)$ is the Yang integral transform of a function $g(\tau)$, then $Y\{{}^c D_\tau^\lambda g(\tau); w\} = \frac{G(w)}{w^\lambda} - \sum_{k=0}^{n-1} w^{k-\lambda+1} g^{(k)}(0)$, where $n - 1 < \lambda \leq n, n \in \mathbb{N}$*

Proof. By Laplace-Yang duality property, we have

$$Y\{{}^c D_\tau^\lambda g(\tau); w\} = L\left\{{}^c D_\tau^\lambda g(\tau); \frac{1}{w}\right\}$$

$$= \left(\frac{1}{w}\right)^\lambda F\left(\frac{1}{w}\right) - \sum_{k=0}^{n-1} \left(\frac{1}{w}\right)^{(\lambda-k-1)} g^{(k)}(0)$$

$$Y\{{}^c D_\tau^\lambda g(\tau); w\} = \frac{G(w)}{w^\lambda} - \sum_{k=0}^{n-1} w^{k-\lambda+1} g^{(k)}(0)$$

Remark 2.6. The Yang integral transform of Caputo time fractional derivative with order λ of the function $g(y, \tau)$ is given by

$$Y\{{}^{c}D_{\tau}^{\lambda}g(y,\ \tau); w\} = \frac{G(y,\ w)}{w^{\lambda}} - \sum_{k=0}^{n-1} w^{k-\lambda+1} g^{(k)}(y,\ 0), \text{ where}$$

$$n - 1 < \lambda < n,\ n \in \mathbb{N}$$

2.1 Homotopy Perturbation Technique (HPT)

Let us investigate the general nonlinear differential equation

$$R(\mathcal{Z}) - g(r) = 0,\ r \in \Omega, \tag{1}$$

having the boundary constraints

$$B\left(\mathcal{Z},\ \frac{\partial \mathcal{Z}}{\partial n}\right) = 0,\ r \in \partial\Omega, \tag{2}$$

where R is a general differential operator, $g(r)$ is a given analytical function, B is a boundary operator, $\partial\Omega$ is the boundary of the domain Ω and n denotes the normal to the boundary $\partial\Omega$. Now splitting up the differential operator R into the linear operator L and the nonlinear operator N, the equation (1) becomes

$$L(\mathcal{Z}) + N(\mathcal{Z}) - g(r) = 0. \tag{3}$$

Using the homotopy technique, we form a homotopy $h(r,\ q) : D \times [0,\ 1] \to \mathbb{R}$ satisfying

$$P(h,\ q) = (1 - q)[L(h) - L(Z_0)] + q[R(h) - g(r)] = 0\ q \in [0,\ 1],\ r \in D$$

(or)

$$P(h,\ q) = L(h) - L(\mathcal{Z}_0) + pL(\mathcal{Z}_0) + q[N(h) - g(r)] = 0, \tag{4}$$

where the embedding parameter $q \in [0,\ 1]$ is considered as an extending parameter, $\mathcal{Z}_0$ is the initial approximation of (1), which satisfies the boundary constraints

$$P(h,\ 0) = L(h) - L(\mathcal{Z}_0) = 0$$

$$P(h,\ 1) = R(h) - g(r) = 0. \tag{5}$$

The process of altering q from 0 to 1 is the same as changing $h(r,\ q)$ from

$\mathcal{Z}_0$ to $\mathcal{Z}$, which is called the deformation in topology and $L(h) - L(\mathcal{Z}_0)$, $R(h) - g(r)$ are known as homotopic. Using classical perturbation technique, the power series solution of (4) in q is given as

$$h = h_0 + qh_1 + q^2 h_2 + \ldots \tag{6}$$

and assuming $q = 1$, it leads to the approximation solution of (1), that is

$$\mathcal{Z} = \lim_{q \to 1} h = h_0 + h_1 + h_2 + \ldots \tag{7}$$

The incorporation of the Homotopy technique and the perturbation technique is known as the HPT, which eliminates the limitations of the standard perturbation techniques. Moreover, the proposed technique can take the full advantage of the standard perturbation techniques.

For the most cases, the series solution (7) is convergent. Howsoever, the convergent rate depends upon the nonlinear operator $N(h)$, whose the second derivative of $N(h)$ with respect to h must be small, because the parameter q may be relatively large, i.e. $q \to 1$ and the norm of $L^{-1}\left(\frac{\partial N}{\partial h}\right)$ must be smaller than one, in order that the series solution is convergent.

3. Analysis of the Proposed Technique

3.1 Homotopy Perturbation Yang integral transform technique

To demonstrate the fundamental idea of the proposed technique, we examine a general nonlinear non-homogeneous partial differential equation

$$ {}^{c}D_{\tau}^{\lambda}\mathcal{Z}(y, \tau) + L_{\mathcal{Z}}(y, \tau) + N_{\mathcal{Z}}(y, \tau) = g(y, \tau),\ n - 1 < \lambda \leq n; n \in \mathbb{N} \tag{8}$$

having the initial constraints

$$\mathcal{Z}(y, 0) = h(y)\ \mathcal{Z}_{\tau}(y, 0) = f(y),$$

where ${}^{c}D_{\tau}^{\lambda}$ is the caputo fractional differential operator of order λ, L is a linear differential operator of order less than λ, N is a nonlinear differential operator and $g(y, \tau)$ is the given source term.

Applying the Yang integral transform on both sides of (8) and using the differentiation property, we arrive

$$\frac{G(y,\, w)}{w^{\lambda}} - \sum_{k=0}^{n-1} w^{k-\lambda+1} \mathcal{Z}^{(k)}(y,\, 0) + Y\{L(\mathcal{Z})\} + Y\{N(\mathcal{Z})\} = Y\{g(y,\, \tau)\}$$

and so

$$G(y,\, w) = \sum_{k=0}^{n-1} w^{k+1} \mathcal{Z}^{(k)}(y,\, 0) - w^{\lambda}[Y\{L(\mathcal{Z})\} + Y\{N(\mathcal{Z})\}] + w^{\lambda}[Y\{g(y,\, \tau)\}] \quad (9)$$

Performing the inverse Yang integral transform on (9), we attain,

$$\mathcal{Z}(y,\, \tau) = k(y,\, \tau) - Y^{-1}\{w^{\lambda}[Y\{L(\mathcal{Z}) + N(\mathcal{Z})\}]\} \quad (10)$$

where $k(y,\, \tau) = \sum_{k=0}^{n-1} w^{k+1} \mathcal{Z}^{(k)}(y,\, 0) + w^{\lambda}[Y\{g(y,\, \tau)\}]$ represents the term arising from the given source term and the initial constraints.

By the HPT,

$$\mathcal{Z}(y,\, \tau) = \sum_{n=0}^{\infty} q^n \mathcal{Z}_n \text{ and} \quad (11)$$

the non-linear term can be decomposed as

$$N\mathcal{Z}(y,\, \tau) = \sum_{n=0}^{\infty} q^n P_n(\mathcal{Z}), \quad (12)$$

where $P_n(\mathcal{Z})$ are J. H. He's polynomials that are given by

$$P_n(\mathcal{Z}_0,\, \mathcal{Z}_1,\, \ldots,\, \mathcal{Z}_n) = \frac{1}{n!}\frac{\partial^n}{\partial q^n}\left[N\left(\sum_{i=0}^{\infty}(q^i \mathcal{Z}_i)\right)\right]_q = 0 \;\; n = 0,\, 1,\, 2,\, \ldots$$

Using (11) and (12) in (10), we get

$$\sum_{n=0}^{\infty} q^n \mathcal{Z}_n(y,\, \tau) = k(y,\, \tau) - q\left(Y^{-1}\left\{w^{\lambda} Y\left\{L \sum_{n=0}^{\infty} q^n \mathcal{Z}_n(y,\, \tau) + \sum_{n=0}^{\infty} q^n P_n(\mathcal{Z})\right\}\right\}\right)$$

which is the combined form of the Yang integral transform with the HPT using He's polynomials.

Comparing the coefficients of similar powers of q on both sides, we attain

$$q^0 : \mathcal{Z}_0(y, \tau) = k(y, \tau)$$

$$q^1 : \mathcal{Z}_1(y, \tau) = -Y^{-1}\{w^{\lambda} Y\{L_{\mathcal{Z}_0}(y, \tau) + P_0(\mathcal{Z})\}\}$$

$$q^2 : \mathcal{Z}_2(y, \tau) = -Y^{-1}\{w^{\lambda} Y\{L_{\mathcal{Z}_1}(y, \tau) + P_1(\mathcal{Z})\}\}$$

and so forth.

Then the exact solution $\mathcal{Z}(y, \tau)$ is approximated by the truncated series,

$$\mathcal{Z}(y, \tau) = \lim_{n\to\infty} \sum_{n=0}^{N} \mathcal{Z}_n(y, \tau).$$

4. Applications

To elucidate the applicability and the efficiency of the proposed technique, some nonlinear partial differential equations of fractional order involving caputo fractional derivatives are discussed.

Example 4.1. Let us first examine the nonlinear time fractional gas dynamic equation

$$ {}^{c}D_{\tau}^{\lambda}\mathcal{Z}(y, \tau) + \frac{1}{2}(\mathcal{Z}^2)_y - \mathcal{Z}(1-\mathcal{Z}) = 0,\ 0 < \lambda \leq 1,\ \tau > 0,\ y \in \mathbb{R} \tag{13}$$

having the initial constraint

$$\mathcal{Z}(y, 0) = e^{-y}.$$

Taking the Yang integral transform on both sides of (13), we attain

$$\frac{G(y, w)}{w^{\lambda}} - \sum_{k=0}^{n-1} w^{k-\lambda+1}\mathcal{Z}^{(k)}(y, 0) = -Y\left\{\frac{1}{2}(\mathcal{Z}^2)_y - \mathcal{Z}(1-\mathcal{Z})\right\}$$

$$G(y, w) = k(y, \tau) - w^{\lambda} Y\left\{\frac{1}{2}(\mathcal{Z}^2)_y - \mathcal{Z}(1-\mathcal{Z})\right\} \tag{14}$$

Performing the inverse Yang integral transform on (14), we arrive

$$\mathcal{Z}(y,\ \tau) = e^{-y} - Y^{-1}\left\{w^{\lambda}Y\left\{\frac{1}{2}(\mathcal{Z}^2)_y - \mathcal{Z}(1-\mathcal{Z})\right\}\right\}$$

According to HPT, we attain

$$\sum_{n=0}^{\infty} q^n \mathcal{Z}_n(y,\ \tau) =$$

$$e^{-y} - q\left(Y^{-1}\left\{w^{\lambda}Y\left\{\frac{1}{2}\left(\sum_{n=0}^{\infty} q^n P_n(\mathcal{Z})\right) - \left(\sum_{n=0}^{\infty} q^n \mathcal{Z}_n(y,\ \tau)\right) + \left(\sum_{n=0}^{\infty} q^n Q_n(\mathcal{Z})\right)\right\}\right\}\right) \quad (15)$$

where $P_n(\mathcal{Z})$ and $Q_n(\mathcal{Z})$ are He's polynomials whose the first few components are given by

$$P_0(\mathcal{Z}) = (\mathcal{Z}_0^2)_y \qquad Q_0(\mathcal{Z}) = \mathcal{Z}_0^2$$

$$P_1(\mathcal{Z}) = 2(\mathcal{Z}_0\mathcal{Z}_1)_y \qquad Q_1(\mathcal{Z}) = 2\mathcal{Z}_0\mathcal{Z}_1$$

$$P_2(\mathcal{Z}) = (\mathcal{Z}_1^2 + 2\mathcal{Z}_0\mathcal{Z}_2)_y \qquad Q_2(\mathcal{Z}) = \mathcal{Z}_1^2 + 2\mathcal{Z}_0\mathcal{Z}_2.$$

Comparing the coefficients of similar powers of q on both sides of (15), we attain

$$q^0 : \mathcal{Z}_0(y,\ \tau) = e^{-y}$$

$$q^1 : \mathcal{Z}_1(y,\ \tau) = -Y^{-1}\left\{w^{\lambda}\left[Y\left\{\frac{1}{2}P_0(\mathcal{Z}) - \mathcal{Z}_0 + Q_0(\mathcal{Z})\right\}\right]\right\}$$

$$= -Y^{-1}\left\{w^{\lambda}\left[Y\left\{\frac{1}{2}(\mathcal{Z}_0^2) - \mathcal{Z}_0 + \mathcal{Z}_0^2\right\}\right]\right\}$$

$$= -Y^{-1}\left\{w^{\lambda}\left[Y\left\{\frac{1}{2}(e^{-2y})_x - e^{-y} + e^{-2y}\right\}\right]\right\}$$

$$= -Y^{-1}\{w^{\lambda+1}\}(-e^{-y})$$

$$= e^{-y}\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}$$

$$q^2 : \mathcal{Z}_2(y,\ \tau) = -Y^{-1}\left\{w^{\lambda}\left[Y\left\{\frac{1}{2}P_1(\mathcal{Z}) - \mathcal{Z}_1 + Q_1(\mathcal{Z})\right\}\right]\right\}$$

$$= e^{-y} \frac{\tau^{2\lambda}}{\Gamma(2\lambda + 1)}$$

and so forth.

Then the exact solution in series form is attained by

$$\mathcal{Z}(y, \tau) = \lim_{n \to \infty} \sum_{n=0}^{N} \mathcal{Z}_n(y, \tau)$$

$$\mathcal{Z}(y, \tau) = e^{-y} \left\{ 1 + \frac{\tau^{\lambda}}{\Gamma(\lambda + 1)} + \frac{\tau^{2\lambda}}{\Gamma(2\lambda + 1)} + \dots \right\}$$

$$= e^{-y} E_{\lambda}(\tau^{\lambda})$$

When $\lambda = 1$, the exact solution of (13) is attained by

$$\mathcal{Z}(y, \tau) = \lim_{n \to \infty} \sum_{n=0}^{N} \mathcal{Z}_n(y, \tau)$$

$$= e^{-y} \left\{ 1 + \frac{\tau^{\lambda}}{\Gamma(\lambda + 1)} + \frac{\tau^{2}}{\Gamma(2\lambda + 1)} + \dots \right\}$$

$$\mathcal{Z}(y, \tau) = e^{\tau - y}$$

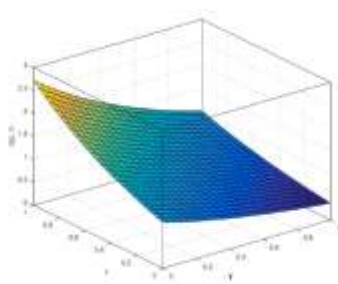

Figure 1. The exact solution for $\lambda = 1$ of example 4.1

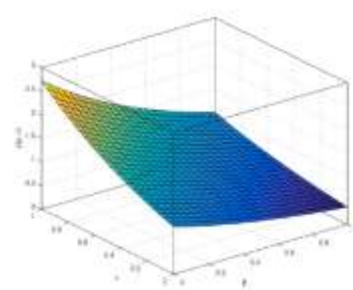

Figure 2. The approximate solution for $\lambda = 1$ of example 4.1

Example 4.2. Let us examine the nonlinear time fractional advection equation

$$ {}^{c}D_{\tau}^{\lambda}\mathcal{Z}(y,\ \tau) + \mathcal{Z}\mathcal{Z}_{y} = 0,\ 0 < \lambda \le 1 \tag{16} $$

having the initial constraint

$$ \mathcal{Z}(y,\ 0) = -y. $$

Applying the Yang integral transform on both sides of (16), we attain

$$ \frac{G(y,\ w)}{w^{\lambda}} - \sum_{k=0}^{n-1} w^{k-\lambda+1}\mathcal{Z}^{(k)}(y,\ 0) = -Y\{\mathcal{Z}\mathcal{Z}_{y}\} \text{ and so} $$

$$ G(y,\ w) = k(y,\ \tau) - w^{\lambda}Y\{\mathcal{Z}\mathcal{Z}_{y}\} \tag{17} $$

Performing the inverse Yang integral transform on (17), we arrive

$$ z(y,\ \tau) = -y - Y^{-1}\{w^{\lambda}[Y\{\mathcal{Z}\mathcal{Z}_{y}\}]\} $$

According to HPT, we attain

$$ \sum_{n=0}^{\infty} q^{n}\mathcal{Z}_{n}(y,\ \tau) = -y - q\left(Y^{-1}\left\{w^{\lambda}\left[Y\left\{\sum_{n=0}^{\infty} q^{n}P_{n}(\mathcal{Z})\right\}\right]\right\}\right), \tag{18} $$

where $P_{n}(\mathcal{Z})$ are He's polynomials whose the first few components are given by

$$ P_{0}(\mathcal{Z}) = \mathcal{Z}_{0}\mathcal{Z}_{0y} $$

$$ P_{1}(\mathcal{Z}) = \mathcal{Z}_{0}\mathcal{Z}_{1y} + \mathcal{Z}_{1}\mathcal{Z}_{0y} $$

$$ P_{2}(\mathcal{Z}) = \mathcal{Z}_{0}\mathcal{Z}_{2y} + \mathcal{Z}_{1}\mathcal{Z}_{1y} + \mathcal{Z}_{2}\mathcal{Z}_{0y}. $$

Comparing the coefficients of similar powers of q on both sides of (18), we attain

$$ q^{0} : \mathcal{Z}_{0}(y,\ \tau) = -y $$

$$ q^{1} : \mathcal{Z}_{1}(y,\ \tau) = -Y^{-1}\{w^{\lambda}[Y\{P_{0}(\mathcal{Z})\}]\} $$

$$ = -Y^{-1}\{w^{\lambda}[Y\{\mathcal{Z}_{0}\mathcal{Z}_{0y}\}]\} $$

$$= -Y^{-1}\{w^{\lambda}[Y\{(-y)(-y)_y\}]\}$$

$$= -Y^{-1}\{w^{\lambda+1}\}(y)$$

$$= -\frac{y\tau^{\lambda}}{\Gamma(\lambda+1)}$$

$$q^2 : \mathcal{Z}_2(y, \tau) = -Y^{-1}\{w^{\lambda}[Y\{P_1(\mathcal{Z})\}]\}$$

$$= -Y^{-1}\{w^{\lambda}[Y\{\mathcal{Z}_0\mathcal{Z}_{1y} + \mathcal{Z}_1\mathcal{Z}_{0y}\}]\}$$

$$= -\frac{2y\tau^{2\lambda}}{\Gamma(2\lambda+1)}$$

$$q^3 : \mathcal{Z}_3(y, \tau) = -Y^{-1}\{w^{\lambda}[Y\{P_2(\mathcal{Z})\}]\}$$

$$= -Y^{-1}\{w^{\lambda}[Y\{\mathcal{Z}_0\mathcal{Z}_{2y} + \mathcal{Z}_1\mathcal{Z}_{1y} + \mathcal{Z}_2\mathcal{Z}_{0y}\}]\}$$

$$= -\frac{y\tau^{3\lambda}}{\Gamma(3\lambda+1)}\left\{4 + \frac{\Gamma(2\lambda+1)}{\Gamma(\lambda+1)}\right\}$$

and so forth.

Then the exact solution in series form is attained by

$$\mathcal{Z}(y, \tau) = \lim_{n\to\infty}\sum_{n=0}^{N}\mathcal{Z}_n(y, \tau)$$

$$Z(y, \tau) = -y\left\{1 + \frac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \frac{2\tau^{2\lambda}}{\Gamma(2\lambda+1)} + \frac{\tau^{3\lambda}}{\Gamma(3\lambda+1)}\left\{4 + \frac{\Gamma(2\lambda+1)}{\Gamma(\lambda+1)^2}\right\} + \dots\right\}$$

When $\lambda = 1$, the exact solution of (16) is attained by

$$\mathcal{Z}(y, \tau) = \lim_{n\to\infty}\sum_{n=0}^{N}\mathcal{Z}_n(y, \tau)$$

$$= -y\left\{1 + \frac{\tau}{\Gamma(1)} + \frac{2\tau^2}{\Gamma(2)} + \frac{\tau^3}{\Gamma(3)}\left\{4 + \frac{\Gamma(2)}{\Gamma(1)^2}\right\} + \dots\right\}$$

$$= -y(1 + \tau + \tau^2 + \tau^3 + \ldots)$$

$$= -y\left(\frac{1}{1-\tau}\right)$$

$$Z(y,\ \tau) = \frac{y}{\tau - 1}$$

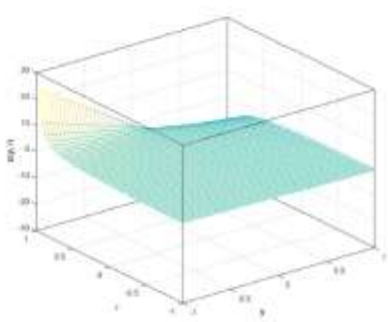

Figure 3. The exact solution for $\lambda = 1$ of example 4.2.

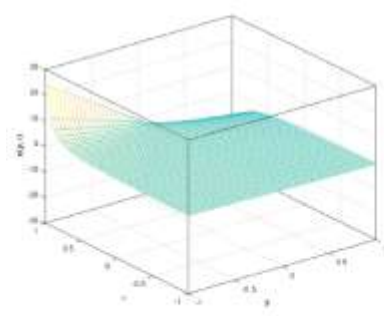

Figure 4. The approximate solution for $\lambda = 1$ of example 4.2

Example 4.3. Let us examine the nonlinear time fractional Korteweg-De Vries equation of the form

$$^{c}D_{\tau}^{\lambda}\mathcal{Z}(y,\ \tau) - 6\mathcal{Z}\mathcal{Z}_{y} + \mathcal{Z}_{yyy} = 0,\ 0 < \lambda \leq 1 \tag{19}$$

having the initial constraint

$$\mathcal{Z}(y,\ 0) = \frac{1}{6}(y - 1)$$

Applying the Yang integral transform on both sides of (19), we attain

$$\frac{G(y,\ w)}{w^{\lambda}} - \sum_{k=0}^{n-1} w^{k-\lambda+1}\mathcal{Z}^{(k)}(y,\ 0) = Y\{6_{\mathcal{Z}\mathcal{Z}_y} - \mathcal{Z}_{yyy}\}$$

$$G(y,\ w) = k(y,\ \tau) + w^{\lambda}Y\{6_{\mathcal{Z}\mathcal{Z}_y} - \mathcal{Z}_{yyy}\} \tag{20}$$

Performing the inverse Yang integral transform on (20), we arrive

$$\mathcal{Z}(y,\ \tau) = \frac{1}{6}(y - 1) + Y^{-1}\{w^{\lambda}Y\{6_{\mathcal{Z}\mathcal{Z}_y} - \mathcal{Z}_{yyy}\}\}$$

According to HPT, we attain

$$\sum_{n=0}^{\infty} q^n \mathcal{Z}_n(y,\ \tau) = \frac{1}{6}(y-1) + q\left(Y^{-1}\left\{w^{\lambda} Y\left\{6\sum_{n=0}^{\infty} q^n P_n(\mathcal{Z}) - \sum_{n=0}^{\infty} q^n \mathcal{Z}_n(y,\ \tau)\right\}\right\}\right), \tag{21}$$

where $P_n(\mathcal{Z})$ are He's polynomials whose the first few components are given by

$$P_0(\mathcal{Z}) = \mathcal{Z}_0 \mathcal{Z}_{0y}$$

$$P_1(\mathcal{Z}) = \mathcal{Z}_0 \mathcal{Z}_{1y} + \mathcal{Z}_1 \mathcal{Z}_{0y}$$

$$P_2(\mathcal{Z}) = \mathcal{Z}_0 \mathcal{Z}_{2y} + \mathcal{Z}_1 \mathcal{Z}_{1y} + \mathcal{Z}_2 \mathcal{Z}_{0y}.$$

Comparing the coefficients of similar powers of q on both sides of (21), we attain

$$q^0 : \mathcal{Z}_0(y,\ \tau) = \frac{1}{6}(y-1)$$

$$q^1 : \mathcal{Z}_1(y,\ \tau) = -Y^{-1}\{w^{\lambda}[Y\{P_0(\mathcal{Z}) - \mathcal{Z}_{0yyy}\}]\}$$

$$= -Y^{-1}\{w^{\lambda}[Y\{6_{\mathcal{Z}_0 \mathcal{Z}_{0y}} - \mathcal{Z}_{0yyy}\}]\}$$

$$= -Y^{-1}\left\{w^{\lambda}\left[Y\left\{6\left(\frac{1}{6}(y-1)\left(\frac{1}{6}(y-1)_y\right)\right) - \frac{1}{6}(y-1)_{yyy}\right\}\right]\right\}$$

$$= -Y^{-1}\{w^{\lambda+1}\}\left(\frac{1}{6}(y-1)\right)$$

$$= \frac{1}{6}(y-1)\frac{\tau^{\lambda}}{\Gamma(\lambda+1)}$$

$$q^2 : \mathcal{Z}_2(y,\ \tau) = -Y^{-1}\{w^{\lambda}[Y\{P_1(\mathcal{Z}) - \mathcal{Z}_{1yyy}\}]\}$$

$$= -Y^{-1}\{w^{\lambda}[Y\{6(\mathcal{Z}_0 \mathcal{Z}_{1y} + \mathcal{Z}_1 \mathcal{Z}_{0y}) - \mathcal{Z}_{1yyy}\}]\}$$

$$= \frac{1}{6}(y-1)\frac{2\tau^{2\lambda}}{\Gamma(2\lambda+1)}$$

$$q^3 : Z_3(y, \tau) = -Y^{-1}\{w^{\lambda}[Y\{P_2(Z) - Z_{2yyy}\}]\}$$

$$= -Y^{-1}\{w^{\lambda}[Y\{6(Z_0 Z_{2y} + Z_1 Z_{1y} + Z_2 Z_{0y}) - Z_{2yyy}\}]\}$$

$$= \frac{1}{6}(y-1)\frac{\tau^{3\lambda}}{\Gamma(3\lambda+1)}\left\{4 + \frac{\Gamma(2\lambda+1)}{\Gamma(\lambda+1)^2}\right\}$$

and so forth.

Then the exact solution in series form is attained by

$$Z(y, \tau) = \lim_{n\to\infty}\sum_{n=0}^{N} Z_n(y, \tau)$$

$$= \frac{1}{6}(y-1)\left\{1 + \frac{\tau^{\lambda}}{\Gamma(\lambda+1)} + \frac{2\tau^{2\lambda}}{\Gamma(2\lambda+1)} + \frac{\tau^{3\lambda}}{\Gamma(3\lambda+1)}\left\{4 + \frac{\Gamma(2\lambda+1)}{\Gamma(\lambda+1)^2}\right\} + \dots\right\}$$

When $\lambda = 1,$ the exact solution of (19) is attained by

$$Z(y, \tau) = \lim_{n\to\infty}\sum_{n=0}^{N} Z_n(y, \tau)$$

$$= \frac{1}{6}(y-1)\left\{1 + \frac{\tau}{\Gamma(1)} + \frac{2\tau^2}{\Gamma(2)} + \frac{t^3}{\Gamma(3)}\left\{4 + \frac{\Gamma(2)}{\Gamma(1)^2}\right\} + \dots\right\}$$

$$= \frac{1}{6}(y-1)(1 + \tau + \tau^2 + \tau^3 + \dots)$$

$$Z(y, \tau) = \frac{1}{6}\left(\frac{y-1}{1-\tau}\right).$$

5. Conclusion

This paper develops an elegant incorporation of the HPT, the Yang integral transform and J. H. He's polynomials to solve nonlinear time partial differential equations of fractional order. Splitting up the nonlinear terms in fractional differential equations using Adomian polynomials is simple but computing Adomian polynomials is very complicated.

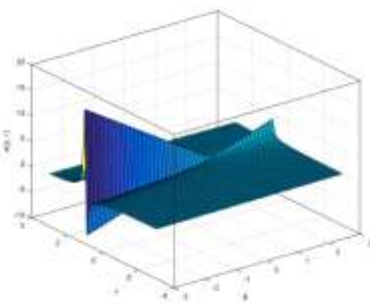

Figure 5. The exact solution for $\lambda = 1$ of example 4.3.

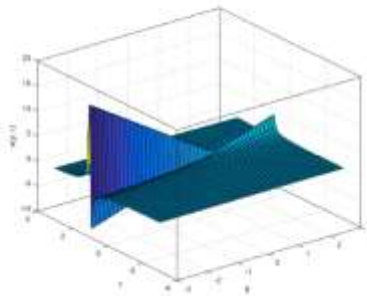

Figure 6. The approximate solution for $\lambda = 1$ of example 4.3.

This deficiency is overcome by J. H. He polynomials in the proposed technique. The proposed technique is also effective in lowering the amount of computational work as compared to the traditional approaches while still retaining good accuracy of the numerical results. In addition, the proposed technique generates solutions in a faster convergent series that yields the closed form solutions.

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