Advances and Applications in Mathematical Sciences
Volume 21, Issue 6, April 2022, Pages 3469-3477

SEPARATION AXIOMS ON SOFT TOPOLOGY

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Abstract

In this paper we characterize soft separation axioms by using the concepts in the parametrized family of topologies induced by the soft topology.

1. Introduction

In general topology separation axioms play a vital role to classify topological spaces. Separation axioms will be used to study some topological properties. For example, compact subspace of a Hausdorff space is closed, closed subset of a compact Hausdorff space is compact and so on. In this paper, by introducing new separation axioms in soft topological spaces, we will find the invariance properties of soft topologies.

Muhammad Shabir and Munazza Naz studied soft separation axioms. Georgiou defined and studied some soft separation axioms, soft θ-continuity and soft connectedness in soft spaces. Sabir Hussain and Bashir Ahmad redezfined and explored several properties of soft T_i, $i = 0, 1, 2$, soft regular, soft T_3, soft normal and soft T_4 axioms using soft points defined by Zorlutuna.

2020 Mathematics Subject Classification: 54-XX.

Keywords: Soft ρ-T_0, Soft ρ-T_1, Soft ρ-T_2, Soft ρ-Hausdroff space, Soft topology.

Received October 10, 2021; Accepted December 9, 2021

2. Preliminaries

2.1 Definition. Let X be an initial universe and E be a set of parameters. The pair (F, E) is called a soft set over X, where $F : E \to 2^X$ is a mapping. Conveniently (F, E) is represented by F_E which is known as a soft set over (X, E). If the parameter space E is fixed, we use F instead of F_E. $S(X, E)$ denotes the collection of all soft subsets of X with parameter set $E \cdot S(X, E\} = \{F : E \to 2^X\}$.

If $F \in S(X, E)$ then we write $F = \{(e, F(e)) : e \in E\}$.

2.2 Definition. Let F and G be any two soft sets over a common universe X with a parameter space E. Then F is a soft subset of G denoted by $F^s \subseteq G$ if $F(e) \subseteq G(e)$ for all $e \in E$.

$F = G$ if $F(e) = G(e)$ for all $e \in E$. The soft complement of F, denoted by F^c, is defined as $F^c = \{(e, F^c(e)) : e \in E\} = \{(e, X \setminus F(e)) : e \in E\}$. That is $F^c(e) = X \setminus F(e)$.

Maji and Biswas introduced the notions of Null soft set and Absolute soft set. They also defined the soft union and soft intersection of soft sets that lead to the definition of soft topology on the family of soft sets over (X, E).

2.3 Definition. A soft set $\varnothing_E = \{(e, \varnothing) : e \in E\}$ said to be a NULL soft set and $X_E = \{(e, X) : e \in E\}$ is an absolute soft set.

2.4 Definition. Let $\{F_j : j \in \Omega\}$ be a family of soft sets over (X, E). The soft union and the soft intersection of soft sets are defined as $F = {}^s\bigcup\{F_j : j \in \Omega\}$ and $G = {}^s\bigcap\{F_j : j \in \Omega\}$ where $F(e) = \bigcup\{F_j(e) : j \in \Omega\}$ and $G(e) = \bigcap\{F_j : j \in \Omega\}$ for every $e \in E$.

2.5 Definition. Let F be a soft set over X and $x \in X$. Then $x \in F$ whenever $x \in F(e)$ for every $e \in E$.

The notions of a soft point and a quasi-soft sets are respectively studied by Zorlutuna, Akdag [67] and Evanzalin [19]. These two notions are very much useful to study a link between topology and soft topology.

2.6 Definition. Let $x \in X$ and $e \in E$. The soft set X_e is called a soft point over X if $x_e(e) = \{X\}$ and $X_e(a) = \varnothing$ for every $a \in E \setminus \{e\}$.

2.7 Definition. Let $A \subseteq X$ and $e \in E$. The soft set A_e is called a quasi-soft set over X if $A_e(e) = A$ and $A_e(a) = \varnothing$ for every $a \in E \setminus \{e\}$.

It is easy to see that every soft point is a quasi-soft set. The converse is not true.

Shabir Hussain and Bashir Ahmad [10] introduced the notion of soft topology on $S(X, E)$. Following this topologists extended some of the notions in point set topology to soft topology. In particular they studied the family of topologies induced by a soft topology.

2.8 Definition. Let ${}^s\tau$ be a sub collection of $S(X, E)$. Then ${}^s\tau$ is said to be a soft topology on X if

(i) $\varnothing_E$ and X_E belong to ${}^s\tau$.

(ii) ${}^s\tau$ is closed under finite intersection.

(iii) ${}^s\tau$ is closed under arbitrary union.

If ${}^s\tau$ is a soft topology on X then the triplet $(X, {}^s\tau, E)$ is called a soft topological space over (X, E) and ${}^s\tau$ is a soft topology over (X, E). The members of ${}^s\tau$ are called the soft open sets in $(X, {}^s\tau, E)$. The soft complements of soft open sets are known as soft closed sets. The soft interior and soft closure of Soft set can be defined in the usual way.

The next lemma gives the parametrized family of topologies induced by the soft topology.

2.9 Lemma [8]. *Let $(X, {}^s\tau, E)$ be a soft topological space. Then for each $e \in E$, the collection of subsets $F(e)$ of X, $F \in {}^s\tau$ is a topology on X, denoted by $({}^s\tau)e$.*

The family $\{({}^{s}\tau)e : e \in E\}$ is called a parametrized family of topologies induced by the soft topology ${}^{s}\tau$.

Throughout this chapter $\kappa(E, {}^{s}\tau)$ is the topology on X generated by $\bigcup\{({}^{s}\tau)_e : e \in E\}$.

3. Soft ρ - T_i where $i \in \{0, 1, 2\}$ and $P \in \{b, {}^{*}b, b^{*}\}$

In this section some new soft-T_i axioms are introduced via b-open, ${}^{*}b$-open and $b^{\#}$-open sets and their basic properties are investigated.

3.1 Definition. The soft space $(X, {}^{s}\tau, E)$ is

(i) Soft ρ - T_0 if for any two distinct elements x, y of X, there exist soft p-open sets F and G such that $(x \in F$ and $y \notin F)$ or $(y \in G$ and $x \notin G)$.

(ii) Soft ρ - T_1 if for any two distinct elements x, y of X, there exist soft p-open sets F and G such that $(x \in F$ and $y \notin F)$ and $(y \in G$ and $x \notin G)$.

(iii) Soft ρ - T_2 if for any two distinct elements x, y of X, there exist disjoint soft p-open sets F and G such that $x \in F$ and $y \notin G$.

The following lemma will be useful in sequel.

3.2 Lemma. *Let A be a subset of X, F be a soft set in $S(X, E)$ and $e \in E$. Then*

(i) $A \subseteq F(e)$ *if and only if* $Ae^{s} \subseteq F$.

(ii) $F(e) \subseteq A$ *if and only if* $F^{s} \subseteq A^{e}$.

Proof. Fix $e \in E$ and $A \subseteq F(e)$. Then $A_e(\alpha) = \begin{cases} A \text{ if } \alpha = e \\ \phi \text{ otherwise} \end{cases}$ that implies $A_e(\alpha) \subseteq F(\alpha)$ for every $a \in E$ so that $A_e{}^{s} \subseteq F$ Conversely, let

${A_e}^s \subseteq F$ that implies $A_e(\alpha) \subseteq F(\alpha)$.

In particular $A_e(\alpha) \subseteq F(\alpha)$. That is $A \subseteq F(e)$. This proves (i).

Also $A_e(\alpha) = \begin{cases} A \text{ if } \alpha = e \\ \phi \text{ otherwise} \end{cases}$ that implies $F(\alpha) \subseteq A^e(\alpha)$ for every $a \in E$ so that $F^s \subseteq A^e$. Conversely, let $F^s \subseteq A^e$. That implies $F(\alpha) \subseteq A^e(\alpha)$. In particular $F(e) \subseteq A^e(a)$. That is $F(e) \subseteq A$. This proves(ii).

3.3 Definition. A subset B of X is p-open in $(X, \kappa(E, {}^s\tau))$ if it is ρ-open in $(X, ({}^s\tau)e)$ for some e.

3.4 Proposition. *The soft space $(X, {}^s\tau, E)$ is soft $p\text{-}T_0$ if and only if for all x, y in X with $x \neq y$, there are p-open sets $F(\alpha)$ and $G(\beta)$ in $(X, \kappa(E, {}^s\tau))$ such that $(x \in F(e)$ for all e and $y \notin F(a))$ or $(y \notin G(e)$ for all e and $x \notin G(\beta))$.*

Proof. The soft space $(X, {}^s\tau, E)$ is soft $p\text{-}T_0$ if and only if for all x, y in X with $x \neq y$, there are soft p-open sets F and G such that $(x \in F$ and $y \notin F)$ or $(y \in G$ and $x \notin G)$.

That is such that $(x \in F(e)$ for all e and $y \notin F(a)$ for some α) or

$(y \in G(e)$ for each e and $x \notin G(\beta)$ for some β)

That is such that $(\{x\}_e{}^s \subseteq F$ for all e and $\{Y\}_a{}^s \subseteq F^c$ for some α) or

$(\{y\}_e{}^s \subseteq G$ for all e and $\{\beta\}_\beta{}^s \subseteq G^c$ for some β)

That is such that $(\{x\}_e{}^s \subseteq F$ for all e and $(\{Y\}_a)^{cs} \supseteq (F^c)^c = F$ for some α) or

$(\{y\} \subseteq G$ for all e and $(\{x\}_\beta)^{cs} \supseteq (G^c)^c = G$ for some β)

That is such that $(\{x\}_e{}^s \subseteq F$ for all e and $(X \setminus \{Y\})^{as} \supseteq F$ for some α) or

$(\{y\}_e \,{}^s\!\subseteq G$ for each e and $(X \setminus \{X\})^{\beta s} \supseteq G$ for some $\beta)$

That is such that $(\{X\}_e \,{}^s\!\subseteq F$ for all e and $F^s \subseteq (X \setminus \{Y\})^{\alpha}$ for some $\alpha)$ or

$(\{y\}_e \,{}^s\!\subseteq G$ for each e and $G^s \subseteq (X \setminus \{X\})^{\beta}$ for some $\beta)$

Then by using Lemma 3.2, $(X, {}^s\tau, E)$ is soft $p\text{-}T_0$ if and only if for all x, y in X with $x \neq y$, there are soft p-open sets F and G such that $(x \in F(e)$ for all e and $F(a) \subseteq X \setminus \{Y\}$ for some $\alpha)$ or $(Y \in G(e)$ for all e and $G(\beta) \subseteq X \setminus \{X\}$ for some $\beta)$ that is iff there are p-open sets $F(\alpha)$ and $G(\beta)$ in $(X, \kappa(E, {}^s\tau))$ such that $(X \in F(\alpha)$ for all e and $y \notin G(e))$ or $(y \in G(e)$ for all e and $X \in G(\beta))$.

3.5 Proposition. *The soft space $(X, {}^s\tau, E)$ is soft $p\text{-}T_1$ if and only if for all x, y in X with $x \neq y$, there are p-open sets $F(\alpha)$ and $G(\beta)$ in $(X, \kappa(E, {}^s\tau))$ such that $(X \in F(e)$ for all e and $y \notin F(\alpha))$ and $(y \in G(e)$ for all e and $X \in G(\beta))$.*

Proof. Analogous to Proposition 3.4.

3.6 Proposition. *The soft space $(X, {}^s\tau, E)$ is soft $p\text{-}T_2$ iff for all x, y in X with $x \neq y$, there are disjoint soft p-open sets F and G such that $\{X\}_e \,{}^s\!\subseteq F$ and $\{y\}_e \,{}^s\!\subseteq G$ for every $e \in E$.*

Proof. The soft space $(X, {}^s\tau, E)$ is soft $p\text{-}T_2$ if and only iff or all x, y in X with $x \neq y$, there are disjoint soft ρ-open sets F and G such that $x \in F$ and $y \in G$ that is such that $x \in F(e)$ and $y \in G(e)$ for every $e \in E$ that is such that $\{x\} \subseteq F(e)$ and $\{y\} \subseteq G(e)$.

Now using Lemma 3.2, $(X, {}^s\tau, E)$ is soft $p\text{-}T_2$ if and only iff or all x, y in X with $x \neq y$, there are soft ρ-open sets F and G such that $\{X\}_e \,{}^s\!\subseteq F$ and $\{y\}_e \,{}^s\!\subseteq G$ for every $e \in E$.

The next two proposition establishes a link between soft $p\text{-}T_i$ in and $p\text{-}T_i$.

3.7 Proposition. *Suppose* $(X, {}^s\tau, E)$ *is a soft space such that* $QS(({}^s\tau)e)$ *is contained in* ${}^s\tau$ *for every* $e \in E$. *If the soft space* $(X, {}^s\tau, E)$ *is soft* $p\text{-}T_2$ *then* $(X, ({}^s\tau)e)$ *is* $p\text{-}T_2$ *for every* $e \in E$ *for every* $e \in E$.

Proof. If $(X, {}^s\tau, E)$ is soft $p\text{-}T_2$ then there are disjoint soft ρ-open sets F and G such that $x \in F$ and $y \in G$ that implies $x \in F(e)$ and $y \in G(e)$ so that by using Proposition 3.1.3, $F(e)$ and $G(e)$ are ρ-open sets in $(X, ({}^s\tau)e)$ that implies $(X, ({}^s\tau)e)$ is $p\text{-}T_2$ for every $e \in E$.

Remark. Soft $p\text{-}T_2$ space is called a soft ρ-Hausdorff space.

Soft ρ-regular spaces where $\rho \in \{b, {}^*b, b^*\}$ in the soft topology are characterized in the following section.

4. Soft ρ-regular spaces where $\rho \in \{b, {}^*b, b^*\}$

4.1 Definition. The soft ρ-Hausdorff (resp. soft Hausdorff) space $(X, {}^s\tau, E)$ is soft ρ-regular (resp. soft regular) if for any element x of X and a soft closed set A with $x \notin A$, there exist disjoint soft ρ-open(resp. soft open) sets F and G such that $x \in F$ and $A \,{}^s\!\subseteq G$. If $(X, {}^s\tau, E)$ is soft regular then the topology ${}^s\tau$ is called a soft regular topology and if it is soft ρ-regular then ${}^s\tau$ is called a soft ρ-regular topology.

4.2 Proposition. *The following are equivalent*

(i) $(X, {}^s\tau, E)$ *is soft regular*.

(ii) *For each* $x \in X$ *and for each soft open set* U *with* $x \in U$, *there is a soft open set* V *with* $x \in V \,{}^s\!\subseteq {}^sClV \,{}^s\!\subseteq U$.

(iii) *For each* $x \in X$ *and for each soft closed set* A *such that* $x \in A$ *there is a soft open set* V *with* $x \in V$ *and* ${}^sClN^s \cap A = \varnothing_E$.

Proof. Suppose $(X, {}^{s}\tau, E)$ is soft regular. Let $x \in X$ and U be a soft open set with $x \in U$. Then $x \notin U^{c}$ and U^{c} is soft closed. Therefore there are disjoint soft open sets V and W such that $x \in V$ and $U^{cs} \subseteq W$. Since V and W are disjoint $V^{s} \cap W = \varnothing_{E}$ that implies $x \in V^{s} \subseteq W^{cs} \subseteq U$ so that $x \in V^{s} \subseteq^{s} ClV^{s} \subseteq W^{s} \subseteq U$. This proves

(i) $\Rightarrow$ (ii).

Now (i) $\Rightarrow$ (ii) follows easily, by taking $U = A^{c}$ in (ii).

Suppose (iii) holds. Let $x \in X$ and AE be a soft closed set with $x \in A$. Then by using (iii), we choose a soft open set V with $x \in V$ and ${}^{s}ClV\, {}^{s}\cap = \varnothing_{E}$ that implies $A\, {}^{s}\subseteq ({}^{s}ClV)^{c}$. Therefore V and $({}^{s}ClV)^{c}$ are the desired soft open sets separating x and A. This proves (iii) $\Rightarrow$ (i).

4.3 Proposition. *The following are equivalent*

(i) $(X, {}^{s}\tau, E)$ *is soft* ρ-*regular*.

(ii) *For each* $x \in X$ *and for each soft open set* U *with* $x \in U$, *there is a soft* ρ-*open set* V *with* $x \in V\, {}^{s}\subseteq {}^{s}ClV\, {}^{s}\subseteq U$.

(iii) *For each* $x \in X$ *and for each soft closed set* A_{E} *such that* $x \in A$ *there is a soft* ρ-*open set* V *with* $x \in V$ *and* ${}^{s}\rho ClV^{s} \cap A = \varnothing_{E}$.

Proof. Suppose $(X, {}^{s}\tau, E)$ is soft ρ-regular. Let $x \in X$ and U be a soft open set with $x \in U$. Then $x \notin U^{c}$ and U^{c} is soft closed. Therefore there are disjoint soft ρ-open sets V and W such that $x \in V$ and $U^{cs} \subseteq W$. Since V and W are disjoint $V^{s} \cap W = \varnothing_{E}$ that implies $x \in V^{s} \subseteq W^{cs} \subseteq U$ so that $x \in V^{s} \subseteq^{s} \rho ClV^{s} \subseteq W^{s} \subseteq U$. This proves

(i) $\Rightarrow$ (ii).

Now (i) $\Rightarrow$ (ii) follows easily, by taking $U = A^{c}$ in (ii).

Suppose (iii) holds. Let $x \in X$ and A be a soft closed set with $x \in A_{E}$.

Then by using (iii), we choose a soft ρ-open set V with $x \in V$ and ${}^{s}\rho ClV \, {}^{s}\bigcap = \varnothing_E$ that implies $A \, {}^{s}\subseteq ({}^{s}\rho ClV)^c$. Therefore V and $({}^{s}\rho ClV)^c$ are the desired soft ρ-open sets separating x and A. This proves (iii) ⇒ (i).

Soft ρ-normal spaces where $\rho \in \{b, {}^{*}b, b^{*}\}$ in the soft topology are characterized in the following section.

5. Conclusion

Soft separation axioms are characterized by using the concepts in the parametrized family of topologies induced by the soft topology.

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