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ON INTERVAL VALUED NEUTROSOPHIC FUZZY MATRICES

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Abstract

Matrices play significant roles in various areas in science and engineering. The problems involving various types of uncertainties cannot be solved by the classical matrix theory. Neutrosophic sets theory was proposed by Florentin Smarandache in 1999, where each element had three associated essential functions, namely the membership function (*T*), the non-membership (*F*) function and the indeterminacy function (*I*) defined on the universe of discourse *X*, the three functions are entirely independent. In this paper, the interval-valued neutrosophic fuzzy matrix (*IVNFM*) is introduced. Some fundamental operations are also presented. The need of the interval-valued neutrosophic fuzzy matrix (*IVNFM*) is explained by an illustration. Illumination of some of the operators are given with the help of the example.

1. Introduction

Academics in economics, sociology, medical science, industrial, atmosphere science and many other numerous fields agree with the vague, imprecise and infrequently lacking information of exhibiting inexact data. As a result, fuzzy set theory was introduced by L. A. Zadeh [15]. Then, the intuitionistic fuzzy sets was developed by K. A. Atanassov [1, 2]. Estimation

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of non-membership values is also not constantly possible for the identical reason as in case of membership values and so, there exists an indeterministic part upon which hesitation persists. As a result, Smarandache et al. [8, 9] has introduced the concept of Neutrosophic Set (NS) which is a generalization of conventional sets, fuzzy set, intuitionistic fuzzy set etc.

The problems concerning various types of hesitations cannot solved by the classical matrix theory. That type of problems are solved by using fuzzy matrix [13, 14]. Fuzzy matrix deals with only membership values. These matrices cannot deal non membership values. Intuitionistic fuzzy matrices (IFMs) introduced first time by Khan, Shyamal and Pal [12]. But, essentially it is difficult to measure the membership or non membership value as a point. So, we consider the membership value as an interval and also in the case of non membership values, it is not nominated as a point, it can be considered as an interval. Interval valued Intuitionistic fuzzy matrices was considered by Madhumangal pal et al [13]. But, the indeterminate values cannot be considered by the Intuitionistic fuzzy matrices. Hence, the concept is extended to interval valued neutrosophic fuzzy matrices (IVNFMs) and some basic operators on IVNFMs are introduced. The interval-valued neutrosophic fuzzy determinant (IVNFD) is also defined. A real life problem on IVNFM is presented. Interpretation of some of the operators are given with the help of this example.

In this work, some definitions are discussed in section 2. Section 3 dealt with the operations of interval valued neutrosophic matrices. Properties of interval valued neutrosophic matrices are given in section 4. The importance of IVNFM is discussed in section 5. Concluding remarks are given in section 6.

2. Definition and Preliminaries

In this section, we first define the neutrosophic fuzzy matrix (NFM) based on the definition of neutrosophic fuzzy sets introduced by Smarandache [8, 9]. The intuitionistic fuzzy matrices are introduced by M. Pal et al. [13]. The same concept is extended to neutrosophic fuzzy matrices here.

Definition 1. Neutrosophic fuzzy matrix (NFM): An neutrosophic fuzzy matrix (NFM) A of order $m \times n$ is defined as $A = [X_{ij}, \langle a_{ij\mu}, a_{ij\lambda}, a_{ijv} \rangle]_{m \times n}$, where $a_{ij\mu}, a_{ij\lambda}, a_{ijv}$ are called truth, indeterminacy and falsity of X_{ij} in A, which maintaining the condition $0 \leq a_{ij\mu} + a_{ij\lambda} + a_{ijv} \leq 3$. For simplicity, we write $A = [X_{ij}, a_{ij}]_{m \times n}$ or simply $[a_{ij}]_{m \times n}$ where $a_{ij} = \langle a_{ij\mu}, a_{ij\lambda}, a_{ijv} \rangle$.

Using the concept of neutrosophic fuzzy sets and interval valued fuzzy sets, we define interval valued neutrosophic fuzzy matrices as follows:

Definition 2. Interval-valued neutrosophic fuzzy matrix (IVNFM): An interval valued neutrosophic fuzzy matrix (IVNFM) A of order $m \times n$ is defined as $A = [X_{ij}, \langle a_{ij\mu}, a_{ij\lambda}, a_{ijv} \rangle]_{m \times n}$, where $a_{ij\mu}, a_{ij\lambda}$ and a_{ijv} are the subsets of $[0, 1]$ which are denoted by $a_{ij\mu} = [a_{ij\mu L}, a_{ij\lambda U}]$, $a_{ij\lambda} = [a_{ij\lambda L}, a_{ij\lambda U}]$ and $a_{ijv} = [a_{ij\mu L}, a_{ijvU}]$ which maintaining the condition $0 \leq \sup a_{ij\mu} + \sup a_{ij\lambda} + \sup a_{ijv} \leq 3$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Definition 3. Interval-valued neutrosophic fuzzy determinant (IVIFD): An interval valued neutrosophic fuzzy determinant (IVNFD) function $f : M \to F$ is a function on the set M (of all $n \times n$ IVNFMs) to the set F, where F is the set of elements of the form $\langle [a_{\mu L}, a_{\mu U}], [a_{\lambda L}, a_{\lambda U}], [a_{vL}, a_{vU}] \rangle$, maintaining the condition $0 \leq a_{\mu U} + a_{\lambda U} + a_{vU} \leq 3$, $0 \leq a_{\mu L} \leq a_{\mu U} \leq 1$ and $0 \leq a_{\lambda L} + a_{\lambda U} \leq 1$, $0 \leq a_{vL} \leq a_{vU} \leq 1$ such that $A \subset M$ then $f(A)$ or $|A|$ or $\det(A)$ belongs to F and is given by

$$|A| = \sum_{\sigma \in s_n} \prod_{i=1}^{n} \langle [a_{i\sigma(i)\mu L}, a_{i\sigma(i)\mu U}], [a_{i\sigma(i)\lambda L}, a_{i\sigma(i)\lambda U}], [a_{i\sigma(i)vL}, a_{i\sigma(i)vU}] \rangle$$

and s_n denotes the symmetric group of all permutations of the symbols $\{1, 2, \ldots, n\}$.

Definition 4. The adjoint IVNFM of an IVNFM: The adjoint IVNFM of an IVNFM A of order $n \times n$, is denoted by adj. A and is defined by adj. $A = [A_{ji}]$ where A_{ji} is the determinant of the IVNFM A of order

$(n-1)\times(n-1)$ formed by suppressing row j and column i of the IVNFM A. In other words, A_{ji} can be written in the form

$$\sum_{\sigma\in s_{n_i n_j}}\prod_{t\in n_j}\langle[a_{t\sigma(t)\mu L},\ a_{t\sigma(t)\mu U}],\ [a_{t\sigma(t)\lambda L},\ a_{t\sigma(t)\lambda U}],\ [a_{t\sigma(t)\upsilon L},\ a_{t\sigma(t)\upsilon U}]\rangle$$

where, $n_j = \{1, 2, \ldots, n\}\setminus\{j\}$ and $s_{n_i n_j}$ is the set of all permutations of set n_j over the set n_i. Depending on the values of diagonal elements, the unit IVNFM are classified into two types: (i) a-unit IVNFM and (ii) r-unit IV NFM.

Definition 5. Acceptance unit IVNFM (a-unit IVNFM): A square IVNFM is a-unit IVNFM if all diagonal elements are $\langle[1, 1], [0, 0], [0, 0]\rangle$ and all remaining elements are $\langle[0, 0], [1, 1], [1, 1]\rangle$ and it is denoted by $I_{\langle[0, 0], [1, 1], [1, 1]\rangle}$.

Definition 6. Rejection unit IVNFM (r-unit IVNFM): A square IVNFM is a r-unit IVNFM if all diagonal elements are $\langle[0, 0], [1, 1], [1, 1]\rangle$ and all remaining elements are $\langle[1, 1], [0, 0], [0, 0]\rangle$ and it is denoted by $I_{\langle[1, 1], [0, 0], [0, 0]\rangle}$. Similarly, three types of null IVNFMs are defined on its elements.

Definition 7. Complete null IVNFM (c-null IVNFM): An IVNFM is a c-null IVNFM if all the elements are $\langle[0, 0], [0, 0], [0, 0]\rangle$.

Definition 8. Acceptance null IVNFM (a-null IVNFM): An IVNFM is a a-null IVNFM if all the elements are $\langle[0, 0], [1, 1], [1, 1]\rangle$.

Definition 9. Rejection null IVNFM (r-null IVNFM): An IVNFM is a r-null IVNFM if all the elements are $\langle[1, 1], [0, 0], [0, 0]\rangle$.

3. Some operations on IVNFMs

Let $A = [\langle[a_{ij\mu L},\ a_{ij\mu U}],\ [a_{ij\lambda L},\ a_{ij\lambda U}],\ [a_{ij\upsilon L},\ a_{ij\upsilon U}]\rangle]$, $B = [\langle[b_{ij\mu L},\ b_{ij\mu U}],\ [b_{ij\lambda L},\ b_{ij\lambda U}],\ [b_{ij\upsilon L},\ b_{ij\upsilon U}]\rangle]$, be two IVNFMs. Then,

(i) $\langle[a_{ij\mu L}, a_{ij\mu U}],[a_{ij\lambda L}, a_{ij\lambda U}],[a_{ij\upsilon L}, a_{ij\upsilon U}]\rangle + \langle[b_{ij\mu L}, b_{ij\mu U}],[b_{ij\lambda L}, b_{ij\lambda U}],$

$[b_{ij\upsilon L}, b_{ij\upsilon U}]\rangle = [\max(a_{ij\mu L}, a_{ij\mu L}), \max(a_{ij\mu U}, a_{ij\mu U})]$
$\langle[\min(a_{ij\lambda L}, b_{ij\lambda L}), \min(a_{ij\lambda U}, b_{ij\lambda U})]\rangle [\min(a_{ij\upsilon L}, b_{ij\upsilon L}), \min(a_{ij\upsilon U}, b_{ij\upsilon U})]$

(ii) $\langle[a_{ij\mu L}, a_{ij\mu U}], [a_{ij\lambda L}, a_{ij\lambda U}], [a_{ij\upsilon L}, a_{ij\upsilon U}]\rangle \cdot \langle[b_{ij\mu L}, b_{ij\mu U}], [b_{ij\lambda L}, b_{ij\lambda U}],$
$[b_{ij\upsilon L}, b_{ij\upsilon U}]\rangle = [\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})]$
$\langle[\min(a_{ij\lambda L}, b_{ij\lambda L}), \min(a_{ij\lambda U}, b_{ij\lambda U})]\rangle [\min(a_{ij\upsilon L}, a_{ij\lambda L}), \min(a_{ij\upsilon U}, a_{ij\upsilon U})]$
$[\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})],$

(iii) $A + B = \langle[\min(a_{ij\mu L}, b_{ij\mu L}), \min(a_{ij\lambda U}, b_{ij\lambda U})]\rangle, [\min(a_{ij\upsilon L}, b_{ij\lambda L}),$
$\min(a_{ij\upsilon U}, b_{ij\upsilon U})], [\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})]$

(iv) $A \cdot B = \langle[\min(a_{ij\lambda L}, b_{ij\lambda L}), \min(a_{ij\lambda U}, b_{ij\lambda U})]\rangle, [\min(a_{ij\upsilon L}, b_{ij\lambda L}),$
$\min(a_{ij\upsilon U}, b_{ij\upsilon U})],$

(v) $\overline{A} = [\langle[a_{ij\upsilon L}, a_{ij\upsilon U}], [1 - a_{ij\lambda L}, 1 - a_{ij\lambda U}], [a_{ij\mu L}, a_{ij\mu U}]\rangle]$. (complement of A)

(vi) $A^T = \langle[a_{ji\mu L}, a_{ji\mu U}], [a_{ji\lambda L}, a_{ji\lambda U}], [a_{ji\upsilon L}, a_{ji\upsilon U}]\rangle_{n\times m}$ (Transpose of A)

(vii) $A \oplus B = [[a_{ji\mu L} + b_{ji\mu L} - a_{ij\mu L} \cdot b_{ij\mu L}, a_{ij\mu U} + b_{ij\mu U} - a_{ij\mu U} \cdot b_{ij\mu U}],$
$\langle[a_{ij\lambda L} \cdot b_{ij\lambda L}, a_{ij\lambda U} \cdot b_{ij\lambda U}]\rangle [a_{ij\upsilon L} \cdot b_{ij\upsilon L}, a_{ij\upsilon U} \cdot b_{ij\upsilon U}]]$

(viii) $$A \odot B = \begin{bmatrix} [a_{ij\mu L} \cdot b_{ij\mu L}, a_{ij\mu U} \cdot b_{ij\mu U}] \\ \langle[a_{ij\lambda L} + b_{ij\lambda L} - a_{ij\lambda L} \cdot b_{ij\lambda L}, a_{ij\lambda U} + b_{ij\lambda U} - a_{ij\lambda U} \cdot b_{ij\lambda U}]\rangle \\ \langle[a_{ij\upsilon L} + b_{ij\upsilon L} - a_{ij\upsilon L} \cdot b_{ij\upsilon L}, a_{ij\upsilon U} + b_{ij\upsilon U} - a_{ij\upsilon U} \cdot b_{ij\upsilon U}]\rangle \end{bmatrix}$$

(ix) $$A @ B = \begin{bmatrix} \left[\frac{a_{ij\mu L} + b_{ij\mu L}}{2}, \frac{a_{ij\mu U} + b_{ij\mu U}}{2}\right] \\ \left\langle\left[\frac{a_{ij\lambda L} + b_{ij\lambda L}}{2}, \frac{a_{ij\lambda U} + b_{ij\lambda U}}{2}\right]\right\rangle \\ \left[\frac{a_{ij\upsilon L} + b_{ij\upsilon L}}{2}, \frac{a_{ij\upsilon U} + b_{ij\upsilon U}}{2}\right] \end{bmatrix}$$

(x) $A\$B = \begin{bmatrix} [\sqrt{a_{ij\mu L} + b_{ij\mu L}}, \sqrt{a_{ij\mu U} + b_{ij\mu U}}] \\ \langle [\sqrt{a_{ij\lambda L} + b_{ij\lambda L}}, \sqrt{a_{ij\lambda U} + b_{ij\lambda U}}] \rangle \\ [\sqrt{a_{ij\upsilon L} + b_{ij\upsilon L}}, \sqrt{a_{ij\upsilon U} + b_{ij\upsilon U}}] \end{bmatrix}$

(xi) $A \# B = \begin{bmatrix} \left[\frac{2a_{ij\mu L} \cdot b_{ij\mu L}}{a_{ij\mu L} + b_{ij\mu L}}, \frac{2a_{ij\mu U} \cdot b_{ij\mu U}}{a_{ij\mu U} + b_{ij\mu U}}\right] \\ \left\langle \left[\frac{2a_{ij\lambda L} \cdot b_{ij\lambda L}}{a_{ij\lambda L} + b_{ij\lambda L}}, \frac{2a_{ij\lambda U} \cdot b_{ij\lambda U}}{a_{ij\lambda U} + b_{ij\lambda U}}\right] \right\rangle \\ \left[\frac{2a_{ij\upsilon L} \cdot b_{ij\upsilon L}}{a_{ij\upsilon L} + b_{ij\upsilon L}}, \frac{2a_{ij\upsilon U} \cdot b_{ij\upsilon U}}{a_{ij\upsilon U} + b_{ij\upsilon U}}\right] \end{bmatrix}$

(xii) $A * B = \begin{bmatrix} \left[\frac{a_{ij\mu L} + b_{ij\mu L}}{2(a_{ij\mu L} \cdot b_{ij\mu L} + 1)}, \frac{a_{ij\mu U} + b_{ij\mu U}}{2(a_{ij\mu U} \cdot b_{ij\mu U} + 1)}\right] \\ \left[\frac{a_{ij\lambda L} + b_{ij\lambda L}}{2(a_{ij\lambda L} \cdot b_{ij\lambda L} + 1)}, \frac{a_{ij\lambda U} + b_{ij\lambda U}}{2(a_{ij\lambda U} \cdot b_{ij\lambda U} + 1)}\right] \\ \left[\frac{a_{ij\upsilon L} + b_{ij\upsilon L}}{2(a_{ij\upsilon L} \cdot b_{ij\upsilon L} + 1)}, \frac{a_{ij\upsilon U} + b_{ij\upsilon U}}{2(a_{ij\upsilon U} \cdot b_{ij\upsilon U} + 1)}\right] \end{bmatrix}$

(xiii) $A \le B$ iff $a_{ij\mu L} \le b_{ij\mu L}$, $a_{ij\mu U} \le b_{ij\mu U}$, $a_{ij\lambda L} \ge b_{ij\lambda L}$, $a_{ij\lambda U} \ge b_{ij\lambda U}$, $a_{ij\upsilon L} \ge b_{ij\upsilon L}$, $a_{ij\upsilon U} \ge b_{ij\upsilon U}$

(xiv) $A = B$ iff $A \le B$ and $B \le A$.

In the following section, we consider a daily life problem which can be studied using IVNFMs in better way.

4. Importance of Interval Valued Neutrosophic Fuzzy Matrices (IVNFMs)

A network consisting of four important cities (vertices) in a country is considered. They are connected by highways (edges). The number neighboring to an edge characterizes the distance between the cities (vertices). The above network can be represented with the help of a classical matrix $A = [a_{ij}]$, $i, j = 1, 2, \ldots, n$, where, n is the total number of nodes. The ij^{th} element a_{ij} of A is defined as

$$a_{ij} = \begin{cases} 0, \text{if } i = j \\ \infty, \text{the vertices } i \text{ and } j \text{ are not directly connected by an adge} \\ w_{ij}, w_{ij} \text{ is the distance of the road connecting } i \text{ and } j \end{cases}$$

Thus the adjacent matrix of the network of is

$$\begin{bmatrix} 0 & 15 & 20 & 35 \\ 15 & 0 & 55 & 40 \\ 20 & 55 & 0 & 75 \\ 35 & 40 & 75 & 0 \end{bmatrix}$$

Since the distance between two vertices is identified, precisely, so the above matrix is obviously a conventional matrix. Generally, the distance between two cities are crisp value, so the corresponding matrix is crisp matrix. Now, we study the crowdness of the roads connecting cities. It is clear that the crowdness of a road clearly, is a fuzzy quantity. The amount of crowdness depends on the decision makers temperament, practices, environments, etc. i.e., finally depends on the decision maker. The measurement of crowdness as a point is a difficult task for the decision maker. So, here we consider the amount of crowdness as an interval instead of a point. The aloneness is considered as an interval and also the indeterminacy is considered as an interval. The crowdness, indeterminacy and the aloneness of a network cannot be represented as a crisp matrix, it can be represented appropriately by a matrix which we designate by interval-valued neutrosophic fuzzy matrices (IVNFMs). The matrix representation of the traffic crowdness, indeterminacy and aloneness of the network of is shown in the following IVNFM.

$$\begin{bmatrix} \langle[0,0],[1,1],[1,1]\rangle & \langle[.1,.3],[.2,.4],[.2,.5]\rangle & \langle[.2,.4][.3,.5],[.1,.5]\rangle & \langle[.3,.4],[.2,.5],[.5,.6]\rangle \\ \langle[.1,.3],[.2,.4],[.2,.5]\rangle & \langle[0,0],[1,1],[1,1]\rangle & \langle[.7,.8],[.2,.4],[0,.1]\rangle & \langle[.3,.5],[.3,.6],[.4,.5]\rangle \\ \langle[.2,.4],[.3,.5],[.1,.5]\rangle & \langle[.7,.8],[.2,.4],[0,.1]\rangle & \langle[0,0],[1,1],[1,1]\rangle & \langle[.5,.6],[.1,.3],[.2,.3]\rangle \\ \langle[.3,.4],[.2,.5],[.5,.6]\rangle & \langle[.3,.5],[.3,.6],[.4,.5]\rangle & \langle[.5,.6],[.1,.3],[.2,.3]\rangle & \langle[0,0],[1,1],[1,1]\rangle \end{bmatrix}.$$

To explain the meaning of the operators defined earlier we consider two IVNFMs A and B. Let A and B represent respectively the crowdness, indeterminacy and loneliness of the network at two time instances t_1 and t_2. Now, the IVNFM $A + B$ represents the maximum amount of traffic crowdness, the minimum of indeterminacy and the minimum amount of aloneness of the network between the time instances t_1 and t_2. $A \cdot B$ represents the minimum amount of traffic crowdness, minimum amount of indeterminacy and the maximum amount of loneliness of the network. $\overline{A}$ matrix represents the aloneness, confidence and crowdness of the network. $A @ B$, $A\$B$ and $A \# B$ reveals the arithmetic mean, geometric mean and harmonic mean of the crowdness, indeterminacy and aloneness in between the two time instances t_1 and t_2 of the network. To illustrate the operators $A \cdot B$, $A + B$ and $|A|$, we consider a network consisting two vertices and two edges. The crowdness, indeterminacy and aloneness of the network are observed at two different time instances t_1 and t_2. The matrices A_{t_1} and A_{t_2} represent the status of the network at t_1 and at t_2. The number adjacent to the sides represents the crowdness, indeterminacy and aloneness of the roads at two different instances of the same network.

Let

$$A_{t_1} = \begin{bmatrix} \langle [0,\ 0],\ [1,\ 1],\ [1,\ 1] \rangle & \langle [.1,\ .3],\ [.2,\ .4],\ [.2,\ .5] \rangle \\ \langle [.1,\ .3],\ [.2,\ .4],\ [.2,\ .5] \rangle & \langle [0,\ 0],\ [1,\ 1],\ [1,\ 1] \rangle \end{bmatrix}$$

and

$$A_{t_2} = \begin{bmatrix} \langle [0,0],[1,1],[1,1] \rangle & \langle [.2,.4],[.3,.5],[.1,.5] \rangle \\ \langle [.2,.4],[.3,5],[.1,.5] \rangle & \langle [0,0],[1,1],[1,1] \rangle \end{bmatrix}.$$

Then $A_{t_1} + A_{t_2} = \begin{bmatrix} \langle [0,0],[1,1],[1,1] \rangle & \langle [.2,.4],[.2,.4],[.1,.5] \rangle \\ \langle [.2,.4],[.2,.4],[.1,.5] \rangle & \langle [0,0],[1,1],[1,1] \rangle \end{bmatrix}$ and

$$A_{t_1} \cdot A_{t_2} = \begin{bmatrix} \langle [0,0],[1,1],[1,1] \rangle & \langle [.1,.3],[.3,.5],[.2,.5] \rangle \\ \langle [.1,.3],[.3,.5],[.2,.5] \rangle & \langle [0,0],[1,1],[1,1] \rangle \end{bmatrix}$$

$$|A_{t_1}| = \langle [0,0],[1,1],[1,1] \rangle \cdot \langle [0,0],[1,1],[1,1] \rangle + \langle [.1,.3],[.2,.4],[.2,.5] \rangle$$

$$\langle [.1,.3],[.2,.4],[.2,.5] \rangle = \langle [0,0],[1,1],[1,1] \rangle + \langle [.1,.3],[.2,.4],[.2,.5] \rangle = \langle [.1,.3],[.2,.4],[.2,.5] \rangle.$$

5. Properties of Interval Valued Neutrosophic Fuzzy Matrices (IVNFMs)

In this section some properties of IVNFMs are presented. IVNFMs satisfy the commutative and associative properties over the operators $+, \cdot \oplus,$ and $\odot$. The operator '.' is distributed over '+' in left and right but the left and right distribution laws do not hold for the operators $\oplus$ and $\odot$.

(1) $A + B = B + A$

(2) $A + (B + C) = (A + B) + C$

(3) $A \cdot B = B \cdot A$

(4) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

(5) (i) $A \cdot (B + C) = A \cdot B + A \cdot C$

(ii) $(B + C) \cdot A = B \cdot A + A \cdot C$

(6) $A \oplus B = B \oplus A$

(7) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

(8) $A \odot B = B \odot A$

(9) $A \odot (B \odot C) = (A \odot B) \odot C$

(10) (i) $A \odot (B \oplus C) \neq (A \odot B) \oplus (A \odot C)$

(ii) $(B \oplus C) \odot A \neq (B \odot A) \oplus (C \odot A)$

Proof.

(1) Let $A = [\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\lambda L}, a_{ij\lambda U}], [a_{ijvL}, a_{ijvU}] \rangle]$, $B = [\langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\lambda L}, b_{ij\lambda U}], [b_{ijvL}, b_{ijvU}] \rangle]$ and $C = [\langle [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\lambda L}, c_{ij\lambda U}], [c_{ijvL}, c_{ijvU}] \rangle]$.

$$A + B = \langle \begin{array}{l} [\max(a_{ij\mu L}, b_{ij\mu L}), \max(a_{ij\mu U}, b_{ij\mu U})], \\ [\min(a_{ij\lambda L}, b_{ij\lambda L}), \min(a_{ijvU}, b_{ijvU})] \\ [\min(a_{ijvL}, b_{ij\lambda L}), \min(a_{ijvU}, b_{ijvU})] \end{array} \rangle$$

$$B + A = \langle \begin{array}{l} [\max(b_{ij\mu L}, a_{ij\mu L}), \max(b_{ij\mu U}, a_{ij\mu U})], \\ [\min(b_{ij\lambda L}, a_{ij\lambda L}), \min(b_{ij\lambda U}, a_{ij\lambda U})], \\ [\min(b_{ijvL}, a_{ij\lambda L}), \min(b_{ijvU}, a_{ijvU})] \end{array} \rangle$$

Therefore, $A + B = B + A$. Similarly, (2), (3), (4), (5), (6), (7), (8) and (9) can be proved.

$$(10)\ \ B \oplus C = \begin{bmatrix} [b_{ij\mu L} + c_{ij\mu L} - b_{ij\mu L} \cdot c_{ij\mu L},\, b_{ij\mu U} + c_{ij\mu U} - b_{ij\mu U} \cdot c_{ij\mu U}], \\ \langle [b_{ij\lambda L} \cdot c_{ij\lambda L},\, b_{ij\lambda U} \cdot c_{ij\lambda U}] \rangle, \\ [b_{ij\upsilon L} \cdot c_{ij\upsilon L},\, b_{ij\upsilon U} \cdot c_{ij\upsilon U}] \end{bmatrix}$$

$$A \odot (B \oplus C) = \begin{bmatrix} [a_{ij\mu L} \cdot (b_{ij\mu L} + c_{ij\mu L} - b_{ij\mu L} \cdot c_{ij\mu L}),\, a_{ij\mu U} \cdot (b_{ij\mu U} + c_{ij\mu U} - b_{ij\mu U} \cdot c_{ij\mu U})] \\ \langle [a_{ij\lambda L} + b_{ij\lambda L} \cdot c_{ij\lambda L} - a_{ij\lambda L} \cdot b_{ij\lambda L} \cdot c_{ij\lambda L},\, a_{ij\lambda U} + b_{ij\lambda U} \cdot c_{ij\lambda U} - a_{ij\lambda U} \cdot b_{ij\lambda U} \cdot c_{ij\lambda U})] \rangle \\ [a_{ij\upsilon L} + b_{ij\lambda U} \cdot c_{ij\lambda U} - a_{ij\upsilon L} \cdot b_{ij\upsilon L},\, a_{ij\upsilon U} + b_{ij\upsilon U} \cdot c_{ij\upsilon U} - a_{ij\upsilon U} \cdot b_{ij\upsilon U} \cdot c_{ij\upsilon U}] \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} [a_{ij\mu L} \cdot b_{ij\mu L},\, a_{ij\mu U} \cdot b_{ij\mu U}], \\ \langle [a_{ij\lambda L} + b_{ij\lambda L} - a_{ij\lambda L} \cdot b_{ij\lambda L},\, a_{ij\lambda U} + b_{ij\lambda U} - a_{ij\lambda U} \cdot b_{ij\lambda U}] \rangle \\ [a_{ij\upsilon L} + b_{ij\upsilon L} - a_{ij\upsilon L} \cdot b_{ij\upsilon L},\, a_{ij\upsilon U} + b_{ij\upsilon U} - a_{ij\upsilon U} \cdot b_{ij\upsilon U}] \end{bmatrix},$$

$$A \odot C = \begin{bmatrix} [a_{ij\mu L} \cdot c_{ij\mu L},\, a_{ij\mu U} \cdot c_{ij\mu U}], \\ \langle [a_{ij\lambda L} + c_{ij\lambda L} - a_{ij\lambda L} \cdot c_{ij\lambda L},\, a_{ij\lambda U} + c_{ij\lambda U} - a_{ij\lambda U} \cdot c_{ij\lambda U}] \rangle \\ [a_{ij\upsilon L} + c_{ij\upsilon L} - a_{ij\upsilon L} \cdot c_{ij\upsilon L} + c_{ij\upsilon U} - a_{ij\upsilon U} \cdot c_{ij\upsilon U}] \end{bmatrix}.$$

Now,

$$(A \odot B) \oplus (A \odot C) = \begin{bmatrix} [a_{ij\mu L}(b_{ij\mu L} + c_{ij\mu L}) - a^2_{ij\mu L} \cdot b_{ij\mu L} \cdot c_{ij\mu L}, \\ a_{ij\mu U}(b_{ij\mu U} + c_{ij\mu U}) - a^2_{ij\mu U} \cdot b_{ij\mu U} \cdot c_{ij\mu U}], \\ \langle [(a_{ij\upsilon L} + b_{ij\upsilon L} - a_{ij\upsilon L} \cdot b_{ij\upsilon L}) \cdot (a_{ij\lambda L} + c_{ij\lambda L} - a_{ij\lambda L} \cdot c_{ij\lambda L}), \\ (a_{ij\lambda U} + b_{ij\lambda U} - a_{ij\lambda U} \cdot b_{ij\lambda U}) \cdot (a_{ij\lambda U} + c_{ij\lambda U} - a_{ij\lambda U} \cdot c_{ij\lambda U})] \rangle \\ [(a_{ij\upsilon L} + b_{ij\upsilon L} - a_{ij\upsilon L} \cdot b_{ij\upsilon L}) \cdot (a_{ij\upsilon L} + c_{ij\upsilon L} - a_{ij\upsilon L} \cdot c_{ij\upsilon L}), \\ (a_{ij\upsilon U} + b_{ij\upsilon U} - a_{ij\upsilon U} \cdot b_{ij\upsilon U}) \cdot (a_{ij\upsilon U} + b_{ij\upsilon U} - a_{ij\upsilon U} \cdot b_{ij\upsilon U})] \end{bmatrix}.$$

So, $A \odot (B \oplus C) \neq (A \odot B) \oplus (A \odot C)$. Similarly, $(B \oplus C) \odot A \neq (B \odot A) \oplus (C \odot A)$ can be proved.

Property 1. Let A be an IVNFM of any order then, $A + A = A$.

Proof. Let $A = [\langle [a_{ij\mu L},\, a_{ij\mu U}],\, [a_{ij\lambda L},\, a_{ij\lambda U}],\, [a_{ij\upsilon L},\, a_{ij\upsilon U}] \rangle]$.

$$[\max(a_{ij\mu L},\, a_{ij\mu L}),\, \max(a_{ij\mu U},\, a_{ij\mu U})],$$

Then, $A + B = \langle [\min(a_{ij\lambda L}, a_{ij\lambda L}), \min(a_{ij\lambda U}, a_{ij\lambda U})] \rangle = [\langle [a_{ij\mu L}, a_{ij\mu U}],$ $[a_{ij\lambda L}, a_{ij\lambda U}], [a_{ij\upsilon L}, a_{ij\upsilon U}] \rangle] = A$

$[\min(a_{ij\upsilon L}, a_{ij\lambda L}), \min(a_{ij\upsilon U}, a_{ij\upsilon U})]$.

Property 2 If A be an IVNFM of any order then, $A + I_{\langle[0,0],[0,0],[0,0]\rangle} \geq A$ where, $I_{\langle[0,0],[0,0],[0,0]\rangle}$ is the null IVNFM of same order.

Proof. Let $A = [\langle[a_{ij\mu L}, a_{ij\mu U}], [a_{ij\lambda L}, a_{ij\lambda U}], [a_{ij\upsilon L}, a_{ij\upsilon U}]\rangle]$ and $I_{\langle[0,0],[0,0],[0,0]\rangle} = \langle[0, 0], [0, 0], [0, 0]\rangle$.

$$[\max(a_{ij\mu L}, 0), \max(a_{ij\mu U}, 0)],$$

Then, $A + I_{\langle[0,0],[0,0],[0,0]\rangle} = \langle[\min(a_{ij\lambda L}, 0), \min(a_{ij\lambda U}, 0)]\rangle = [\langle[a_{ij\mu L}, a_{ij\mu U}][0, 0], [0, 0]\rangle]$ $[\min(a_{ij\upsilon L}, 0), \min(a_{ij\upsilon U}, 0)]$.

Therefore, $A + I_{\langle[0,0],[0,0],[0,0]\rangle} \geq A$.

Some more properties on determinant and adjoint of IVNFM are presented below.

Property 3. If A be a square IVNFM then $|A| = |A^T|$.

Proof. Let $A = [\langle[a_{ij\mu L}, a_{ij\mu U}], [a_{ij\lambda L}, a_{ij\lambda U}], [a_{ij\upsilon L}, a_{ij\upsilon U}]\rangle]$. Then $A^T = [\langle[a_{ji\mu L}, a_{ji\mu U}], [a_{ji\lambda L}, a_{ji\lambda U}], [a_{ji\upsilon L}, a_{ji\upsilon U}]\rangle]$.

$$\langle[a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\lambda L}, a_{\sigma(1)1\lambda U}], [a_{\sigma(1)1\upsilon L}, a_{\sigma(1)1\upsilon U}]\rangle.$$

Now,

$$|A^T| = \sum_{\sigma \in s_n} \langle[a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}], [a_{\sigma(2)2\lambda L}, a_{\sigma(2)2\lambda U}], [a_{\sigma(2)2\upsilon L}, a_{\sigma(2)2\upsilon U}]\rangle$$

$$\ldots \langle[a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\lambda L}, a_{\sigma(n)n\lambda U}], [a_{\sigma(n)n\upsilon L}, a_{\sigma(n)n\upsilon U}]\rangle.$$

Let ψ be the permutation of $\{1, 2, \ldots, n\}$ such that $\psi\sigma = I$, the identity permutation. Then $\psi = \sigma - 1$. As σ runs over the whole set of permutations, so does ψ. Let $\sigma(i) = j$, $i = \sigma^{-1}(j) = \psi(j)$.

Therefore, $a_{\sigma(i)i\mu L} = a_{j\psi(j)\mu L}$, $a_{\sigma(i)i\mu U} = a_{\sigma(i)i\mu U}$, $a_{\sigma(i)i\lambda L} = a_{j\psi(j)\lambda L}$, $a_{\sigma(i)i\lambda L} = a_{j\psi(j)\lambda L}$, $a_{\sigma(i)i\upsilon L} = a_{j\psi(j)\upsilon L}$, $a_{\sigma(i)i\upsilon L} = a_{j\psi(j)\upsilon L}$. As i goes over the set

$\{1, 2, \ldots, n\}$, j does so.

Now, $\langle [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\lambda L}, a_{\sigma(1)1\lambda U}], [a_{\sigma(1)1vL}, a_{\sigma(1)1vU}] \rangle$
$\langle [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}], [a_{\sigma(2)2\lambda L}, a_{\sigma(2)2\lambda U}], [a_{\sigma(2)2vL}, a_{\sigma(2)2vU}] \rangle \cdots$
$\langle [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\lambda L}, a_{\sigma(n)n\lambda U}], [a_{\sigma(n)nvL}, a_{\sigma(n)nvU}] \rangle$

$\langle [a_{1\psi(1)\mu L}, a_{1\psi(1)1\mu U}], [a_{1\psi(1)\lambda L}, a_{\psi(1)1\lambda U}], [a_{1\psi(1)vL}, a_{1\psi(1)vU}] \rangle$
$= \langle [a_{2\psi(2)\mu L}, a_{2\psi(2)\mu U}], [a_{2\psi(2)\lambda L}, a_{2\psi(2)\lambda U}], [a_{2\psi(2)vL}, a_{2\psi(2)vU}] \rangle \cdots$
$\langle [a_{n\psi(n)\mu L}, a_{n\psi(n)\mu U}], [a_{n\psi(n)\lambda L}, a_{n\psi(n)\lambda U}], [a_{n\psi(n)vL}, a_{n\psi(n)vU}] \rangle$

$\langle [a_{\sigma(1)1\mu L}, a_{\sigma(1)1\mu U}], [a_{\sigma(1)1\lambda L}, a_{\sigma(1)1\lambda U}], [a_{\sigma(1)1vL}, a_{\sigma(1)1vU}] \rangle$

Therefore, $| A^T | = \sum_{\sigma \in s_n} \langle [a_{\sigma(2)2\mu L}, a_{\sigma(2)2\mu U}], [a_{\sigma(2)2\lambda L}, a_{\sigma(2)2\lambda U}],$
$[a_{\sigma(2)2vL}, a_{\sigma(2)2vU}] \rangle \cdots = \langle [a_{\sigma(n)n\mu L}, a_{\sigma(n)n\mu U}], [a_{\sigma(n)n\lambda L}, a_{\sigma(n)n\lambda U}], [a_{\sigma(n)nvL}, a_{\sigma(n)nvU}] \rangle$

$\langle [a_{1\psi(1)\mu L}, a_{1\psi(1)1\mu U}], [a_{1\psi(1)\lambda L}, a_{1\psi(1)1\lambda U}], [a_{1\psi(1)vL}, a_{1\psi(1)vU}] \rangle$

$\sum_{\mu \in s_n} \langle [a_{2\psi(2)\mu L}, a_{2\psi(2)\mu U}], [a_{2\psi(2)\lambda L}, a_{2\psi(2)\lambda U}], [a_{2\psi(2)vL}, a_{2\psi(2)vU}] \rangle \cdots$
$| A | = \langle [a_{n\psi(n)\mu L}, a_{n\psi(n)\mu U}], [a_{n\psi(n)\lambda L}, a_{n\psi(n)\lambda U}], [a_{n\psi(n)vL}, a_{n\psi(n)vU}] \rangle.$

Property 4. If A and B be two square IVNFMs and $A \leq B$, then, adj $A \leq adj\ B$.

Proof. Let $C = [\langle [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\lambda L}, c_{ij\lambda U}], [c_{ijvL}, c_{ijvU}] \rangle] = A$, $D = [\langle [d_{ij\mu L}, d_{ij\mu U}], [d_{ij\lambda L}, d_{ij\lambda U}], [d_{ijvL}, d_{ijvU}] \rangle] = adj\ B$.

where,

$$\langle [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\lambda L}, c_{ij\lambda U}], [c_{ijvL}, c_{ijvU}] \rangle = \sum_{\sigma \in s_{n_i n_j}} \prod_{t \in n_j} \langle [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}],$$

$$[a_{t\sigma(t)\lambda L}, a_{t\sigma(t)\lambda U}], [a_{t\sigma(t)vL}, a_{t\sigma(t)vU}] \rangle$$

and

$$\langle [d_{ij\mu L}, d_{ij\mu U}], [d_{ij\lambda L}, d_{ij\lambda U}], [d_{ijvL}, d_{ijvU}] \rangle = \sum_{\sigma \in s_{n_i n_j}} \prod_{t \in n_j} \langle [b_{t\sigma(t)\mu L}, b_{t\sigma(t)\mu U}],$$

$$[b_{t\sigma(t)\lambda L}, b_{t\sigma(t)\lambda U}], [b_{t\sigma(t)vL}, b_{t\sigma(t)vU}] \rangle$$

It is clear that $\langle [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\lambda L}, c_{ij\lambda U}], [c_{ij\upsilon L}, c_{ij\upsilon U}]\rangle \leq \langle [d_{ij\mu L}, d_{ij\mu U}], [d_{ij\lambda L}, d_{ij\lambda U}], [d_{ij\upsilon L}, d_{ij\upsilon U}]\rangle$. Since, $a_{t\sigma(t)\mu L} \leq b_{t\sigma(t)\mu L}$, $a_{t\sigma(t)\mu L} \leq b_{t\sigma(t)\mu U}$, $a_{t\sigma(t)\lambda L} \geq b_{t\sigma(t)\lambda L}$, $a_{t\sigma(t)\lambda L} \geq a_{t\sigma(t)\lambda U}$ and $a_{t\sigma(t)\upsilon L} \geq b_{t\sigma(t)\upsilon U}$ for all $t \neq j$, $\sigma(t) \neq \sigma(j)$.

Therefore $C \leq D$, i.e., $adj\ A \leq adj\ B$.

Property 5. For a square IVNFM A, $adj(A^T) = (adj\ A)^T$.

Proof. Let $B = adj\ A$, $C = adj\ A^T$.

Therefore,

$$\langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\lambda L}, b_{ij\lambda U}], [b_{ij\upsilon L}, b_{ij\upsilon U}]\rangle = \sum_{\sigma \in s_{n_i n_j}} \prod_{t \in n_j} \langle [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\lambda L}, a_{t\sigma(t)\lambda U}], [a_{t\sigma(t)\upsilon L}, a_{t\sigma(t)\upsilon U}]\rangle$$ and

$$\langle [c_{ij\mu L}, c_{ij\mu U}], [c_{ij\lambda L}, c_{ij\lambda U}], [c_{ij\upsilon L}, c_{ij\upsilon U}]\rangle = \sum_{\sigma \in s_{n_i n_j}} \prod_{t \in n_j} \langle [a_{t\sigma(t)\mu L}, a_{t\sigma(t)\mu U}], [a_{t\sigma(t)\lambda L}, a_{t\sigma(t)\lambda U}], [a_{t\sigma(t)n L}, a_{t\sigma(t)\upsilon U}]\rangle = \langle [b_{ij\mu L}, b_{ij\mu U}], [b_{ij\lambda L}, b_{ij\lambda U}], [b_{ij\upsilon L}, b_{ij\upsilon U}]\rangle.$$

Therefore, $adj(A^T) = (adj\ A)^T$.

Property 6. For an IVNFM A, $|A| = |adj\ A|$.

Proof. $A = [\langle [A_{ij\mu L}, A_{ij\mu U}], [A_{ij\lambda L}, A_{ij\lambda U}], [A_{ij\upsilon L}, A_{ij\upsilon U}]\rangle]$ where, $[\langle [A_{ij\mu L}, A_{ij\mu U}], [A_{ij\lambda L}, A_{ij\lambda U}], [A_{ij\upsilon L}, A_{ij\upsilon U}]\rangle]$ is the cofactor of the element $[\langle [a_{ij\mu L}, a_{ij\mu U}], [a_{ij\lambda L}, a_{ij\lambda U}], [a_{ij\upsilon L}, a_{ij\upsilon U}]\rangle]$ in the IVNFM A. Therefore,

$$|adj\ A| = \sum_{\sigma \in s_n} \langle [A_{1\sigma(1)\mu L}, A_{1\sigma(1)\mu U}], [A_{1\sigma(1)\lambda L}, A_{1\sigma(1)\lambda U}], [A_{1\sigma(1)\upsilon L}, A_{1\sigma(1)\upsilon U}]\rangle \langle [A_{2\sigma(2)\mu L}, A_{2\sigma(2)\mu U}], [A_{2\sigma(2)\lambda L}, A_{2\sigma(2)\lambda U}], [A_{2\sigma(2)\upsilon L}, A_{2\sigma(2)\upsilon U}]\rangle \cdots \langle [A_{n\sigma(n)\mu L}, A_{n\sigma(n)\mu U}], [A_{n\sigma(n)\lambda L}, A_{n\sigma(n)\lambda U}], [A_{n\sigma(n)\upsilon L}, A_{n\sigma(n)\upsilon U}]\rangle$$

$$= \sum_{\sigma \in s_n} \prod_{i=1}^{n} \langle [A_{i\sigma(i)\mu L}, A_{i\sigma(i)\mu U}], [A_{i\sigma(i)\lambda L}, A_{i\sigma(i)\lambda U}], [A_{i\sigma(i)\upsilon L}, A_{i\sigma(i)\upsilon U}]\rangle$$

$$=\sum_{\sigma\in s_n}\prod_{i=1}^{n}\sum_{\theta\in s_{n_i n_{\sigma(i)}}}\prod_{t\in n_j}\langle[a_{t\theta(t)\mu L}, a_{t\theta(t)\mu U}], [a_{t\theta(t)\lambda L}, a_{t\theta(t)\lambda U}], [a_{t\theta(t)\upsilon L}, a_{t\theta(t)\upsilon U}]\rangle$$

$$=\sum_{\sigma\in s_n}[(\prod_{t\in n_1}\langle[a_{t\theta_1(t)\mu L}, a_{t\theta_1(t)\mu U}], [a_{t\theta_1(t)\lambda L}, a_{t\theta_1(t)\lambda U}], [a_{t\theta_1(t)\upsilon L}, a_{t\theta_1(t)\upsilon U}]\rangle)$$

$$(\prod_{t\in n_2}\langle[a_{t\theta_2(t)\mu L}, a_{t\theta_2(t)\mu U}], [a_{t\theta_2(t)\lambda L}, a_{t\theta_2(t)\lambda U}], [a_{t\theta_2(t)\upsilon L}, a_{t\theta_2(t)\upsilon U}]\rangle)\ldots$$

$$(\prod_{t\in n_n}\langle[a_{t\theta_n(t)\mu L}, a_{t\theta_n(t)\mu U}], [a_{t\theta_n(t)\lambda L}, a_{t\theta_n(t)\lambda U}], [a_{t\theta_n(t)\upsilon L}, a_{t\theta_n(t)\upsilon U}]\rangle)]$$

$$=\sum_{\sigma\in s_n}[((\langle[a_{2\theta_1(2)\mu L}, a_{2\theta_1(2)\mu U}], [a_{2\theta_1(2)\lambda L}, a_{2\theta_1(2)\lambda U}], [a_{2\theta_1(2)\upsilon L}, a_{2\theta_1(2)\upsilon U}]\rangle$$

$$\langle[a_{3\theta_1(3)\mu L}, a_{3\theta_1(3)\mu U}], [a_{3\theta_1(3)\lambda L}, a_{3\theta_1(3)\lambda U}], [a_{3\theta_1(3)\upsilon L}, a_{3\theta_1(3)\upsilon U}]\rangle$$

$$\langle[a_{n\theta_1(n)\mu L}, a_{n\theta_1(n)\mu U}], [a_{n\theta_1(n)\lambda L}, a_{n\theta_1(n)\lambda U}], [a_{n\theta_1(n)\upsilon L}, a_{n\theta_1(n)\upsilon U}]\rangle)$$

$$\langle[a_{1\theta_2(1)\mu L}, a_{1\theta_2(1)\mu U}], [a_{1\theta_2(1)\lambda L}, a_{1\theta_2(1)\lambda U}], [a_{1\theta_2(1)\upsilon L}, a_{1\theta_2(1)\upsilon U}]\rangle$$

$$\langle[a_{3\theta_2(3)\mu L}, a_{3\theta_2(3)\mu U}], [a_{3\theta_2(3)\lambda L}, a_{3\theta_2(3)\lambda U}], [a_{3\theta_2(3)\upsilon L}, a_{3\theta_2(3)\upsilon U}]\rangle$$

$$\langle[a_{n\theta_2(n)\mu L}, a_{n\theta_2(n)\mu U}], [a_{n\theta_2(n)\lambda L}, a_{n\theta_2(n)\lambda U}], [a_{n\theta_2(n)\upsilon L}, a_{n\theta_2(n)\upsilon U}]\rangle)\ldots$$

$$((\langle[a_{1\theta_n(1)\mu L}, a_{1\theta_n(1)\mu U}], [a_{1\theta_n(1)\lambda L}, a_{1\theta_n(1)\lambda U}], [a_{1\theta_n(1)\upsilon L}, a_{1\theta_n(1)\upsilon U}]\rangle\ldots$$

$$\langle[a_{2\theta_n(2)\mu L}, a_{2\theta_2(2)\mu U}], [a_{2\theta_n(2)\lambda L}, a_{2\theta_n(2)\lambda U}], [a_{2\theta_n(2)\upsilon L}, a_{2\theta_n(2)\upsilon U}]\rangle$$

$$\langle[a_{n-1\theta_2(n-1)\mu L}, a_{n-1\theta_2(n-1)\mu U}], [a_{n-1\theta_2(n-1)\lambda L}, a_{n-1\theta_2(n-1)\lambda U}], [a_{n-1\theta_2(n-1)\upsilon L},$$

$a_{n-1\theta_2(n-1)\upsilon U}]\rangle)]$ where, $f_{\hat{\theta}\theta}\in[1, 2, \ldots, n]\setminus\{\hat{\theta}\}$, $\hat{\theta}=1, 2, \ldots, n$.

But since,

$$\langle[a_{\hat{\theta}\theta f_{\hat{\theta}}(\hat{\theta})\mu L}, a_{\hat{\theta}\theta f_{\hat{\theta}}(\hat{\theta})\mu U}], [a_{\hat{\theta}\theta f_{\hat{\theta}}(\hat{\theta})\lambda L}, a_{\hat{\theta}\theta f_{\hat{\theta}}(\hat{\theta})\lambda U}], [a_{\hat{\theta}\theta f_{\hat{\theta}}(\hat{\theta})\upsilon L}, a_{\hat{\theta}\theta f_{\hat{\theta}}(\hat{\theta})\upsilon U}]\rangle$$

$$=\langle[a_{n\sigma(n)\mu L}, a_{n\sigma(n)\mu U}], [a_{n\sigma(n)\lambda L}, a_{n\sigma(n)\lambda U}], [a_{n\sigma(n)\upsilon L}, a_{n\sigma(n)\upsilon U}]\rangle.$$

Therefore

$$\langle[a_{1\sigma(1)\mu L}, a_{1\sigma(1)\mu U}], [a_{1\sigma(1)\lambda L}, a_{1\sigma(1)\lambda U}], [a_{1\sigma(1)\upsilon L}, a_{1\sigma(1)\upsilon U}]\rangle$$

$$|adj\, A|=\sum_{\sigma\in s_n}\langle[a_{2\sigma(2)\mu L}, a_{2\sigma(2)\mu U}], [a_{2\sigma(2)\lambda L}, a_{2\sigma(2)\lambda U}], [a_{2\sigma(2)\upsilon L}, a_{2\sigma(2)\upsilon U}]\rangle$$

$\langle [a_{n\sigma(n)\mu L}, a_{n\sigma(n)\mu U}], [a_{n\sigma(n)\lambda L}, a_{n\sigma(n)\lambda U}], [a_{n\sigma(n)\upsilon L}, a_{n\sigma(n)\upsilon U}] \rangle = | A |.$

Conclusion

In this work, Interval valued neutrosophic fuzzy matrices are introduced based on M. Pal et al. [13]. Then the operations on Interval valued neutrosophic fuzzy matrices are discussed. Some properties on them are conferred with the real time example. The same operations and properties can be extended to various Neutrosophic numbers.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87-96.

[2] K. Atanassov, Operations over interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 64 (1994), 159-174.

[3] H. Hashimoto, Canonical form of a transitive matrix, Fuzzy Sets and Systems 11 (1983), 157-162.

[4] R. Hemasinha, N. R. Pal and J. C. Bezdek, Iterates of fuzzy circulant matrices, Fuzzy Sets and Systems 60 (1993), 199-206.

[5] S. K. Khan and M. Pal, Some operations on intuitionistic fuzzy matrices, Acta Ciencia Indica, XXXII M (2) (2006), 515-524.

[6] K. H. Kim and F. W. Roush, Generalised fuzzy matrices, Fuzzy Sets and Systems 4 (1980), 293-315.

[7] W. Kolodziejczyk, Canonical form of a strongly transitive fuzzy matrix, Fuzzy Sets and Systems 22 (1987), 292-302.

[8] S. Broumi, F. Smarandache and M. Dhar, On fuzzy soft matrix based on reference function, Information Engineering and Electronic Business 2 (2013), 52-59.

[9] F. Smarandache, Neutrosophic set, A generalisation of the intuitionistic fuzzy sets, Inter. J. Pure Appl. Math. 24 (2005), 287-297.

[10] J. B. Kim, Determinant theory for fuzzy and boolean matrices, Congressus Numerantium (1988), 273-276.

[11] M. Pal, Intuitionistic fuzzy determinant, V. U. J. Physical Sciences 7 (2001), 87-93.

[12] M. Pal, S. K. Khan and A. K. Shyamal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets 8(2) (2002), 51-62.

[13] Madhumangal Pal and Susanta K. Khan, Interval valued intuitionistic fuzzy matrices, NIFS 11 (2005), 16-27.

[14] M. Z. Ragab and E. G. Emam, The determinant and adjoint of a square fuzzy matrix, Fuzzy Sets and Systems 61 (1994), 297-307.

[15] M. G. Ragab and E. G. Emam, On the min-max composition of fuzzy matrices, Fuzzy Sets and Systems 75 (1995), 83-92.

[16] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965), 338-353.