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FUZZY *s*-DOMINATING ENERGY

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Abstract

The energy of a graph is defined as the sum of the absolute values of eigenvalues of its adjacency matrix. The absolute value of the largest eigenvalue is called the spectral radius of the graph. This article introduces *s*-dominating energy in simple connected crisp graphs and extends the same to connected fuzzy graphs. Also *s*-dominating energy of a complete fuzzy graph is determined and bounds on fuzzy *s*-dominating energy are acquired.

1. Introduction

Eigenvalues and Eigen vectors of matrices have huge real life applications. Steiner domination number in crisp graphs has been studied from [7]. Also domination in fuzzy graphs was studied from [2]. The close relation between eigenvalues of dominating matrix and dominating energy are expounded in [3], [4] and [5]. The different types of energies of fuzzy graphs are explicated in [1] and [8]. These studies lead us to introduce Steiner dominating energy (i.e.) *s*-dominating energy in crisp graphs and is then extended to fuzzy graphs.

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2. *s*-Dominating Energy (Steiner dominating energy)

This section introduces Steiner dominating energy (s-dominating energy) in crisp graphs and some properties are discussed. If $\widetilde{G}$ is a simple connected crisp graph with n nodes and $\widetilde{S}$ is the minimum Steiner dominating set of $\widetilde{G}$, then the Steiner dominating matrix is defined as $A_s(\widetilde{G}) = [\widetilde{s}_{ij}]$ where

$$\widetilde{s}_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } v_i \in \widetilde{S} \\ 1 & \text{if } v_i v_j \in E(\widetilde{G}) \\ 0 & \text{otherwise} \end{cases}$$

The fuzzy characteristic polynomial of $A_s(\widetilde{G})$ is given by $f_n(\widetilde{G}, \lambda) = \det(\lambda I - A_s(\widetilde{G}))$. The eigenvalues of the matrix $A_s(\widetilde{G})$ are called the s-dominating eigenvalues of $\widetilde{G}$. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the s-dominating eigenvalues in the order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ then the s-dominating energy of G is defined as

$$E_s(\widetilde{G}) = \sum_{i=1}^{n} | \lambda_i |.$$

Also $| \lambda_1 |$ is called the s-dominating spectral radius of $\widetilde{G}$. The following results can be verified easily.

Observations:

- There is a close relation between the s-dominating spectral radius of $\widetilde{G}$ and the Steiner domination number.
- For complete graphs K_n, the s-dominating energy is equal to the Steiner domination number which is equal to the number of nodes n. This also equals the s-dominating spectral radius of K_n.
- From table 1, for star graphs $K_{1,n}$, $E_s(K_{1,n}) \geq \gamma_s(\widetilde{G}) \geq \lambda_1$.

Table 1. s-dominating energies, s-dominating eigenvalues and ranks of complete bipartite graphs.

$K_{1,n}$	$\gamma_s(\widetilde{G})$	$E_s(K_{1,n})$	λ_1	Rank $((E_s(K_{1,n}))$	Eigenvalues
$K_{1,2}$	2	4	2	3	2,1.-1
$K_{1,3}$	3	5.6	2.3	4	2.3, -1.3, 1,1
$K_{1,4}$	4	7	2.5	5	2.5, -1.5, 1,1,1
$K_{1,5}$	5	8.583	2.791	6	2.79, -1.79,1,1,1,1
$K_{1,6}$	6	10	3	7	3, -2, 1,1,1,1,1
$K_{1,7}$	7	11.38	3.193	8	3.19,-2.19,1,1,1,1,1,1
$K_{1,8}$	8	12.74	3.372	9	3.37,-2.37,1,1,1,1,1,1,1
$K_{1,9}$	9	14.08	3.5413	10	3.54, -2.54,1,1,1,1,1,1,1,1

For complete bipartite graphs $K_{m,n}$ where $m \geq 2$, $\lambda_1 > \gamma_s(\widetilde{G})$.

3. Fuzzy *s*-dominating Energy

Next s-dominating energy is defined in connected fuzzy graphs.

3.1. Definition (fuzzy *s*-dominating energy)

Let $G(V, \sigma, \mu)$ be a connected fuzzy graph and S be a minimum fuzzy Steiner dominating set of G. The minimum fuzzy s-dominating matrix is defined as the $n \times n$ matrix $A_s^f(G) = [s_{ij}]$ where

$$s_{ij} = \begin{cases} \sigma(v_i) & \text{if } i = j \text{ and } v_i \in S \\ \mu(v_i, v_j) & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

3.2. Definition

The fuzzy characteristic polynomial of $A_s^f(G)$ is given by $f_n(G, \lambda) = \det(\lambda I - A_s^f(G))$. The eigenvalues of the matrix $A_s^f(G)$ is called the s-dominating eigenvalues of G and these eigenvalues form the s-dominating spectrum of G. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the s-dominating eigenvalues in the order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$ then the s-dominating energy of G is defined as

$$E_s^f(G) = \sum_{i=1}^{n} | \lambda_i |.$$

Theorem 1. *Let $G(V, \sigma, \mu)$ be a simple connected fuzzy graph with node set $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and arc set E. Let $S = \{u_1, u_2, u_3, \ldots, u_k\}$ be the minimum fuzzy Steiner dominating set (i.e.) γ^{fs}- set of G and $A_S^f(G) = [s_{ij}]$ be the fuzzy s-dominating matrix of G. Let $D = | \det A_s^f(G) |$. If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are the fuzzy s-dominating eigenvalues of G. Then the following results hold*

(i) $\sum_{i=1}^{n} \lambda_i = \gamma^{fs}$

(ii) $\sum_{i=1}^{n} \lambda_i^2 \approx s + 2t$

(iii) $\sqrt{s + 2t + 2\frac{1}{n(n-1)} D^{2/n}} \leq [E_S^f(G)] \leq \sqrt{n(s + 2t)}$

where $E_s^f(G)$ is the fuzzy s-dominating matrix of G, $s = \sum_{i=1}^{k} (\sigma(u_i))^2$ is the sum of the squares of node weights of S and $t = \sum_{i \neq j} (\mu(v_i, v_j)^2)$ is sum of the squares of arc weights of G.

Proof. Let $G(V, \sigma, \mu)$ be the fuzzy graph as given in the hypothesis of the theorem. Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the fuzzy s-dominating eigenvalues of G. We have the result that sum of the fuzzy s-dominating eigenvalues of G is the trace of $A_S^f(G)$ which is equal to the fuzzy cardinality of S, the fuzzy Steiner domination number γ^{fs} (i.e.) $\sum_{i=1}^{n} \lambda_i = \gamma^{fs}$. Also we know that sum of squares of s-dominating eigenvalues is the trace of $[A_s^f(G)]^2$.

Hence

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}s_{ij}s_{ji}\\ &= \sum_{i=1}^{n}s_{ii}^2 + \sum_{i\neq j}s_{ij}s_{ji}\\ &= \sum_{i=1}^{n}s_{ii}^2 + 2\sum_{i<j}s_{ij}^2\\ &\approx \sum_{i=1}^{n}(\sigma(u_i))^2 + 2\sum_{i\neq j}(\mu(v_i, v_j))^2\\ &\approx s + 2t.\end{aligned}$$

Thus (i) and (ii) hold.

Now to prove (iii) consider the Schwartz inequality, $(\sum_{i=1}^{n} x_i y_i)^2 \leq (\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2)$.

In this inequality taking $x_i = 1$ and $y_i = |\lambda_i|$, we get

$$\begin{aligned}\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 &\leq \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}|\lambda_i|^2\right)\\ &\Rightarrow [E_s^f(G)]^2 \leq n\left(\sum_{i=1}^{n}\lambda_i^2\right)\\ &\Rightarrow [E_s^f(G)]^2 \leq n(s + 2t)\\ &\Rightarrow [E_s^f(G)] \leq \sqrt{n(s + 2t)}. \qquad (1)\end{aligned}$$

Now we shall prove the lower bound. We know that always geometric mean is less than or equal to the arithmetic mean. Therefore $\frac{1}{n}\sum_{i=1}^{n}|\lambda_i| \geq [\prod_{i=1}^{n}|\lambda_i|]^{1/n}$.

$$\Rightarrow \frac{1}{n(n-1)} \sum_{\substack{i=j \\ i \neq j}}^{n} |\lambda_i||\lambda_j| \geq \left[\prod_{\substack{i=1 \\ i \neq j}}^{n} |\lambda_i||\lambda_j| \right]^{1/n(n-1)}$$

$$= \left[\prod_{i=1}^{n} |\lambda_i|^{2(n-1)} \right]^{1/n(n-1)}$$

$$= \left[\prod_{i=1}^{n} |\lambda_i| \right]^{2/n}$$

$$= \left| \left[\prod_{i=1}^{n} \lambda_i \right] \right|^{2/n}$$

$$= | \det(A_S^f(G)) |^{2/n} = D^{2/n}.$$

$$\Rightarrow \sum_{\substack{i=1 \\ i \neq j}}^{n} |\lambda_i||\lambda_j| \geq \frac{1}{n(n-1)} D^{2/n}.$$

Now

$$[E_s^f(G)]^2 = \left(\sum_{i=1}^{n} |\lambda_i^2| \right)^2$$

$$= \sum_{i=1}^{n} |\lambda_i|^2 + 2 \sum_{i \neq j} |\lambda_i||\lambda_j|$$

$$= \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{i \neq j} |\lambda_i||\lambda_j|$$

$$\geq s + 2t + 2 \frac{1}{n(n-1)} D^{2/n}$$

$$\Rightarrow [E_S^f(G)] \geq \sqrt{s + 2t + 2 \frac{1}{n(n-1)} D^{2/n}}. \quad (2)$$

From (1) and (2) $\sqrt{s + 2t + 2\frac{1}{n(n-1)}D^{2/n}} \leq [E_S^f(G)] \leq \sqrt{n(s+2t)}$.

Hence (iii) holds.

Theorem 2. *The fuzzy s-dominating energy of a complete fuzzy graph is approximately equal to the fuzzy Steiner domination number* (*i.e.*) $E_S^f(G) \approx \gamma^{fs}(G)$.

Proof. Let K^{σ} be a complete fuzzy graph with n nodes. We have the result that for a complete fuzzy graph $\gamma^{fs} = p = \sum_{u \in V(G)} \sigma(u)$ where p is the order of the fuzzy graph G. Hence the only Steiner dominating set is $V(G)$.

Let $v_1, v_2, v_3, \ldots, v_n$ be the nodes of G such that $\sigma(v_1) \geq \sigma(v_2) \geq \sigma(v_3) \geq \ldots \geq \sigma(v_n)$.

Now the fuzzy s-dominating matrix is

$$A_S(G) = \begin{bmatrix} \sigma(v_1) & \sigma(v_1) & \sigma(v_1) & \sigma(v_n) \\ \sigma(v_2) & \sigma(v_2) & \sigma(v_2) & \\ \sigma(v_3) & \sigma(v_3) & \sigma(v_3) & \\ \sigma(v_n) & \sigma(v_n) & & \sigma(v_n) \end{bmatrix}.$$

Clearly in the characteristic polynomial of $A_S(G)$, the coefficient of λ is $\sigma(v_1) + \sigma(v_2) + \sigma(v_3) + \ldots + \sigma(v_n)$ which is the sum of the eigenvalues. Hence the fuzzy s-dominating energy is equal to $\sum_{u \in V(G)} \sigma(u) = p = \gamma^{fs}(G)$. Hence the theorem.

References

[1] Anjali Narayanan and Sunil Mathew, Energy of a fuzzy graph, Annals of Fuzzy Mathematics and Informatics ISSN: 2093-9310 (printed version), 2287-6235 (Online version) 6(3) (2013), 455-465.

[2] Benjier H. Arriola, A note on some domination of fuzzy graphs, Far East Journal of Applied Mathematics (2017), 113-123.

[3] Clemens Barand and Norbert Seifter, eigenvalues and domination in graphs, Mathematica Slovoca 46(1) (1996), 33-39.

[4] S. M. Hosamani, P. V. Patil and S. H. Malghan, A note on the minimum dominating

energy of a graph, Palestine Journal of Mathematics 7(2) (2018), 663-666.

[5] M. R. Rajesh Kanna, B. N. Dharmendra and G. Sridhara, The minimum dominating energy of a graph, International Journal of Pure and Applied Mathematics 707-718, ISSN : 1311-8080 (printed version), 1314-3395 (Online version) 85(4) (2013).

[7] K. Ramalakshmi and K. Palani, On Steiner domination number of graphs, International Journal of Mathematics Trends and Technology 4 (2017).

[8] Sadegh Rahimi Sharbaf and Fatmem Fayazi, Laplacian energy of a fuzzy graph, Iranian Journal of Mathematical Chemistry 5(1) (2014), 1-10.