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COMMON FIXED POINT RESULTS FOR HYBRID PAIR OF MAPPINGS VIA SOME PROPERTIES IN *G*-METRIC SPACES

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Abstract

In this paper, we introduce the notion of compatibility for hybrid pair of mappings in framework of *G*-metric spaces. Firstly, we prove common fixed point theorem for hybrid pair of mappings along with the (owc)-property. Secondly, we prove common fixed point theorem for hybrid pair of occasionally coincidentally idempotent mappings satisfying (CLR_f)-property using Hausdorff *G*-distance. Also, we give examples to indicate the usefulness of our main results.

1. Introduction

In 2004, Mustafa and Sims [22] had shown that most of the results concerning Dhage's *D*-metric spaces are invalid and they introduced an improved version of the generalized metric space called *G*-metric spaces. Mustafa et al. [22-24] studied many fixed point results for a self mapping in G-metric space under certain conditions. Chugh et al. [16] obtained some fixed point results for maps satisfying property p in *G*-metric spaces. The study of common fixed point problems in *G*-metric spaces was initiated by Abbas and Rhoades [14]. Thereafter many authors obtained common fixed point results for self mappings satisfying different contractive conditions in

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G-metric spaces. Kaewcharoen and Kaewkhao [1] and Nedal et al. [20] proved fixed point results for single-valued and multivalued mappings in G-metric spaces. After that, some authors used (E.A) and (CLR) properties to prove common fixed point theorems for self mappings in generalized metric spaces. In 2012, Choudhury et al. [3] introduced the notion of compatible for self mappings in G-metric space. Recently, in 2019, A. Farajzadeh [2] proved some fixed point theorems in K-metric type space by introducing some properties and KKM mappings.

In this paper, we introduce compatible and non-compatible mappings for hybrid maps in G-metric space. We use this concept of compatible and non-compatible mappings for particular case of our main results. In our main results, we obtain some common fixed point theorems for hybrid pair of mappings by using (owc)-property and (CLR_f)-property to the setting of Hausdorff G-distance. Examples provided to indicate the usefulness of our main results.

Now we give preliminaries and basic definitions which are used throughout the paper.

In 2006, Mustafa and Sims [23] introduced the concept of G-metric spaces as follows:

Definition 1.1. Let X be a nonempty set and $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (Symmetric in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric or more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. A G-metric space is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Example 1.1. Let (X, d) be a usual metric space. Then the function $G : X \times X \times X \to R^+$ defined by $G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for all $x, y, z \in X$ is a G-metric space.

Definition 1.3. Let (X, G) be a G-metric space. Then a sequence $\{x_n\}$ in X is:

(i) a G-convergent sequence if for any $\varepsilon > 0$, there exist an $x \in X$ and $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \leq N$,

(ii) a G-Cauchy sequence if for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $G(x_n, x_m, x_1) < \varepsilon$, for all $n, m, 1 \leq N$.

Proposition 1.4. *Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence in X. Then the following are equivalent*:

(i) $\{x_n\}$ *is converges to x*,

(ii) $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ *as* $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ *as* $m, n \to \infty$.

Proposition 1.5. *Let (X, G) be a G-metric space. Then for any x, y, z and $a \in X$ it follows that*:

(1) if $G(x, y, z) = 0$, *then* $x = y = z$,

(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(3) $G(x, y, y) \leq 2G(y, x, x)$,

(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(5) $G(x, y, z) \leq \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$

(6) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Proposition 1.6. *Let* (X, G) *be a G-metric space, define* $d_G : X \times X \to R$ *by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x)$$

for all $x, y \in X$. *Then* (X, d_G) *is a metric space. It can be noted that* $G(x, y, y) \leq \frac{2}{3} d_G(x, y)$. *If* (X, G) *is a symmetric G-metric space, then* $d_G(x, y) = 2G(x, y, y)$ *for all* $x, y \in X$. *However, if* (X, G) *is not symmetric, then it follows from the G-metric properties that*

$$\frac{3}{2} G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y) \text{ for all } x, y \in X.$$

Remark 1.1. Let X be a G-metric space, $x \in X$ and $B \subseteq X$. Then for each $y \in B$, we have

$$\begin{aligned} G(x, B, B) &= d_G(x, B) + d_G(B, B) + d_G(x, B) \\ &\leq 2d_G(x, y) \\ &= 2(G(x, x, y) + G(x, y, y)) \\ &\leq 2(G(x, y, y) + G(x, y, y) + G(x, y, y)) \\ &\leq 6G(x, y, y). \end{aligned}$$

In 2011, Kaewcharoen and Kaewkhao [1] established the following concepts:

Let X be a G-metric space and let $CB(X)$ be the family of all nonempty closed bounded subsets of X. Let $H_G(\cdot, \cdot)$ be the Hausdorff G-distance on $CB(X)$, i.e.,

$$H_G(A, B, C) = \max\left\{\sup_{a \in A} G(a, B, C), \sup_{b \in B} G(b, A, C), \sup_{c \in C} G(c, A, B)\right\}.$$

where

$$G(x, B, C) = d_G(x, B) + d_G(B, C) + d_G(x, C)$$

$$d_G(x, B) = \inf\{d_G(x, y), y \in B\},$$

$$d_G(A, B) = \inf \{d_G(a, b), a \in A, b \in B\},$$

$$G(a, b, C) = \inf \{G(a, b, c), c \in C\}.$$

In 2012, Tahat et al. [20] gave the following lemma in G-metric space:

Lemma 1.1. *Let* (X, G) *be a G-metric space and* $A, B \in CB(X)$. *Then for each* $a \in A$, *we have* $G(a, B, B) \leq H_G(A, B, B)$.

In 1976, Jungck [5] proved a common fixed point theorem of commuting mappings in a metric space. Sessa [17] generalized the idea of commuting mappings in 1982, by introducing the concept of weakly commuting mappings. In 1986, Jungck [6] defined the notion of compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting maps are compatible but the converse is not true in metric spaces. These results have been extended to multivalued mappings by Kaneko and Sessa [8]. Pathak [9] generalized the concept of compatibility by defining weak compatibility for hybrid pairs of mappings. Naturally, compatible mappings are weakly compatible but not conversely. Jungck and Rhoades [7] in 2006, coined the idea of occasionally weakly compatible mappings ((owc)-property). Abbas and Rhoades [13] extended the definition of occasionally weakly compatible mappings to the setting of multivalued mappings.

Aamri and Moutawakil [12] in 2002, defined the idea of (E. A) property for self mappings which is a true generalization of non-compatible mappings in metric spaces. Later on Kamran [19] extended the notion of (E.A) property to hybrid pair of mappings. The (E.A) property requires completeness (closedness) for the existence of the fixed point in the underlying subspace. To relaxes the requirement of completeness (closedness), the very first common limit range property with respect to mapping $f((CLR_f)$- property) is introduced by Sintunaravat and Kumam [21] regarding fuzzy metric space after that this property is used in many other spaces which showed the superiority of (CLR_f)- property than (E. A) property. Imdad et al. [15] established (CLR_f)- property for hybrid pair of mappings in symmetric spaces.

The following definitions and results are standard in the theory of hybrid pair of mappings.

Definition 1.7. Let $f : X \to X$ and $T : X \to CB(X)$ be a single valued and multivalued mapping respectively. Then

(i) a point $x \in X$ is a fixed point of f (resp. T) if $x = fx$ (resp. $x \in Tx$). The set of all fixed points of f (resp. T) is denoted by $F(t)$ (resp. $F(T)$)

(ii) a point $x \in X$ is a coincidence point of f and T if $fx \in Tx$. The set of all coincidence point of f and T is denoted by $C(f, T)$.

(iii) a point $x \in X$ is a common fixed point of f and T if $x = fx \in Tx$. The set of all common fixed points of f and T is denoted by $F(f, T)$.

Definition 1.8. Let $f : X \to X$ and $T : X \to CB(X)$ be a single valued and multivalued mapping respectively in metric space. Then a hybrid pair of mappings (f, T) is said to be

(i) compatible [18] if $fTx \in CB(X)$ for all $x \in X$ and $\lim_{n \to \infty} H(Tfx_n, fTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to +\infty} Tx_n = A \in CB(X) \text{ and } \lim_{n \to \infty} fx_n = t \in A;$$

(ii) non-compatible [8] if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n \to +\infty} Tx_n = A \in CB(X)$ and $\lim_{n \to +\infty} fx_n = t \in A$ but $\lim_{n \to +\infty} H(Tfx_n, fTx_n)$ is either nonzero or nonexistent;

(iii) weakly compatible [9] if $Tfx = fTx$ for each $x \in C(f, T)$;

(iv) occasionally weakly compatible [13] (in short (owc)-property) if $fTx \subseteq Tfx$ for some $x \in C(f, T)$;

(v) satisfy the property $(E \cdot A)$ [19] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \to +\infty} fx_n = t \in A = \lim_{n \to +\infty} Tx_n$ for some $t \in X$ and $A \in CB(X)$;

(vi) satisfy common limit range property with respect to the mapping f (in short (CLR_f)- property) [15] if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to +\infty} fx_n = fu \in A = \lim_{n \to +\infty} Tx_n,$$

for some $u \in X$ and $A \in CB(X)$.

(vii) coincidentally idempotent [11] if $ffv = fv$ for every $v \in C(f, T)$. that is, f is idempotent at the coincidence points of f and T;

(viii) occasionally coincidentally idempotent [10] if $ffv = fv$ for some $v \in C(f, T)$.

The following example (taken from [25]) showing the relationship of occasionally coincidentally idempotent with other notions described in the previous definition.

Example 1.2. Let $X = \{1, 2, 3\}$ (with the standard metric),

$$f : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \; T : \begin{pmatrix} 1 & 2 & 3 \\ \{1\} & \{1, 3\} & \{1, 3\} \end{pmatrix}.$$

Then, it is straight forward to observe the following:

- $C(f, T) = \{1, 2\}$ and $F(f, T) = \{1\}$.
- (d, T) is neither compatible nor weakly compatible
- (f, T) is not coincidentally idempotent since $ff2 = f3 = 2 \neq 3 = f2$.
- (f, T) is occasionally coincidentally idempotent since $ff1 = 1 = f1$.

Obviously, in this case (f, T) is also non-compatible, but simple modifications of this example can show that occasionally coincidentally idempotent property is independent of this notion, too.

Remark 1.2. In a paper [4], Doric et al. asserted that, the occasionally weak compatibility does not produce new common fixed point results, when involved mappings have a unique point of coincidence and therefore it reduces to weak compatibility in the case of single valued mappings. However, this conclusion does not hold well in the case of hybrid pairs of mappings ([4] Example 2.5). Hence the occasionally weakly compatible property still produces new results for hybrid pairs of mappings.

The following example (taken from [25], Examples 7 and 8) exhibit the relationship between $(E \cdot A)$ property and common limit range property (CLR_f)-property).

Example 1.3. Let us consider $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define $f, g : X \to X$ and $T : X \to CB(X)$ as follow:

$$fx = \begin{cases} 2 - x & if \ 0 \leq x < 1, \\ \frac{9}{5} & if \ 1 \leq x \leq 2, \end{cases} \quad gx = \begin{cases} 2 - x & if \ 0 \leq x < 1, \\ \frac{9}{5} & if \ 1 \leq x \leq 2, \end{cases} \text{ and}$$

$$Tx = \begin{cases} \left[\frac{1}{2}, \frac{3}{2}\right] & if \ 0 \leq x \leq 1 \\ \left[\frac{1}{4}, \frac{1}{2}\right] & if \ 1 < x \leq 2. \end{cases}$$

One can verify that the pair (f, T) enjoys the property $(E \cdot A)$ as considering the sequence $\{x_n\} = \{1 - 1/n\}_{n \in N}$, but not the (CLR_f)- property. On the other hand, the pair (g, T) satisfies the (CLR_g)- property.

Remark 1.3. If a pair (f, T) satisfies the property $(E \cdot A)$ along with the closedness of $f(X)$, then the pair also satisfies the (CLR_f)- property.

In 2012, Choudhury et al. [3] introduced the notion of compatible mappings in G-metric space in case of self mappings as follows:

Definition 1.9. Let f and g be self maps of a G-metric space (X, G). The mappings f and g are said to be compatible if

$$\lim_{n \to +\infty} G(fgx_n, gfx_n, gfx_n) = 0 \text{ or}$$

$$\lim_{n \to +\infty} G(gfx_n, fgx_n, fgx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gx_n = t \in X$.

2. Main Results

Firstly, we introduce the definition of compatible mappings for a pair of hybrid mappings in G-metric spaces as follows:

Definition 2.1. Let (X, G) be a G-metric space with $f : X \to X$ and

$T : X \to CB(X)$. Then a hybrid pair of mappings (f, T) is said to be compatible if $fTx \in CB(X)$ for all $x \in X$,

$$\lim_{n \to +\infty} H_G(Tfx_n, fTx_n, fTx_n) = 0 \text{ and } \lim_{n \to +\infty} H_G(fTx_n, Tfx_n, Tfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to +\infty} Tx_n = A \in CB(X)$ and $\lim_{n \to +\infty} fx_n = t \in A$.

Also, the hybrid pair of mappings (f, T) is said to be non-compatible if $fT \in CB(X)$ for all $x \in X$ and there exists at least one sequence $\{x_n\}$ in X such that $Tx_n = A \in CB(X)$ and $\lim_{n \to +\infty} fx_n = t \in A$ but either $\lim_{n \to +\infty} H_G(Tfx_n, fTx_n, fTx_n) = 0$ or $\lim_{n \to +\infty} H_G(fTx_n, Tfx_n, Tfx_n) = 0$, does or does not exist and if it does it is different from zero.

Here, we prove common fixed point theorem for hybrid pair of mappings along with the (owc)-property.

Theorem 2.1. *Let (X, G) be a symmetric G-metric space. Let $f : X \to X$ and $T : X \to CB(X)$ satisfy the following conditions*:

(i) *The pair (f, T) satisfy the* (*owc*)*-property*,

(ii) *for all $x, y, z \in X$*,

$$H_G(Tx, Ty, Tz) \leq k \max \begin{Bmatrix} G(fx, fy, fz), \dfrac{G(fx, Tx, Tx) + G(fy, Ty, Ty) + G(fx, Tz, Tz)}{3}, \\ \dfrac{G(fx, Ty, Ty) + G(fy, Tz, Tz) + G(fz, Tx, Tx)}{3} \end{Bmatrix},$$

where $0 < k < 1$. Then the mappings f and T have a unique common fixed point in X.

Proof. Since the pair (f, T) satisfy the (owc)-property, there exist $u \in X$ such that

$$fu \in Tu, \; fTu \subseteq Tfu$$

which implies that $ffu \in Tfu$. Now, we prove that *fu* is a fixed point of *f*.

Suppose that $ffu \neq fu$. Then, by using the condition (2.1), we have

$$H_G(Tu, Tfu, Tfu) \leq \max \left\{ \begin{array}{l} G(fu, ffu, ffu), \\ \dfrac{G(fu, Tu, Tu) + G(ffu, Tfu, Tfu) + G(ffu, Tfu, Tfu)}{3}, \\ \dfrac{G(fu, Tfu, fTu) + G(ffu, Tfu, Tfu) + G(ffu, Tu, Tu)}{3} \end{array} \right\}$$

Now for $fu \in Tu$, $ffu \in Tfu$ and in the view of definition of Hausdorff G-distance, we obtain

$$G(fu, ffu, ffu) \leq H_G(Tu, Tfu, Tfu)$$

and using Remark 1.1, we have

$$G(fu, ffu, ffu) \leq H_G(Tu, Tfu, Tfu)$$

$$\leq k \max \left\{ \begin{array}{l} G(fu, ffu, ffu), \dfrac{G(fu, fu, fu) + G(ffu, ffu, ffu) + G(ffu, ffu, ffu)}{3}, \\ \dfrac{G(fu, ffu, ffu) + G(ffu, ffu, ffu) + G(ffu, fu, fu)}{3} \end{array} \right\}$$

$$\leq k \max \left\{ G(fu, ffu, ffu), 0, \frac{G(fu, ffu, ffu) + G(ffu, fu, fu)}{3} \right\}.$$

By using symmetricity of G-metric spaces, we have $G(fu, ffu, ffu)$ $= G(ffu, fu, fu)$ and hence

$$G(fu, ffu, ffu) \leq H_G(Tu, Tfu, Tfu)$$

$$\leq k \max \left\{ G(fu, ffu, ffu), 0, \frac{2}{3}, G(fu, ffu, ffu) \right\}$$

$$\leq kG(fu, ffu, ffu).$$

Since $0 < k < 1$, which implies that $fu = ffu \in Tfu$. Therefore, fu is a common fixed point of f and T. Now we prove that fu is a unique common fixed point of f and T. Assume that $w \neq z$ is another common fixed point of f and T (i.e $w = fw \in Tw$ and $z = fz \in Tz$) and in the view of definition of Hausdorff G-distance, we have

$$G(fz, fw, fw) \leq H_G(Tz, Tw, Tw).$$

From the condition (2.1) and using Remark 1.1, we obtain

$$G(fz, fw, fw) \leq H_G(Tz, Tw, Tw)$$

$$\leq k \max \left\{ \begin{array}{l} G(fz, fw, fw), \dfrac{G(fz, Tz, Tz) + G(fw, Tw, Tw) + G(fw, Tw, Tw)}{3}, \\ \dfrac{G(fz, Tw, Tw) + G(fw, Tw, Tw) + G(fw, Tz, Tz)}{3} \end{array} \right\}$$

$$\leq k \max \left\{ \begin{array}{l} G(fz, fw, fw), \dfrac{G(fz, fz, fz) + G(fw, fw, fw) + G(fw, fw, fw)}{3}, \\ \dfrac{G(fz, fw, fw) + G(fw, fw, fw) + G(fw, fz, fz)}{3} \end{array} \right\}$$

$$\leq k \max \left\{ G(fz, fw, fw), 0, \frac{2}{3}, G(fz, fw, fw) \right\}$$

$$\leq kG(fz, fw, fw).$$

Since $0 < k < 1$, which implies that $fu = fw$. Thus the common fixed point z is unique. This completes the proof.

Now, we give example which validates the result in Theorem 2.1.

Example 2.2. Consider $X = [0, \infty)$ equipped with the G-metric defined by

$$G(x, y, z) = \max \{|x - y|, |y - z|, |x - z|\},$$

and define $f : X \to X$ and $T : X \to CB(X)$ as follows:

$$fx = \begin{cases} 1 & \text{if } x < 2 \\ x & \text{if } x \geq 2, \end{cases} \text{ and } Tx = \begin{cases} \left[0, \dfrac{x}{4}\right] & \text{if } x < 2 \\ \{2\} & \text{if } x \geq 2. \end{cases}$$

Then the pair (f, T) satisfy the (owc)-property for coincidence point $x = 2$ and also we have

$$f(2) \in T(2), \ fT(2) = \{2\} \subseteq \{2\} = Tf(2).$$

Now, we verify that the mappings f and T satisfy the condition (2.1). Without loss of generality, we assume that $0 \leq x \leq y \leq z$. Also we have

$$d_G(x, y) = G(x, y, y) + G(y, x, x)$$

$$= 2|x - y| \text{ for all } x, y \in X.$$

Consider the following possible cases:

Case 1. When $0 \leq x \leq y \leq z < 2$. Then, we have

$$H_G(Tx, Ty, Tz) = H_G\left(\left[0, \frac{x}{4}\right], \left[0, \frac{y}{4}\right], \left[0, \frac{z}{4}\right]\right)$$

$$= \max\left\{\sup_{0 \leq a \leq \frac{x}{4}} G\left(a, \left[0, \frac{y}{4}\right], \left[0, \frac{z}{4}\right]\right), \sup_{0 \leq b \leq \frac{y}{4}} G\left(b, \left[0, \frac{z}{4}\right], \left[0, \frac{x}{4}\right]\right),\right.$$

$$\left.\sup_{0 \leq c \leq \frac{z}{4}} G\left(c, \left[0, \frac{x}{4}\right], \left[0, \frac{y}{4}\right]\right)\right\}.$$

Since $x \leq y \leq z$, so $\left[0, \frac{x}{4}\right] \subseteq \left[0, \frac{y}{4}\right] \subseteq \left[0, \frac{z}{4}\right]$.

This implies that

$$d_G\left(\left[0, \frac{x}{4}\right], \left[0, \frac{y}{4}\right]\right) = d_G\left(\left[0, \frac{y}{4}\right], \left[0, \frac{z}{4}\right]\right) = d_G\left(\left[0, \frac{x}{4}\right], \left[0, \frac{z}{4}\right]\right) = 0.$$

Then for each $0 \leq a \leq \frac{x}{4}$, we have

$$G\left(a, \left[0, \frac{y}{4}\right], \left[0, \frac{z}{4}\right]\right) = d_G\left(a, \left[0, \frac{y}{4}\right]\right) + d_G\left(\left[0, \frac{y}{4}\right], \left[0, \frac{z}{4}\right]\right) + d_G\left(a, \left[0, \frac{z}{4}\right]\right) = 0$$

Also for each $0 \leq b \leq \frac{y}{4}$, we have

$$G\left(b, \left[0, \frac{z}{4}\right], \left[0, \frac{x}{4}\right]\right) = d_G\left(b, \left[0, \frac{z}{4}\right]\right) + d_G\left(\left[0, \frac{z}{4}\right], \left[0, \frac{x}{4}\right]\right) + d_G\left(b, \left[0, \frac{x}{4}\right]\right)$$

$$= \begin{cases} 0 & if \;\; b \leq \frac{x}{4} \\ 2b - \frac{x}{2} & if \;\; b > \frac{x}{4} \end{cases}.$$

This yields that

$$\sup_{0 \leq b \leq \frac{y}{4}} G\left(b, \left[0, \frac{z}{4}\right], \left[0, \frac{x}{4}\right]\right) = \frac{y}{2} - \frac{x}{2}.$$

Moreover, for each $0 \leq c \leq \frac{z}{4}$, we have

$$G\left(c, \left[0, \frac{x}{4}\right], \left[0, \frac{y}{4}\right]\right) = d_G\left(c, \left[0, \frac{x}{4}\right]\right) + d_G\left(\left[0, \frac{x}{4}\right], \left[0, \frac{y}{4}\right]\right) + d_G\left(c, \left[0, \frac{y}{4}\right]\right)$$

$$= \begin{cases} 0 & if \;\; c \leq \frac{x}{4} \\ 2c - \frac{x}{2} & if \;\; \frac{x}{4} < c \leq \frac{y}{4}, \\ 4c - \frac{x}{2} - \frac{y}{2} & if \;\; c > \frac{y}{4}. \end{cases}$$

This yield that

$$\sup_{0 \leq b \leq \frac{y}{4}} G\left(c, \left[0, \frac{x}{4}\right], \left[0, \frac{y}{4}\right]\right) = z - \frac{x}{2} - \frac{y}{2}.$$

Finally, we have

$$H_G(Tx, Ty, Tz) = z - \frac{x}{2} - \frac{y}{2}.$$

In order to verify condition (2.1), it is sufficient to show that

$$H_G(Tx, Ty, Tz) \leq k\left\{\frac{G(fx, Tx, Ty) + G(fy, Ty, Ty) + G(fz, Tz, Tz)}{3}\right\}.$$

Now taking

$$G(fx, Tx, Tx) = G\left(1, \left[0, \frac{x}{4}\right], \left[0, \frac{x}{4}\right]\right)$$

$$= 2d_G\left(1, \left[0, \frac{x}{4}\right]\right)$$

$$= 4\left(1 - \frac{x}{4}\right) = 4 - x.$$

Similarly, we have $G(fy, Ty, Ty) = 4 - y$ and $G(fz, Tz, Tz) = 4 - z$. This implies that

$$\frac{G(fx, Tx, Tx) + G(fy, Ty, Ty) + G(fz, Tz, Tz)}{3} = 4 - \frac{(x + y + z)}{3}$$

We deduce that

$$z - \frac{x}{2} - \frac{y}{2} \leq k\left(4 - \frac{(x + y + z)}{3}\right).$$

Thus for all $0 \leq x \leq y \leq z < 2$, the condition (2.1) is satisfy.

Case 2. When $2 \leq x \leq y \leq z$. Then, we have

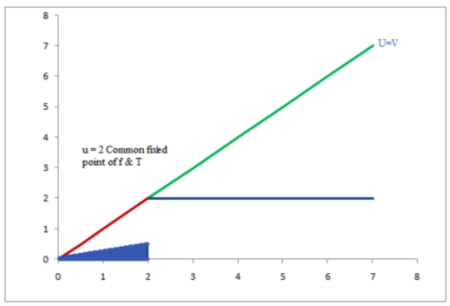
$$H_G(Tx, Ty, Tz) = H_G(\{2\}, \{2\}, \{2\}) = 0$$

and

$$G(fx, Tx, Tx) = G(x, \{2\}, \{2\}) = 2d_G(x, \{2\}) = 0.$$

Thus the condition (2.1) is also satisfied in this case.

Therefore, all the assumptions of the Theorem are fulfilled and further, the point $x = 2$ is a unique common fixed point of the mappings *f* and *T* which is verified by following figure.



In the above figure, lines with green colour represent function $f(u)$, blue colour represents the multivalued function $T(u)$ and red lined represents $u = v$ for fixed point purpose. Clearly, we find that *f* and *T* intersect on the line $u = v$ only at $u = 2$. So, $u = 2$ is a unique common fixed point of the mappings *f* and *T*.

By taking a single valued map *g* instead of *T*, we get following corollary:

Corollary 2.2. *Let* (X, G) *be a symmetric G-metric space. Let* $f, g : X \to X$ *be two single valued mappings satisfy the following conditions*:

(i) *The pair* (f, g) *satisfy the* (*owc*)*-property*,

(ii) *for all* $x, y, z \in X$,

$$G(fx, fy, fz) \leq \max\left\{\begin{matrix} G(gx, gy, gz), \dfrac{G(gx, fx, fx) + G(gy, fy, fy) + G(gz, fz, fz)}{3}, \\ \dfrac{G(gx, fy, fy) + G(gy, fz, fz) + G(gz, fx, fx)}{3} \end{matrix}\right\} \quad (2.2)$$

where $0 < k < 1$. *Then the mappings f and g have a unique common fixed point in X.*

Here, we prove common fixed point theorem for hybrid pair of occasionally coincidentally idempotent mappings satisfying (CLR_f)- property.

Theorem 2.3. *Let* (X, G) *be a symmetric G-metric space. Let* $f : X \to X$ *and* $T : X \to CB(X)$ *satisfy the following conditions*:

(i) *The pair* (f, T) *satisfy the* (CLR_f)-*property*,

(ii) *for all* $x, y, z \in X$,

$$H_G(Tx, Ty, Tz) \leq k \max\left\{\begin{matrix} G(fx, fy, fz), \\ \dfrac{G(fx, Tx, Tx) + G(fy, Ty, Ty) + G(fz, Tz, Tz)}{3}, \\ \dfrac{G(fx, Ty, Ty) + G(fy, Tz, Tz) + G(fz, Tx, Tx)}{3} \end{matrix}\right\}, \quad (2.3)$$

where $0 < k < 1$. *Then the mappings f and T have a coincidence point. Moreover, if the pair* (f, T) *enjoys occasionally coincidentally idempotent property then the pair* (f, T) *has a common fixed point.*

Proof. Since the pair (f, T) satisfy the (CLR_f)- property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to +\infty} fx_n = fu \in A = \lim_{n \to +\infty} Tx_n$$

for some $u \in X$ and $A \in CB(X)$. We assert that $fu \in Tu$. If not then using the condition (2.3), we get

$$H_G(Tx_n, Tu, Tu) \leq k \max\left\{\begin{matrix} G(fx_n, fu, fu), \\ \dfrac{G(fx_n, Tx_n, Tx_n) + G(fu, Tu, Tu) + G(fu, Tu, Tu)}{3}, \\ \dfrac{G(fx_n, Tu, Tu) + G(fu, Tu, Tu) + G(fu, Tx_n, Tx_n)}{3} \end{matrix}\right\}$$

Taking the limit as $n \to \infty$, we have

$$H_G = (A, Tu, Tu) \leq k \max \left\{ \begin{array}{l} G(fu, fu, fu), \\ \dfrac{G(fu, A, A) + G(fu, Tu, Tu) + G(fu, Tu, Tu)}{3}, \\ \dfrac{G(fu, Tu, Tu) + G(fu, Tu, Tu) + G(fu, A, A)}{3} \end{array} \right\}.$$

As $fu \in A$, so by Lemma 1.1, the above inequality implies that

$$G(fu, Tu, Tu) \leq H_G(A, Tu, Tu) \leq k \max \left\{ 0, \frac{2}{3} G(fu, Tu, Tu), \frac{2}{3} G(fu, Tu, Tu) \right\}$$

$$\leq \frac{2k}{3} G(fu, Tu, Tu),$$

which is a contradiction.

Since $0 < k < 1$, which implies that $fu \in Tu$ and hence the pair (f, T) has a coincidence point (i.e., $C(f, T) \neq \phi$).

If the hybrid pair (f, T) is occasionally coincidentally idempotent, then for some $v \in C(f, T)$, we have $ffv = fv \in Tv$. Now we show that $Tv = Tfv$. If not, then using the condition (2.3), we get

$$H_G(Tv, Tfv, Tfv) \leq k \max \left\{ \begin{array}{l} G(fv, ffv, ffv), \\ \dfrac{G(fv, Tv, Tv) + G(ffv, Tfv, Tfv) + G(ffv, Tfv, Tfv)}{3}, \\ \dfrac{G(fv, Tfv, Tfv) + G(ffv, Tfv, Tfv) + G(ffv, Tv, Tv)}{3} \end{array} \right\}$$

$$\leq k \max \left\{ \begin{array}{l} G(fv, fv, fv), \dfrac{G(fv, Tv, Tv) + G(fv, Tfv, Tfv) + G(fv, Tfv, Tfv)}{3}, \\ \dfrac{G(fv, Tfv, Tfv) + G(fv, Tfv, Tfv) + G(fv, Tv, Tv)}{3} \end{array} \right\}.$$

As $fv \in Tv$, so by Lemma 1.1, the above inequality implies that

$$G(fv, Tfv, Tfv) \leq H_G(Tv, Tfv, Tfv) \leq k \max \left\{ 0, \frac{2}{3} G(fv, Tfv, Tfv), \frac{2}{3} G(fv, Tfv, Tfv) \right\}$$

$$\leq \frac{2k}{3} G(fv, Tfv, Tfv),$$

which is a contradiction. Thus we have $fv = ffv \in Tv = Tfv$ which show that fv is a common fixed point of the mappings f and T.

Now, we give example which validates the result in Theorem 2.3.

Example 2.3. Consider $X = [0, \infty)$ equipped with the G-metric defined by

$$G(x, y, z) = \max \{|x - y|, |y - z|, |x - z|\},$$

and define $f : X \to X$ and $T : X \to CB(X)$ as follows:

$$fx = x$$

and

$$Tx = \left[0, \frac{x}{4}\right].$$

Then the mappings f and T satisfy the (CLR_f)-property for the sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$ for each $n \geq 1$. Therefore, we have

$$\lim_{n \to +\infty} fx_n = 0 = \{0\} = \lim_{n \to +\infty} Tx_n.$$

Thus the pair (f, T) satisfy the (CLR_f)-property. Also, we have

$$d_G(x, y) = G(x, y, y) + G(y, x, x)$$

$$= 2|x - y| \text{ for all } x, y \in X.$$

To prove condition (2.3), let $x., y, z \in X$. If $x = y = z = 0$, then

$$H_G(Tx, Ty, Tz) = 0 \leq G(fx, fy, fz).$$

Thus we assume that x, y and z are not all zero. Without loss of generality, we assume that $x \leq y \leq z$. Then from Example 2.2, we have

$$H_G(Tx, Ty, Tz) = H_G\left(\left[0, \frac{x}{4}\right], \left[0, \frac{y}{4}\right], \left[0, \frac{z}{4}\right]\right) = z - \frac{x}{2} - \frac{y}{2}.$$

In order to verify condition (2.3), it is sufficient to show that

$$H_G(Tx, Ty, Tz) \leq \left\{\frac{G(fx, Tx, Tx) + G(fy, Ty, Ty) + G(fz, Tz, Tz)}{3}\right\}.$$

Now taking

$$G(fx, Tx, Tx) = G\left(x, \left[0, \frac{x}{4}\right], \left[0, \frac{x}{4}\right]\right)$$

$$= 2d_G\left(x, \left[0, \frac{x}{4}\right]\right)$$

$$= 4\left(x - \frac{x}{4}\right) = 3x.$$

Similarly, we have $G(fy, Ty, Ty) = 3y$ and $G(fz, Tz, Tz) = 3z$. This implies that

$$\frac{G(fx, Tx, Tx) + G(fy, Ty, Ty) + G(fz, Tz, Tz)}{3} = x + y + z.$$

We deduce that

$$z - \frac{x}{2} - \frac{y}{2} \leq x + y + z.$$

Therefore, the condition (2.3) is satisfied and further, 0 is the coincidence point of the mappings f and T. Also, we have $ffa = fa$ for $a = 0 \in C(f, T)$, that is the hybrid pair of mappings (f, T) is occasionally coincidentally idempotent. Thus all the conditions of Theorem (2.3) are satisfied. Therefore, f and T have a common fixed point in X. In this case, a point 0 is a unique common fixed point of f and T.

In view of Remark 1.3, we have the following corollary:

Corollary 2.4. *Let* (X, G) *be a symmetric G-metric space. Let* $f : X \to X$ *and* $T : X \to CB(X)$ *satisfy the condition* (2.3) *and enjoy the* $(E \cdot A)$ *property along with the closedness of* $f(X)$, *then the mappings f and T have a coincidence point. Moreover, if the pair* (f, T) *enjoys occasionally coincidentally idempotent property then the pair* (f, T) *has a common fixed point.*

Here, we use our newly introduced concept of non-compatibility of hybrid maps. Also, we know that, a non-compatible hybrid pair always satisfies the property $(E \cdot A)$. So, in this regard, we get the following corollary:

Corollary 2.5. *Let* (X, G) *be a symmetric G-metric space. Let* $f : X \to X$ *and* $T : X \to CB(X)$ *satisfy the condition* (2.3). *If the hybrid pair* (f, T) *is non-compatible and* $f(X)$ *a closed subset of X, then the mappings f and T have a coincidence point. Moreover, if the pair* (f, T) *enjoys occasionally coincidentally idempotent property then the pair* (f, T) *has a common fixed point.*

Corollary 2.6. *Let* (X, G) *be a symmetric G-metric space. Let* $f, g : X \to X$ *be two single valued mappings satisfy the following conditions*:

(i) *The pair* (f, g) *satisfy the* (CLR_g)*- property,*

(ii) *for all* $x, y, z \in X$,

$$G(fx, fy, fz) \leq \max \left\{ \begin{array}{l} G(gx, gy, gz), \\ \dfrac{G(gx, fx, fx) + G(gy, fy, fy) + G(gz, fz, fz)}{3}, \\ \dfrac{G(gx, fy, fy) + G(gy, fz, fz) + G(gz, fx, fx)}{3} \end{array} \right\}. \tag{2.4}$$

Then the mappings f and g have a coincidence point. Moreover, if the pair (f, g) enjoys *occasionally coincidentally idempotent property then the pair* (f, g) *has a common fixed point.*

3. Conclusion

We prove some coincidence and common fixed point theorems for hybrid pairs of mappings by using occasionally weakly compatible mapping ((owc)-property) and common limit range property ((CLR)-property) under different contractions in *G*-metric spaces. Our established results here generalize and enrich the already existing theorems in literature.

4. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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