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A STUDY ON SCHNUTE GROWTH MODELS IN FORESTRY

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Abstract

The sole purpose of this paper is to discuss the Schnute Growth model in detail. A comparative study among the various cases of the Schnute growth model also presented in this paper. Classification of growth models and their roles in real life are discussed in this paper. Few methods of estimation are introduced to estimate the parameters of different forms of the Schnute model. Two different data sets are considered in order to fit the models using the newly introduced methods of estimation. A well-established selection criterion is considered to compare the obtained results and to select the best fit model. The results and discussions are also provided thereafter.

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1. Introduction

A mathematical model is a simplified representation of some real systems that can be used to visualize the problems using mathematical languages and to help explain and predict the behavior of the systems. Growth can be defined as variation and development in tissues and organs of an organism over time. Growth model is an abstraction of the natural dynamics of a forest stand and may encompass growth, mortality and other changes in stand composition and structure [4]. For a real-world problem, the first job is to formulate a mathematical model by identifying the independent and dependent variable and making assumptions that simplify the phenomenon enough to make it mathematically controllable. If there is no any physical law associated with the problem, some sets of data may be considered and examine the data in the form of a table in order to discern a pattern. In the second stage, some mathematical conclusions will be derived by using mathematics on the mathematical model. The third step is to take those mathematical conclusions and take them as information about the original real world phenomenon by making predictions. The final step is to test the predictions by checking against new real data. If the prediction doesn't fit well then one need to refine the model or to formulate a new model and start the cycle again. The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior [11]. Growth patterns are traditionally classed in two groups: determinate and indeterminate ones. Indeterminate growth is defined as growth that continues past maturation and may continue to the end of life [5]. The growth models may also be classified as a linear and nonlinear model based on the order of their parameters. Nonlinear models are sometimes called mechanistic models, distinguishing them from linear models, which are typically thought of as empirical models [4]. A mathematical growth equation never gives a complete accurate picture of a physical situation; it is an idealization. A good model simplifies real enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of a model [11].

In this study, we are mainly concerned about Schnute growth models. The model is derived from a concise biological principle. It has both flexibility and versatility. Jon Schnute used this model as a fishery model. In this model property of growth curves, such as asymptotic limits or inflection points, are shown to be incidental. This model is cast in terms of parameters which have stable statistical estimates. The general form of the model is.

$$Y = \left[y_1^b + (y_2^b - y_1^b) \frac{1 - e^{-a(t-t_1)}}{1 - e^{-a(t_2-t_1)}} \right]^{1/b}, \tag{1}$$

Where $y(t)$ is the amount of production and a, b, y_1, y_2 are parameters. The model given by Schnute has four limiting cases and all the cases have a different form. Here in this study we take second case of the Schnute model. This form has three parameters and. The parameter is considered as zero here.

The Schnute growth model is based upon growth acceleration that is the rate of a rate, therefore using the following first order differential equation the differential form of Schnute model can be derived.

$$\frac{dy}{dt} = \eta y^m - ky \tag{2}$$

Where η and k are constants of anabolism and catabolism respectively and the exponent m indicates that the latter are proportional to some power of the state variable y. The change of y is given by the difference between the processes of building up and breaking down.

1.1. Various Limiting cases of Schnute model

The different limiting forms of Schnute model are,

Case 1. When the parameters $a \neq 0$ and $b \neq 0$,

$$y(t) = \left[y_1^b + (y_2^b - y_1^b) \frac{1 - e^{-a(t-t_1)}}{1 - e^{-a(t_2-t_1)}} \right]^{\frac{1}{b}}, \tag{3}$$

Case 2. when the parameters $a \neq 0$ and $b = 0$,

$$y(t) = y_1 \left[\exp\left(\log\left(\frac{y_2}{y_1} \right) \frac{1 - e^{-a(t-t_1)}}{1 - e^{-a(t_2-t_1)}} \right) \right], \tag{4}$$

Case 3. when the parameters $a = 0$ and $b = 0$,

$$y(t) = y_1 + \exp\left[\log\left(\frac{y_2}{y_1}\right)\frac{t - t_1}{t_2 - t_1}\right], \tag{5}$$

Case 4. when the parameters $a = 0$ and $b \neq 0$,

$$y(t) = \left[y_1^b + (y_2^b - y_1^b)\frac{t - t_1}{t_2 - t_1}\right]^{\frac{1}{b}}, \tag{6}$$

It is assumed throughout this paper $y_2 > y_1 > 0$ and $\tau_2 > \tau_1$.

Here, a, b, τ_1, τ_2, y_1, y_2, y_1^b, y_2^b are parameters and t is an independent variable and $y(t)$ is dependent variable.

2. Materials and Methods

In this paper, we are considering to study the model forms of case 1, case 2 and case 3 given in equation (3), (4) and (5) respectively. Few new method of estimations are introduced to estimate the parameters of these models. The newly introduced method of estimations are given in the section 2.1. After fitting the growth models using different methods of estimation, the best fit model is selected based on a selection criterion. The selection criteria consist of five distinct steps are adopted from the paper by Borah and Mahanta [2]. Here, in this paper all the steps from I to V are the steps I, II, III, IV and VI respectively of the paper by Borah and Mahanta [2].

The parameters of the Schnute growth models have been fitted to the cumulative basal area production and the mean diameter at breast height data, originated from the Bowmont Norway spruce thinning experiment, sample plot 3661 [1].

2.1. Methods of Estimation

2.1.1. Estimation of case I (equation (3))

2.1.1.1. Method I: Method of three arbitrary points

In this method, we use three arbitrary points of the given data set. Let n be the no of observation and t_a, t_b and t_c are three arbitrary points. Assume that d be known.

Therefore the required estimation are:

$$B_1 = \frac{1 - \exp\left(\dfrac{\log w_2 - \log w_3}{d}\right)}{\exp(-kt_b) - \exp\left(\dfrac{\log w_2 - \log w_3}{d}\right)\exp(-kt_c)},$$

$$A_1 = \exp\{\log w_3 - d\log(1 - Be^{-kt_c})\},$$

$$d = \frac{\log w_1 - \log A_1}{\log(1 - Be^{-kt_a})}.$$

2.1.1.2. Method II: Method of three partial sums

In this method, we divide the whole observation into three equal parts. Let the no of observation is n then we have to consider an m such that $m = \left[\dfrac{n}{3}\right]$. Now let be the sum of first m observations, S_2 be the sum of second m observations and S_3 be the last m observations. Again assume that the parameter d is known,

Therefore the required estimation are:

$$\Rightarrow A = \frac{1}{d}\left[S_1 + \frac{(S_2 - S_1)^2}{(2S_2 - S_1 - S_3)}\right],$$

$$\Rightarrow A = \frac{1}{d}\left[\frac{2S_1S_2 - S_1^2 - S_1S_3 + S_2^2 - 2S_1S_2 + S_1^2}{2S_2 - S_1 - S_3}\right],$$

$$\Rightarrow A = \frac{1}{d}\left[\frac{S_2^2 - S_1S_3}{(2S_2 - S_1 - S_3)}\right],$$

2.1.1.3. Method III: Method using the concept of least square

In this method, we will use method of least square by considering parameters d and k known, Let, $y = (A - Be^{kt})^d$, $\Rightarrow y^{\frac{1}{d}} = A - Be^{-kt}$.

Let $y^{\frac{1}{d}} = y^b = w$, since $b = \dfrac{1}{d}$. Let $z = e^{-kt}$ and $B_1 = -B$.

Therefore, $w = (A + B_1 z)$.

Now, this is a linear equation and we can use least square method to find A and B_1. After finding A and B_1, the value of k, d can be estimated applying Newton-Raphson method 0 on the equation $y = (A - Be^{kt})^d$.

2.1.1.4. Method IV: Method of two arbitrary points

In this method, we use two arbitrary points of the given data set by assuming k and d are known. Now the model can be written as, $y = (A - Be^{kt})^d \Rightarrow y^{\frac{1}{d}} = (A - Be^{-kt})$.

Taking $y^{\frac{1}{d}} = y', e^{-kt} = z,$ the model can be written as, $y' = (A - Bz)$.

Therefore the required estimation will be

$$\Rightarrow A = \frac{y'_2 z_1 - y'_1 z_2}{z_1 - z_2}.$$

Now applying Newton-Rapshon method on $y = (A - Be^{-kt})^d$, we can find the value of k, d.

2.1.1.5. Method V: Method of three equidistant points

In this method, we use three equidistant point of the given data set. Let n be the no of observation, t_1 be the first observation, t_2 be the $\frac{t_1 + n}{2} th$ observation. Let $m = t_2 - t_1$, then t_3 be the $(t_2 + m)th$ observation. Assume that the parameter d known.

Now,

$$y = (A - Be^{-kt})^d, \Rightarrow y^{\frac{1}{d}} = A - Be^{-kt}, \Rightarrow Y = A - Be^{-kt}.$$

Where, $Y = y^{\frac{1}{d}}$.

Finally we will get the required estimation as follows:

$$B = \frac{(Y_2 - Y_1)^2}{2Y_2 - Y_1 - Y_3}\left(\frac{Y_2 - Y_1}{Y_3 - Y_2}\right)^{\frac{t_1}{m}}$$

$$A = \frac{Y_2^2 - Y_1 Y_3}{2Y_2 - Y_1 - Y_3}$$

$$c = \frac{1}{m}\log\left(\frac{Y_3 - Y_2}{Y_2 - Y_1}\right)$$

$$d = \log\frac{y}{(A - Be^{-kt})}.$$

2.1.2. Estimation of case II (equation 4)

2.1.2.1. Method I: Method of three equidistant points

Here the number of observation is taken to be n. We take three equidistant points t_1, t_2, t_3 from the data set. t_2 be the $\frac{t_1 + n}{2} th$ observation t_1 is the first observation and t_3 be the last observation. Let, $m = t_2 - t_1$.

Therefore the required estimation are:

$$C_p = \frac{1}{m}\log\left(\frac{y_3 - y_2}{y_2 - y_1}\right),$$

$$B_p = \frac{(y_2 - y_1)^2}{y_3 - 2y_2 + y_1}\left(\frac{y_2 - y_1}{y_3 - y_2}\right)^{\frac{t_1}{m}},$$

$$A_p = \frac{y_1 y_3 - y_2^2}{y_3 - 2y_2 + y_1},$$

With, $y = \log y$, $A' = \log A_1$, $B' = -B_1$, $c = -a$.

2.1.2.2. Method II: Method of Partial Sum

In this method we take log on both sides and then the data sets divided into three parts. After that we take three sums S_1, S_2, S_3. Then the required parameters estimates are:

$$A_p = \frac{1}{d}\left(\frac{S_1 S_3 - S_2^2}{S_3 - 2S_2 + S_1}\right),$$

$$B_p = \frac{(S_2 - S_1)^3}{(S_3 - 2S_2 + S_1)^2}\left(1 - \left(\frac{S_2 - S_1}{S_3 - S_2}\right)^{\frac{1}{d}}\right),$$

$$C_p = \frac{1}{d}\log\left(\frac{S_3 - S_2}{S_2 - S_1}\right),$$

With, $y = \log y$, $A' = \log A_1$, $B' = -B_1$, $c = -a$.

2.12.3. Method III: Composite Method 1

In this method, we estimated the value of c using the method of three equidistant points and then for the other two unknown parameters A, B we use the method of least square.

If c is known then we have

$$y = A + BZ, \text{ where, } Z = e^{ct}$$

Now it is a linear equation and we can use the method of least square which is given by

$$\hat{B} = \frac{(n\sum zy - \sum z \sum y)}{n\sum Z^2 - (\sum Z)^2},$$

$$\hat{A} = \bar{y} - \hat{B}\bar{Z},$$

2.1.2.4 Method IV: Composite Method 2

In this method we take the value of c from the method of three partial sum and then to find the other two unknown we apply he least square method same as method 3.

2.1.2.5 Method V: Method of Linear Transformations

For this method we take the parameter A is known. Then to estimate the other two parameters, the sum of residual square ϕ has been minimized, where

$$\phi = \sum_{i=1}^{n} (y_i - f(t_i, B))^2, \tag{7}$$

Where y_i and t_i denote the dependent and independent observations respectively. The sum of residuals is a function of B and a, Now differentiating ϕ, with respect to B and a, two equation will be obtained as

$$f = \phi_B = \sum_{i=1}^{n} \{(y_i - f(t_i, B))^2\} \left[\frac{\delta f(t_i, B)}{\delta B}\right], \tag{8}$$

$$g = \phi_a = \sum_{i=1}^{n} \{(y_i - f(t_i, B))^2\} \left[\frac{\delta f(t_i, B)}{\delta a}\right], \tag{9}$$

Now the Newton-Rapshon method for two variables can be applied to estimate the parameters B and a.

After getting the values of B and a we can estimate the value of A as:

Taking the natural logarithm, *ln* on both side of the model we have

$$\Rightarrow \log\left(\frac{A}{y}\right) = Be^{-at}, \tag{10}$$

Therefore A can be estimated as

$$\Rightarrow A = \left\{\prod_{i=1}^{n} y_i \left(Ae^{Be^{-a}} \frac{e^{-ak} - 1}{e^{-a} - 1}\right)\right\}^{\frac{1}{n}}, \tag{11}$$

2.1.2.6 Method VI: Newton-Raphson method under the assumption that the parameter *a* is known.

In this method the parameter a is taken as known then to estimate the other two parameters we take the sum of the square of the residuals as

$$\phi = \sum_{i=1}^{n} \{y_i - Ae^{-B(Z_i)}\}^2 = R^2, \tag{12}$$

where $Z_i = e^{-at}$.

Now differentiating ϕ with respect to A and B and applying the Newton-

Rapshon method for two variables we can estimate the parameters A and B.

After estimating the parameters A and B the parameter a can be estimated minimizing

$$\phi = \sum_{i=1}^{n} \{y_i - Ae^{-B(Z_i)}\}^2 = R^2, \tag{13}$$

using the Newton-Rapshon method.

2.1.3 Estimation of case IV (equation 5)

2.1.3.1 Two point Method

In this method, we use two arbitrary points of the given data set. Let t_1 and t_2 are two arbitrary observations from the data set.

The required estimation will be:

$$\Rightarrow A = w_1 e^{-\alpha t_1}.$$

2.1.3.2 Method of partial sum

In this method, we will divide the whole observation into two equal parts. Let n be total number of observations.

Therefore the required estimation will be:

$$\Rightarrow A = \frac{1}{d}\left\{S_1 - \frac{d(d+1)}{2}\frac{S_2 - S_1}{d^2}\right\}$$

3. Results and discussions

In the introduction we have discussed about growth models and specially about the Schnute growth model. From that discussion we can say that the Schnute growth model shows good results when it is applied in growth data. Here in this paper also, the three different limiting forms of Schnute growth model are fitted to the two data set from Bowmont Norway spruce thinning experiment. The parameters are estimated by using some newly introduced estimation techniques and a total of 13 methods of estimation are used to estimate the parameters of the Schnute growth model. Also based on five model selection criteria, the results are summarized and are given below.

3.1 For cumulative basal area production data from Norway experiment spruce Thinning Experiment

The estimation of parameters for the Schnute growth models and the summary of statistical analysis to cumulative basal area production are presented in Table 1. The best fit model is selected on the basis of five criteria's: Step 1: Logical and Biological consistency of the parameters. Step 2: Chi-Square Goodness of Fit Test $((\chi))^2$, in this step only those results will be considered which have 95% level of significance with their respective degree of freedom. Then the 3rd step is The Root Mean Square Error (RMSE). Comparing the RMSE we select the best fit result. The least value of RMSE represents the best fit result. The fourth step is Coefficients of determination R^2 and adjusted coefficient of determination (R_a^2). If the value of R^2 is above 0.9, it is accepted as efficient. In this paper only those results are considered which have (R_a^2) value not less than 0.99. Finally, the last step is Approximate R^2 for prediction.

After checking all the criteria for best fit model as mentioned above we found that in case of cumulative basal area production data from Norway spruce Thinning Experiment. Schnute growth model of three parameters $(b = 0)$ with method VI has shown more effective results than the other methods. This method is logically and biologically significant, showing 95% of significance level, least RMSE value, having (R_a^2) value 0.99 and R^2 prediction value is 99.90. All the eliminated results in each step are highlighted accordingly in Table 1

Table 1. Fitting of Schnute growth model for cumulative basal area production data from Norway experiment.

Case	Method	y_1	y_2	a	b	CHI	RMSE	R_a^2	R^2	R_P^2
I	I	5.8284	5.8324	.0641	2.6995	35.31	17.24	0.37	58.20	34.20
	II	92.0027	92.1198	.1356	1.1119	4.15	3.92	0.97	97.84	97.96
	III	37.7837	107	.0758	1.001	.34	1.618	0.99	99.63	99.57
	IV	38.7837	80.7085	.0901	1.0044	.76	2.68	0.98	98.99	98.50

	V	2133.268	2135.712	.0755	7102	11	1.07	1.00	99.84	99.74
II	I	41.4324	83	.5129		.09	0.86	0.99	99.89	99.87
	II	42.3977	82.634	.5633		.15	1.13	0.99	99.81	99.72
	III	40.8141	82.687	.5256		0.8	0.78	0.99	99.91	99.89
	IV	42.2346	82.3416	.5647		.15	1.11	0.99	99.82	99.75
	V	39.4588	82.6409	.5222		.14	1.06	0.99	99.83	99.80
	VI	41.193	82.6345	5316		.07	0.74	0.99	99.92	99.90
IV	I	37.99	119.54			21.54	12.21	0.69	79.06	73.95
	II	44.7487	134.6253			4.81	6.64	0.91	93.81	90.61

3.2 For mean diameter at breast height growth from the Bowmont Norway spruce Thinning Experiment

Again, after checking all the criteria's of selection as we have done in the previous data set, Schnute growth model with three parameters $(b = 0)$ with Method VI has been found to be more suitable for mean diameter at breast height growth from the Bowmont Norway spruce Thinning Experiment. This method is logically and biologically significant, showing 95% of significance level, least RMSE value, having (R_a^2) value 0.99 and R^2 prediction value is 99.79. All the eliminated results in each step are highlighted accordingly in Table 2

Table 2. Fitting of Schnute growth model for mean diameter at breast height growth from the Bowmont Norway spruce Thinning Experiment.

Case	Method	y_1	y_2	a	b	CHI	RMSE	R_a^2	R^2	R_P^2
	I	1.2549	1.2549	0	17.5215	114.07	10.44	-3.41	-194.03	-343.51
	II	21.3082	21.434	0.1365	1.1464	2.22	1.21	0.94	96.00	95.77
I	III	8.4	23.82	0.0759	1.001	0.38	0.78	0.98	98.35	98.14
	IV	8.4814	17.3519	0.0653	1.0055	0.29	0.76	1.00	98.40	97.70
	V	692.4047	696.1801	0.0758	0.6108	0.05	0.31	1.00	99.74	99.59
	I	9.0706	17.8852	0.629		0.03	0.30	1.00	99.75	99.58
	II	8.9107	17.8233	0.637		0.04	0.31	1.00	99.37	99.52

	III	9.0344	173.7936	0.6276		0.03	0.27	1.00	99.79	99.66
II	IV	9.0077	17.7947	0.6221		0.03	0.26	1.00	99.81	99.69
	V	8.104	17.797	0.5748		0.09	0.63	0.99	99.65	99.57
	VI	8.7249	17.7976	0.5963		0.02	0.22	1.00	99.86	99.79
IV	I	8.4	26.5			3.66	2.43	0.76	84.06	79.67
	II	9.451	30.0289			0.93	1.44	0.92	94.40	91.15

In terms of usefulness for growth data analysis we cannot say that the Schnute Growth Model is the most appropriate one, there might be other several models which can give better results. The uncertainty of growth models arise here for which one single model can't be regarded as the best one. In this paper the main focus is to select the most appropriate results for the Schnute growth model fitted to cumulative basal area production and the mean diameter at breast height data, originated from the Bowmont Norway spruce thinning experiment, sample plot 3661. The results presented in this paper clearly show that Schnute growth model with three parameters $(b = 0)$ work more suitably than the other limiting cases of Schnute growth model. For selecting the best fit, statistical tools were used. Undoubtedly, statistical tools need some depth knowledge of mathematics but these can make the process of finding the most appropriate result even easier. Effort has also been made to introduce some new estimation techniques to estimate the model parameters. The new techniques need less computation and can be used to any growth data. Schnute model is not a fully new model in field but its applications and applicability chances are still in dark. The main purpose of this paper is to provide some help for minimizing the complexity of nonlinear Schnute growth model and help the researchers to explore its wide applicability in the forestry and also in other fields.

Conclusion

As though the concept of growth modelling is not a whole new thing but it has not caught general attention how it can be used effectively. If the models are not used in its proper perspective its development will be incomplete. For the beneficial utilization of growth models, step by step modification is very much necessary. The main contribution believed to be delivered by this paper is the thorough study of the diverse nonlinear Schnute growth model and examining its practical applicability in real life fields. Because of its diverse nature and chances of wide applicability it has been chosen to be the prime focus. The growth model in a very flexible way was applied to different growth data and the results were found to be pretty much effective. The results of this paper are thought to help the growth modellers, forest managers, researchers to know this non-linear Schnute growth model in a clearer context and also to have a vivid idea about its application in different fields. To investigate forest management alternatives or other future predictions, the growth model should provide all the necessary information for forest management or field wherever it's applied. Because good prediction can only be expected from a genuine growth model if users' inputs are also genuine. Errors in users supplied data could have a bigger effect on the overall variability of predictions than the contributions from the growth model [8]. Therefore, keeping all these in mind Schnute growth model was fitted to three sets of data which were collected from some reliable sources. A growth model should always be clearer, accurate, concise and appropriate and it should always be well timed but not out-dated. A growth model should be considered as a useful aid to investigate about future predictions and to provide more and better information of its applied field without threatening the perception and experience of the researcher.

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