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ESTABLISHMENT OF EVEN ORDERED A SUBSEQUENCE OF A FAREY SEQUENCE AS A TOPOLOGICAL SPACE

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Abstract

Farey sequence in [0, 1] is a sequence of real numbers generated using median properties. A subsequence of Farey sequence, $\widetilde{F_N}$ Farey N-subsequence has been established as a topological space and a Hausdorff space by appropriately defining basis and open sets. Also the axiom has been discussed with an illustration. A Lebesgue measure of Farey sequence of both odd and even order has been presented in this paper.

1. Introduction

The Farey Sequence usually referred to as the Farey series, maybe a chain of sequences during which every series includes rational numbers that move in estimate from zero to one [2].

In "Extraction of Cantor Middle $\left(\omega = \frac{2}{5}, \frac{3}{7}\right)$ from Non-Reducible Farey

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Subsequence" -Cantor ternary sets and some higher ordered Cantor sets have been extracted from Farey sequences of like order. There are many properties observed on the Farey sequence in "Farey to Cantor" [1]. In "Farey Triangle Graphs and Farey Triangle Matrices" [4] the terms of a Farey sequence have been considered as ordered pairs and a pattern of matrices and graphs have been identified.

It is mentioned in "https://ncatlab.org/nlab" that Cantor sets may be developed into topological space and Hausdorff space. Having identified the terms of F_N, F_{N+1} is written by writing the mediant of all the successive terms of F_N. With slight modification, sequence whose terms are Farey sequences has been establishedas various spaces namely topological space, Hausdorff space and T_1 space. A Hausdorff space is basically a topological space. To form a topology a nonempty set with basis elements should be defined clearly.

2. Preliminaries

Definition 1. Farey sequence [1] A Sequence of rational numbers $\frac{p}{q}$ with $(p, q) = 1$ in $[0, 1]$ and $q \leq n$ is called a Farey Sequence of order n, denoted by F_n.

Example 1.

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2} \frac{1}{1}\right\}$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

Definition 2. Farey *N*-subsequence [1] In a Farey sequence F_N the elements with denominators precisely N are classified as Farey N-subsequence and denoted by $\langle F'_N \rangle$.

$$\langle F'_N \rangle = \left\{\frac{u_i}{N}/0 \le u_i \le N,\ 0 \le i \le N\right\}$$

Example 2. The Farey N-Sequence of order 4 is

$$\langle F'_N \rangle = \left\{\frac{0}{1} = \frac{0}{4} < \frac{1}{4} < \frac{3}{4} < \frac{4}{4} = \frac{1}{1}\right\}$$

Definition 3. Non - Reducible Farey Sequence [1] A subset of the Farey sequence F'_N whose denominator not exceeding N is taken as Non – Reducible Farey Sequence. It is denoted by $\widetilde{F_N}$.

Example 3. The quaternary Non - Reducible Farey Sequence of order 4 is

$$\widetilde{F}_4 = \left\{\frac{0}{1} = \frac{0}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2} = \frac{2}{4}, \frac{2}{3}, \frac{3}{4}, \frac{4}{4} = \frac{1}{1}\right\}$$

Definition 4. Non-Reducible Farey N-Subsequence [1] For F_N, the element of the sequence with denominator N is taken as Non-Reducible Farey N-subsequence. It is denoted by $\widetilde{F}_N$.

Example 4. The Non- Reducible Farey N-Subsequence of order 4 is

$$\widetilde{F}_4 = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = \frac{1}{1}\right\}$$

Non-Reducible Farey N-subsequenceoforder $\widetilde{F}_4$ and its higher powers have been given below.

$$\widetilde{F}_{4^1} = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1\right\} \tag{2.1}$$

$$\widetilde{F}_{4^2} =$$

$$\left\{\frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16}, \frac{5}{16}, \frac{6}{16}, \frac{7}{16}, \frac{8}{16}, \frac{9}{16}, \frac{10}{16}, \frac{11}{16}, \frac{12}{16}, \frac{13}{16}, \frac{14}{16}, \frac{15}{16}, \frac{16}{16} = 1\right\} \tag{2.2}$$

$$\widetilde{F}_{4^3} = \left\{\frac{0}{64}, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \frac{4}{64}, \frac{5}{64}, \frac{6}{64}, \frac{7}{64}, \frac{8}{64}, \frac{9}{64}, \frac{10}{64}, \ldots, \frac{64}{64} = 1\right\} \tag{2.3}$$

3. A Subsequence of Farey Sequence-Topological Space

Theorem 1. *For any integer* $N \geq 3$ *a subsequence of Farey sequence, Farey N-subsequence denoted by* $\widetilde{F}_{N^k}$ *is a topological space.*

Proof. To define a topology on a set first a basis and hence open sets should be described clearly. Here the basis is defined as follows.

Consider $X = \left\{ \frac{i}{N^k}, \quad \begin{matrix} i = 0, 1, 2, 3, \ldots \\ k = 1, 2, 3, 4, \\ N = 3, 4, 5, \ldots \end{matrix} \right\}$

Take $B = \{\widetilde{F}_{N^k}, k = 1, 2, 3, \ldots\}$

Here every element of B is a sequence of real numbers.

Claim. B constitute a basis for X

Clearly $F = \cap \widetilde{F}_{N^k}, k = 1, 2, 3, \ldots \infty = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$

Case (i). If $x \in F$ then choose basis elements as any one of $\widetilde{F}_{n^k}, k \geq 2$

Case (ii). Suppose that x is not in F. X may be any one of the following forms.

$$X = \frac{i}{N^{k+1}}; N^k + 1 \leq \overset{0 \leq i \leq N^k - 1}{i} \leq j * N^k - 1/j = 1, 2, 3, \ldots, N \text{ for } k = 1, 2, 3, \ldots, \infty.$$

Choose basis elements as any one of $\widetilde{F}_{N^k}, k = 2, 3, \ldots, \infty$

Then clearly $B_i \cap B_j$ contains a basis element in which x is a members.

The open sets may taken as a sequence of union of members of B. Then for every elements in U there exists a member in B such that $x \in B \subseteq U$.

Illustration 1. Consider $X = \left\{ \frac{i}{N^k}, \quad \begin{matrix} i = 0, 1, 2, 3, \ldots \\ k = 1, 2, 3, 4, \\ N = 3, 4, 5, \ldots \end{matrix} \right\}$

$B = \{\widetilde{F}_{N^k}, k = 1, 2, 3, \ldots$

Take $N = 4$, $k = 1, 2, 3, \ldots$

$B = \{\widetilde{F}_{4^1}, \widetilde{F}_{4^2}, \widetilde{F}_{4^3}, \ldots\}$

Consider higher ordered Farey sequence

$$\widetilde{F}_{4^1} = \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = 1\right\}$$

$\widetilde{F}_{4^2} =$

$$\left\{\frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \frac{4}{16}, \frac{5}{16}, \frac{6}{16}, \frac{7}{16}, \frac{8}{16}, \frac{9}{16}, \frac{10}{16}, \frac{11}{16}, \frac{12}{16}, \frac{13}{16}, \frac{14}{16}, \frac{15}{16}, \frac{16}{16} = 1\right\}$$

$$\widetilde{F}_{4^3} = \left\{\frac{0}{64}, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \frac{4}{64}, \frac{5}{64}, \frac{6}{64}, \frac{7}{64}, \frac{8}{64}, \frac{9}{64}, \frac{10}{64}, \ldots, \frac{64}{64} = 1\right\}$$

$\vdots$

$\vdots$

$U = \widetilde{F}_{4^2} \cup \widetilde{F}_{4^3}$

Such that $x \in \dfrac{3}{4^2}$ and $x \in \widetilde{F}_{4^2} \subseteq U$

Let $\tau = \{\widetilde{F}_{4^1}, \widetilde{F}_{4^2}, \widetilde{F}_{4^3}, \ldots\}$

Consider $\tau_1 = \widetilde{F}_{4^1}$

$$= \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$$

and $\tau_1 = \widetilde{F}_{4^2}$

$$= \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \ldots, \frac{15}{16}, 1\right\}$$

Then

$$\tau_1 \cup \tau_2 = \widetilde{F}_{4^1} \cup \widetilde{F}_{4^2}$$

$$= \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\} \cup \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \ldots, \frac{15}{16}, 1\right\}$$

$$= \left\{0, \frac{1}{16}, \frac{2}{16}, \frac{3}{16}, \ldots, \frac{15}{16}, 1\right\} \in \tau$$

Therefore the union of the elements of a subset of τ is in τ.

Consider $\tau_1 = \widetilde{F}_{4^3}$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \ldots, \frac{63}{64}, 1\right\}$$

and $\tau_1 = \widetilde{F}_{4^4}$

$$= \left\{0, \frac{1}{256}, \frac{2}{256}, \frac{3}{256}, \ldots, \frac{255}{256}, 1\right\}$$

Then

$$\tau_1 \cup \tau_2 = \widetilde{F}_{4^3} \cup \widetilde{F}_{4^4}$$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \ldots, \frac{63}{64}, 1\right\} \cup \left\{0, \frac{1}{256}, \frac{2}{256}, \frac{3}{256}, \ldots, \frac{255}{256}, 1\right\}$$

$$= \left\{0, \frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \ldots, \frac{63}{64}, 1\right\} \in \tau$$

Therefore the intersection of the elements of any finite subcollection of τ is in τ. In well known that in a Hausdorff space every pair of elements is separated by open sets the following is the theorem of Farey N-subsequenceas Hausdorff space.

Theorem 2. *For set consisting of rational numbers of* $\frac{i}{N^k}$ *the form the basis defined in the above theorem forms a Hausdorff space.*

Proof. A Hausdroff space is in fact a topological space. To define a topology a basis should be describe in the basis for the topology is defined as above. Here an open set is taken in the form

$$\widetilde{W}_{N^k} = X - \bigcup_{j=1}^{k-1} \widetilde{F}_{N^j} \text{ where } \begin{matrix} k = 1, 2, 3, \ldots \\ N = 3, 4, 5, \ldots \end{matrix}$$

Consider the points $x_1 = \frac{i}{N^r}$ and $x_2 = \frac{i}{N^t}$, $r, t = 1, 2, 3, \ldots$ clearly $\frac{i}{N^r} \in \widetilde{F}_{N^r}$ and $\frac{i}{N^t} \in \widetilde{F}_{N^t}$ which are disjoint by their construction. Therefore any two disjoint points of X have disjoint neighbourhoods. Therefore X is a Hausdorff space.

Illustration 2. $\widetilde{W}_{N^k} = X - \bigcup_{j=1}^{k-1} \widetilde{F}_{N^j}$ where $\begin{matrix} k = 1, 2, 3, \ldots \\ N = 3, 4, 5, \ldots \end{matrix}$

Take $N = 4$, $k = 1, 2, 3, \ldots$

Consider $S = \widetilde{W}_{4^1}, \widetilde{W}_{4^2}, \widetilde{W}_{4^3}, \ldots$

$s_1 = \frac{61}{4^3}$ and $s_2 = \frac{251}{4^4}$ are distinct points of S.

Then there exist neighborhoods

$D_1 = \widetilde{W}_{4^3}$ and $D_2 = \widetilde{W}_{4^4}$ of s_1 and s_2 that are also disjoint.

Therefore the topological space S is called a Hausdorff space.

Corollary 1. *On the same construction above the topological space X also satisfies T_1 axioms.*

Illustration 3. $\widetilde{W}_{N^k} = X - \bigcup_{j=1}^{k-1} \widetilde{F}_{N^j}$ where $\begin{matrix} k = 1, 2, 3, \ldots \\ N = 3, 4, 5, \ldots \end{matrix}$

Take $N = 4$, $k = 1, 2, 3, \ldots$

Consider $T = \widetilde{W}_{4^1}, \widetilde{W}_{4^2}, \widetilde{W}_{4^3}, \ldots$

Given two points $q_1 = \frac{15}{4^2}$ and $q_2 = \frac{517}{4^5}$ of T

There exist an open set $I_1 = \widetilde{W}_{4^2}$ and $I_2 = \widetilde{W}_{4^5}$ of T

Therefore $q_1 \in \widetilde{W}_{4^2}$ and $q_1 \notin \widetilde{W}_{4^5}$

$\therefore q_2 \in \widetilde{W}_{4^5}$ and $q_2 \notin \widetilde{W}_{4^2}$.

Theorem 3. $F = \cup F_N$, F_N *is a Farey sequence is bounded by* 0 *and* 1. *The subsequence* $V = \cup F_{N^k}$ *of F it has convergent subsequence.*

Proof. Consider the Farey sequence F_N, where $N = 1, 2, 3, \ldots$ for all positive integers N, F_N is a bounded sequence and is bounded by 0 and 1.

The set F is defined above is also bounded by 0 and 1.

Now the subsequence of V namely $\left\{\frac{k}{N^k} / k = 1, 2, 3, \ldots\right\}$ it is a convergence sequence and converges to 0. This is because $\frac{1}{N^k} \to 0$ as $k \to \infty$ and for all positive integers N.

The Farey N subsequence of order 4 can be depicted in the graph as follows X axis = Farey N-subsequence

Y axis Y axis = Integer

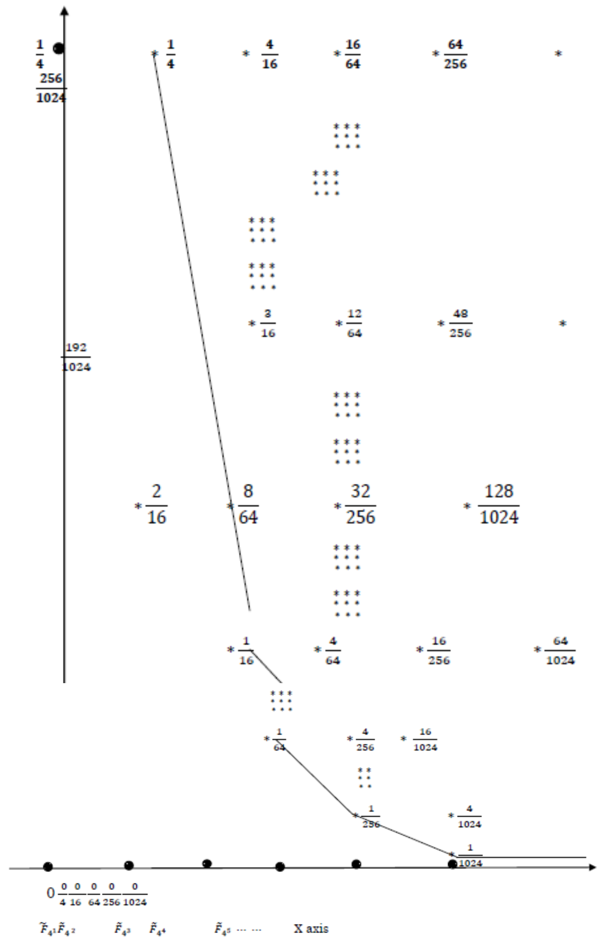

Figure 1.

From the above graph, it is clear that the curve resembles inverse exponential curve.

5. Conclusion

We have established the Farey N-Subsequence and also we have shown their topology space, Hausdorff space and space.

References

[1] A. Gnanam, P. Balamurugan and C. Dinesh, Farey Grids, IOSR Journal of Mathematics (IOSR -JM) 12(6) (2016), 70-73.

[2] A. Gnanam and C. Dinesh, Extraction of Cantor Middle $\left(\omega = \frac{2}{5}, \frac{3}{7}\right)$ from Non-Reducible Farey Subsequence, International Journal of Scientific Engineering and Research (IJSER) 4(2) (2016), 18-20.

[3] A. Gnanam and C. Dinesh, Modified and Non-Reducible Farey Matrix, International Journal of Scientific Research in Science, Engineering and Technology (IJSRSET) 2(2) (2016), 89-93.

[4] Ivan Niven, Herbert S. Zuckerman and Hugh L. Montgomery, An Introduction to the Theory of Numbers, John Wiley and Sons, Inc, 5th Edition, (2011).

[5] James R. Munkres, Topology, Pearson Education Limited, 2nd Edition, (2003).

[6] John E. Hutchinson, Fractals and Self Similarity, Australian National University 30 (1980), 1- 27.

[7] Md. Shaniful Islam Khan and Md. ShahidulIalam, An Exploration of the Generalized Cantor Set, International Journal of Scientific and Technology Research 2(7) (2013), 50-54.

[8] Shigeru Kanemitsu, TakakoKuzumaki and Masami Yoshimoto, Some sums involving Farey fractions II, Journal of the Mathematical Society of Japan 52(4) (2000), 915-947.

[9] S. Deepa and A. Gnanam, Extraction of Cantor One -Third Set from Stern Brocot Sequence, International Journal of Mathematics and Computer Science 15(4) (2020), 1149-1153.

[10] Sameen Ahmed Khan, Farey Sequences and resistor network Proc. Indian Acad. Sci. (Math. Sci.) 122(2) (2012), 153-162.

[11] Sanjoy Pratihar and Partha Bhowmickm, Farey Sequence and its augmentation for Applications to image analysis, International Journal of Applied Mathematics and Computer Science 27(3) (2017), 637-658.

[12] Vladimir Sukhoy and Alexander Stoytchev, Formulas and algorithms for the length of a Farey Sequence, Scientific reports 11 (2021), 1-18.