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THE PROPERTIES OF ONE PARAMETER POWER EXPONENTIAL MEAN AND ITS INVARIANTS

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Abstract

In this paper, studied and verified the properties, various kinds of Schur convexities of one parameter "Power exponential mean" and its invariants.

1. Introduction

The idea of Mathematical methods is presented and concentrated by Greek Mathematicians depending on extents and their significance in the fourth century A. D in the Pythagorean School. Later on a few creators contributed and built up this field considering applications to different parts of science and innovation. Lately, Lokesha et al. gotten the connection among

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arrangement and significant means [13]. Greek Means and other means[9], answer for an open issue by Rooin including means [4], presented and contemplated Gnan mean in two and n variables [3, 11], considered homogeneous capacities as an application got a few disparities including means [8, 10], initially examined Oscillatory means, Greeks formula for means, its speculations and a few inequality results were found in [6, 7, 12]. The great number of Schur convexity results and properties of means are examined by Nagaraja et al. ([14]-[24]) and creators of [1, 2, 5]. In [27], Yang proposed the Power exponential mean of structure $Z(a, b) = (a^a b^b)^{\frac{1}{a+b}}$.

Definition 1.1 [25, 27]**.** A mean is a mapping; $M : R_+^2 \to R_+$, which satisfy $Min(x, y)$ less than $M(x, y)$ less than $Max\,(x, y)$.

Properties of means it states that each mean is reflexive, that is,

$M(x_1, x_1) = x_1$ for all $x_1 > 0$.

Lemma 1.1 [2]. *The following are the Schur convexity conditions of the symmetric function* ϕ.

$$(y_1 - y_2)\left(\frac{\partial\varphi}{\partial y_1} - \frac{\partial\varphi}{\partial y_2}\right) \geq 0(\leq 0). \tag{1.1}$$

$$(\ln y_1 - \ln y_2)\left(y_1 \frac{\partial\varphi}{\partial y_1} - y_2 \frac{\partial\varphi}{\partial y_2}\right) \geq 0(\leq 0). \tag{1.2}$$

$$(y_1 - y_2)\left(y_1^2 \frac{\partial\varphi}{\partial x_1} - y_2^2 \frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0). \tag{1.3}$$

Definition 1.2. For all real values p and q, the parameter $\alpha > 0$, then one parameter "Power exponential mean" is defined as:

$$Z(p, q, \alpha) = \begin{cases} \left(p^{p^\alpha} q^{q^\alpha}\right)^{\frac{1}{p^\alpha + q^\alpha}} & \text{when } \alpha \neq 0 \\ \sqrt{pq} & \text{when } \alpha = 0 \\ \left(p^p q^q\right)^{\frac{1}{p+q}} & \text{when } \alpha = 1. \end{cases}$$

2. Properties of one Parameter "Power Exponential Mean"

Property 2.1. Symmetry, $Z(p, q; \alpha) = Z(q, p; \alpha)$.

Consider, $Z(p, q, \alpha) = (q^{q^\alpha} p^{p^\alpha})^{\frac{1}{p^\alpha + q^\alpha}} = (p^{p^\alpha} q^{q^\alpha})^{\frac{1}{q^\alpha + p^\alpha}} = Z(p, q, \alpha)$.

This shows that $Z(p, q, \alpha)$ is symmetric.

Property 2.2. Homogeneous, $Z(pt, qt; \alpha) = tZ(p, q, \alpha)$.

Consider, $Z(pt, qt, \alpha) = (pt^{pt^\alpha} qt^{qt^\alpha})^{\frac{1}{pt^\alpha + qt^\alpha}} = (pt^{p\alpha} qt^{q\alpha})^{\frac{1}{pt + qt}}$.

$$= (q^{q^\alpha} p^{p^\alpha})^{\frac{1}{q^\alpha + p^\alpha}} = (t^{q^\alpha + p^\alpha})^{\frac{1}{q^\alpha + p^\alpha}} = tZ(p, q, \alpha).$$

This proves that $Z(p, q; \alpha)$ is Homogeneous.

Proposition 2.1. *Verify that* $Z(p, q, \alpha) = (p^{p^\alpha} q^{q^\alpha})^{\frac{1}{p^\alpha + q^\alpha}}$ *is a mean.*

For $a < b$. Consider, $= (p^{p^\alpha + q^\alpha})^{\frac{1}{p^\alpha + q^\alpha}} - (p^{p^\alpha} q^{q^\alpha})^{\frac{1}{q^\alpha + p^\alpha}} = p^{p^\alpha} p^{q^\alpha} - p^{p^\alpha} p^{q^\alpha} = p^{p^\alpha}(p^{q^\alpha} - q^{q^\alpha})$. Therefore, $p - Z(p, q; \alpha) < 0$. Since $(P < q)$. This implies $p < Z(p, q; \alpha)$ and $Z(p, q; \alpha) - q = (p^{p^\alpha} + q^{q^\alpha})^{\frac{1}{p^\alpha + q^\alpha}} - q = (p^{p^\alpha} q^{q^\alpha})^{\frac{1}{q^\alpha + p^\alpha}} - (q^{q^\alpha + p^\alpha})^{\frac{1}{q^\alpha + p^\alpha}} = p^{p^\alpha} q^{q^\alpha} - q^{p^\alpha} q^{q^\alpha} = q^{q^\alpha}(p^{p^\alpha} - q^{p^\alpha})$. Therefore, $Z(p, q; \alpha) - q < 0$, since $(p < q)$. This implies $Z(p, q; \alpha) < q$. Thus, $Z(p, q; \alpha)$ is satisfies the property of mean.

Proposition 2.2. *Prove that* $Z(p, q; \alpha)$ *is monotonically increasing with respect to the parameter* α.

Consider, $Z(p,q;\alpha+1)-Z(p,q;\alpha)=(p^{p^{\alpha+1}}q^{q^{\alpha+1}})^{\frac{1}{q^{\alpha+1}+p^{\alpha+1}}}-(p^{p^{\alpha}}q^{q^{\alpha}})^{\frac{1}{q^{\alpha}+p^{\alpha}}}$

$$= \frac{1}{q^{\alpha+1}+p^{\alpha+1}}[q^{\alpha+1}\log q + p^{\alpha+1}\log p] - \frac{1}{q^{\alpha}+p^{\alpha}}[q^{\alpha}\log q + p^{\alpha}\log p]$$

$$= \frac{p^{\alpha}\log p(pq^{\alpha}-q^{\alpha+1}) + q^{\alpha}\log q(qp^{\alpha}-p^{\alpha+1})}{(p^{\alpha+1}+q^{\alpha+1})(p^{\alpha}+q^{\alpha})}$$

$$= p^{\alpha}q^{\alpha}\log p(p-q) - q^{\alpha}p^{\alpha}\log p(p-q) = p^{\alpha}q^{\alpha}(q-p)(\log q - \log p) > 0.$$

Since $p < q$. Therefore, $Z(p, q; \alpha+1) - Z(p, q; \alpha) > 0$.

Property 2.3. The one parameter "Power exponential mean" $Z(p, q; \alpha)$ is Schur convex if $\alpha \le \frac{2}{3}$ and otherwise Schur concave.

Let $Z = Z(p, q; \alpha) = (p^{p^{\alpha}} - q^{q^{\alpha}})^{\frac{1}{p^{\alpha}+q^{\alpha}}}$ taking log on both sides then $\log Z = \frac{q^{\alpha}\log q + p^{\alpha}\log p}{p^{\alpha}+q^{\alpha}}$. The derivative of Z with respect to p and q gives

$$\frac{\partial Z}{\partial p} = (Z)p^{\alpha-1}\left[\frac{p^{\alpha}+q^{\alpha}-\alpha q^{\alpha}(\log q - \log p)}{(p^{\alpha}+q^{\alpha})^2}\right] \tag{2.1}$$

and

$$\frac{\partial Z}{\partial p} = (Z)q^{\alpha-1}\left[\frac{p+q-\alpha p(\log p - \log q)}{(q^{\alpha}+p^{\alpha})^2}\right]. \tag{2.2}$$

The quantity,

$$(p-q)\left(\frac{\partial Z}{\partial p}-\frac{\partial Z}{\partial q}\right) = (p-q)(Z)\left[\frac{(q^{\alpha}+p^{\alpha})(p^{\alpha}q-q^{\alpha}p)+\alpha(p+q)(pq)^{\alpha}(\log p-\log q)}{pq(q^{\alpha}+p^{\alpha})^2}\right].$$

Put $p = e^{-y}$ and $q = e^{y}$. Thus, $(p-q)\left(\frac{\partial Z}{\partial p}-\frac{\partial Z}{\partial q}\right) = (8-12\alpha)y^2 + \left(-\frac{4\alpha^2}{3}+4\alpha^2-10\alpha+\frac{20}{3}\right)y^4 + \ldots$

$\Rightarrow [(8 - 12\alpha)y^2] > 0, \Rightarrow \alpha \le \frac{2}{3}, (+ve).$

Therefore, $Z(p, q; \alpha)$ is Schur convex $\Rightarrow \alpha > \frac{2}{3}, (-ve).$ Hence the proof.

Property 2.4. The one parameter Power exponential mean $Z(p, q; \alpha)$ Schur geometrically convex. From equations (2.1) *and* (2.2),

$$p\frac{\partial Z}{\partial p} - q\frac{\partial Z}{\partial q} = (Z)\left[\frac{p^{2\alpha} - q^{2\alpha} 2\alpha(pq)^{\alpha}(\log p - \log q)}{(p^{\alpha} + q^{\alpha})^2}\right].$$

Thus, $(\ln p - \ln q)\left(p\frac{\partial Z}{\partial p} - q\frac{\partial Z}{\partial q}\right) = (\ln p - \ln q)(Z)\left[\frac{p^{2\alpha} - q^{2\alpha} + 2\alpha(pq)^{\alpha}(\log p - \log q)}{(p^{\alpha} + q^{\alpha})^2}\right]$

$= (-ve)(-ve) = (+ve) > 0.$ Hence the proof.

Property 2.5. The one parameter "Power exponential mean" $Z(p, q; \alpha)$ is Schur harmonically convex. From equations (2.1) and (2.2),

$$\left[\frac{(q^{\alpha} + p^{\alpha})(p^{\alpha+1} - q^{\alpha+1}) - \alpha(p + q)(pq)^{\alpha}(\log p - \log q)}{(q^{\alpha} + p^{\alpha})^2}\right].$$

Thus, $p^2\frac{\partial Z}{\partial p} - q^2\frac{\partial Z}{\partial q} = (Z)(q - p)\left(p^2\frac{\partial Z}{\partial p} - q^2\frac{\partial Z}{\partial q}\right)$

$$= (q - p)(Z)\left[\frac{(q^{\alpha} + p^{\alpha})(p^{\alpha+1} - q^{\alpha+1}) - \alpha(p + q)(pq)^{\alpha}(\log p - \log q)}{(q^{\alpha} + p^{\alpha})^2}\right]$$

$= (-ve)(-ve) = (+ve) > 0.$

Hence the proof.

3. Properties of one Parameter Invariant Power Exponential Mean

Definition 3.1. For all real values p and q, the parameter $\alpha > 0$. The one parameter invariant Power exponential mean is defined as*:*

$$Z^i(p, q; \alpha) = \begin{cases} (p^{q^{\alpha}} q^{p^{\alpha}})^{\frac{1}{p^{\alpha}+q^{\alpha}}} & \text{when } \alpha \neq 0 \\ \sqrt{pq} & \text{when } \alpha = 0 \\ (p^q q^p)^{\frac{1}{p+q}} & \text{when } \alpha = 1. \end{cases}$$

Property 3.1. Symmetry. To prove $Z^i(p, q; \alpha) = Z^i(q, p; \alpha)$.

Consider, $Z^i(q, p; \alpha) = (q^{p^{\alpha}} q^{q^{\alpha}})^{\frac{1}{p^{\alpha}+q^{\alpha}}} = (p^{p^{\alpha}} q^{q^{\alpha}})^{\frac{1}{p^{\alpha}+q^{\alpha}}} = Z^i(p, q; \alpha)$

Property 3.2. Homogeneous. *To prove* $Z^i(pt, qt; \alpha) = tZ^i(p, q; \alpha)$.

Consider, $Z^i(pt, qt, \alpha) = ((pt)^{qt^{\alpha}} (qt)^{pt^{\alpha}})^{\frac{1}{(pt)^{\alpha}+(qt)^{\alpha}}}$

$$= (p^{p^{\alpha}} t^{q^{\alpha}} q^{q^{\alpha}} t^{p^{\alpha}})^{\frac{1}{p^{\alpha}+q^{\alpha}}} = (q^{p^{\alpha}} p^{q^{\alpha}})^{\frac{1}{p^{\alpha}+q^{\alpha}}} (t^{(p^{\alpha}+q^{\alpha})})^{\frac{1}{(p^{\alpha}+q^{\alpha})}} = tZ^i(p, q; \alpha).$$

Proposition 3.1. *Verify that* $Z^i(p, q; \alpha)$ *is a mean.*

For $p < q$ Consider,

$$p - z^i(p, q; \alpha) = p - (q^{p^{\alpha}} p^{q^{\alpha}})^{\frac{1}{q^{\alpha}+p^{\alpha}}} = (p^{p^{\alpha}+q^{\alpha}})^{\frac{1}{p^{\alpha}+q^{\alpha}}} - (q^{p^{\alpha}} p^{q^{\alpha}})^{\frac{1}{p^{\alpha}+q^{\alpha}}}$$

$$= p^{p^{\alpha}} p^{q^{\alpha}} - p^{q^{\alpha}} q^{p^{\alpha}} = p^{q^{\alpha}} (p^{p^{\alpha}} - q^{q^{\alpha}}).$$

Therefore, $p - Z^i(p, q; \alpha) < 0$, Since $p < q$, implies $p < Z^i(p, q, \alpha)$ and

$$Z^i(p, q, \alpha) - q = (q^{p^{\alpha}} p^{q^{\alpha}})^{\frac{1}{q^{\alpha}+p^{\alpha}}} - q = (q^{p^{\alpha}} p^{q^{\alpha}})^{\frac{1}{q^{\alpha}+p^{\alpha}}} - (q^{p^{\alpha}+q^{\alpha}})^{\frac{1}{q^{\alpha}+p^{\alpha}}}$$

$$= p^{q^{\alpha}} q^{p^{\alpha}} - q^{p^{\alpha}} q^{q^{\alpha}} = q^{q^{\alpha}} (q^{p^{\alpha}} - q^{p^{\alpha}}).$$

Therefore, $Z^i(p, q; \alpha) - q < 0$, since $(p < q)$. This implies $Z^i(p, q; \alpha) < q$. Thus $Z^i(p, q; \alpha)$ is a mean.

Proposition 3.2. *Prove that* $Z^i(p, q; \alpha)$ *is monotonically decreasing with respect to the parameter* α.

Consider, $Z^i(p,q;\alpha+1)-Z^i(p,q;\alpha)=(q^{p^{\alpha+1}}p^{q^{\alpha+1}})^{\frac{1}{q^{\alpha+1}p^{\alpha+1}}}-(q^{p^\alpha}p^{q^\alpha})^{\frac{1}{q^\alpha+p^\alpha}}$

$$=\frac{1}{q^{\alpha+1}+p^{\alpha+1}}[p^{\alpha+1}\log q+q^{\alpha+1}\log p]-\frac{1}{q^\alpha+p^\alpha}[p^\alpha\log q+q^\alpha\log p]$$

$=\dfrac{q^\alpha\log(qp^\alpha-p^{\alpha+1})-p^\alpha\log(q^{\alpha+1}-pq^\alpha)}{(q^{\alpha+1}+p^{\alpha+1})(q^\alpha+p^\alpha)}=q^\alpha p^\alpha(q-p)(\log p-\log q)<0.$ Since, $p<q$. Therefore $Z^i(p,q;\alpha+1)-Z^i(p,q;\alpha)<0$.

Property 3.3. The one parameter invariant Power exponential mean is Schur convex.

Let, $Z^i(p,q;\alpha)=(p^{q^\alpha}q^{p^\alpha})^{\frac{1}{p^\alpha+q^\alpha}}$ take log on both sides

$$\log Z^i=\frac{p^\alpha\log q+q^\alpha\log p}{q^\alpha+p^\alpha}$$

$$\frac{\partial Z^i}{\partial p}=(Z^i)p^\alpha\left[\frac{p^\alpha+q^\alpha+\alpha p^\alpha(\log q-\log p)}{p(q^\alpha+p^\alpha)^2}\right] \tag{3.1}$$

and

$$\frac{\partial Z^i}{\partial q}=(Z^i)p^\alpha\left[\frac{p^\alpha+q^\alpha-\alpha q^\alpha(\log q-\log p)}{q(q^\alpha+p^\alpha)^\alpha}\right]. \tag{3.2}$$

Thus, $(p-q)\left(\dfrac{\partial Z^i}{\partial p}-\dfrac{\partial Z^i}{\partial q}\right)=(p-q)(Z^i)$

$$\left[\frac{(q^{2\alpha+1}-p^{2\alpha+1})+\alpha(q+p)(pq)^\alpha(\log q-\log p)+(q-p)}{pq(q^\alpha+p^\alpha)^2}\right].$$

Since, $p-q<0$, implies $(p-q)\left(\dfrac{\partial Z^i}{\partial p}-\dfrac{\partial Z^i}{\partial q}\right)<0$. Hence the proof.

Property 3.4. The one parameter invariant Power exponential mean $Z^i(p,q;\alpha)$ Schur geometrically concave from equations (3.1) and (3.2), on simplification leads to;

$$\left(p\frac{\partial Z^i}{\partial p} - q\frac{\partial Z^i}{\partial p}\right) = (Z^i)\left[\frac{(q^{2\alpha} - p^{2\alpha}) - 2\alpha(pq)^{\alpha}(\log p - \log q)}{(p^{\alpha} + q^{\alpha})^2}\right].$$

Thus,

$$(\ln p - \ln q)\left(p\frac{\partial Z^i}{\partial p} - q\frac{\partial Z^i}{\partial p}\right) = (Z^i)\left[\frac{(q^{2\alpha} - p^{2\alpha}) + 2\alpha(pq)^{\alpha}(\log p - \log q)}{(p^{\alpha} + q^{\alpha})^2}\right]$$

$$= (-ve)(+ve) = (-ve) < 0.$$

Hence the proof.

Property 3.5. The one parameter invariant Power exponential mean $Z^i(p, q; \alpha)$ is Schur harmonically convex if $\alpha \leq \frac{1}{2}$ and otherwise Schur harmonically concave.

From equations (3.1) and (3.2), $\left(p^2\frac{\partial Z^i}{\partial p} - q^2\frac{\partial Z^i}{\partial p}\right) = (Z^i)$

$$\left[\frac{-(q^{\alpha} - p^{\alpha})(qp^{\alpha} - pq^{\alpha}) + \alpha(q + p)(pq)^{\alpha}(\log q - \log p)}{(q^{\alpha} + p^{\alpha})^2}\right].$$

Put $p = e^{-y}$ and $q = e^{y}$.

Thus,

$$(p - q)\left(p^2\frac{\partial Z^i}{\partial p} - q^2\frac{\partial Z^i}{\partial q}\right) = (8 - 16\alpha)y^2 - \frac{8}{3}(2\alpha^3 - 3\alpha^2 + 4\alpha - 1)y^4 + \ldots$$

$$\Rightarrow [(8 - 16\alpha)y^2] > 0. \Rightarrow \alpha \leq \frac{1}{2}, (+ve).$$

Therefore, $Z^i(p, q; \alpha)$ is Schur harmonically convex $\Rightarrow \alpha > \frac{1}{2}, (-ve)$. Hence the proof.

4. Application to Mean Inequality and Best Possible Values

For positive real numbers p and q, then

$$Z^i(p,\, q) = (p^q,\, q^p)^{\frac{1}{p+q}},\ Z(p,\, q) = (p^q,\, q^p)^{\frac{1}{p+q}},$$

$$G(p,\, q) = \sqrt{pq},\ A(p,\, q) = \frac{p+q}{2},\ H_e(p,\, q) = \frac{p+\sqrt{pq}+q}{3},$$

$$H(p,\, q) = \frac{2pq}{p+q},\ C^i(p,\, q) = \frac{pq(p+q)}{p^2+q^2},\ C(p,\, q) = \frac{p^2+q^2}{p+q},\ Z(p,\, q,\, \alpha)$$

$$= (p^{p^\alpha},\, q^{q^\alpha})^{\frac{1}{p^\alpha+q^\alpha}} Z^i(p,\, q;\, \alpha) = (p^{q^\alpha},\, q^{a^\alpha})^{\frac{1}{p^\alpha+q^\alpha}}$$

are respectively called invariant Power exponential mean, Power exponential mean, Geometric, Arithmetic, Heron, Harmonic, invariant contra harmonic, Contra harmonic, one parameter power exponential and its invariant means. Put $p = 1$ and $q = x$, then by using Taylor's series expansion,

$$Z(1,\, x) = (x^x)^{\frac{1}{1+x}} = 1 + \frac{(x-1)}{2} + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \ldots, \tag{4.1}$$

$$Z^i(1,\, x) = (x)^{\frac{1}{1+x}} = 1 + \frac{(x-1)}{2} - \frac{1}{8}(x-1)^2 + \frac{3}{16}(x-1)^3 + \ldots, \tag{4.2}$$

$$A(1,\, x) = \frac{1+x}{2} = 1 + \frac{x-1}{2} \ldots \tag{4.3}$$

$$G(1,\, x) = \sqrt{x} = 1 + \frac{(x-1)}{2} - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 + \ldots \tag{4.4}$$

$$H(1,\, x) = 1 + \frac{(x-1)}{2} - \frac{1}{4}(x-1)^2 + \frac{1}{8}(x-1)^3 + \ldots, \tag{4.5}$$

$$H_e(1,\, x) = \frac{1+\sqrt{x}+1}{2} = 1 + \frac{(x-1)}{2} - \frac{1}{24}(x-1)^2 + \frac{1}{48}(x-1)^3 + \ldots, \tag{4.6}$$

$$C(1,\, x) = \frac{1+x^2}{1+2} = 1 + \frac{(x-1)}{2} + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^3 + \ldots, \tag{4.7}$$

$$C^i(1,\, x) = \frac{x(1+x)}{1+x^2} = 1 + \frac{(x-1)}{2} - \frac{1}{2}(x-1)^2 + \frac{1}{4}(x-1)^3 + \ldots, \tag{4.8}$$

$$Z(1, x; \alpha) = (x^{x^\alpha})^{\frac{1}{1+x^\alpha}} = 1 + \frac{(x-1)}{2} + \frac{(2\alpha-1)}{8}(x-1)^2 + \frac{(1-2\alpha)}{16}(x-1)^3 + \ldots, \tag{4.9}$$

$$Z^i(1, x; \alpha) = (x)^{\frac{1}{1+x^\alpha}} = 1 + \frac{(x-1)}{2} - \frac{(2\alpha+1)}{8}(x-1)^2 + \frac{(2\alpha+1)}{16}(x-1)^3 + \dots \quad (4.10)$$

From equations (4.1) to (4.8), it was seen that;

$$C^i(p, q) \le Z^i(p, q) \le H(p, q) \le G(p, q) \le H_e(p, q) \le A(p, q) \le Z(p, q) \le C(p, q).$$

From equations (4.9) and (4.10) the following conclusions are drawn.

(i) $G(p, q) \le Z(p, q; \alpha) \le C(p, q)$ (ii) $C^i(p, q) \le Z^i(p, q; \alpha) \le G(p, q)$.

From equations (4.1) to (4.10), it was observed that:

(1) $Z(p, q; 0) = G(p, q) = Z^i(p, q; 0)$ (2) $Z\left(p, q; \frac{1}{3}\right) = H_e(p, q)$ (3) $Z\left(p, q; \frac{1}{2}\right) = A(p, q)$ (4) $Z(p, q; 1) = Z(p, q)$ (5) $Z\left(p, q; \frac{3}{2}\right) = C(p, q)$ (6) $Z^i\left(p, q; \frac{1}{2}\right) = H(p, q)$ (7) $Z^i(p, q; 1) = Z^i(p, q)$ (8) $Z^i\left(p, q; \frac{3}{2}\right) = C^i(p, q)$.

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