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A STUDY ON $^{**}g\alpha$-OPEN AND $^{**}g\alpha$-CLOSED MAPS IN TOPOLOGICAL SPACE

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Abstract

This paper focused on various results obtained from $^{**}g\alpha$-continuous functions in topological spaces. $^{**}g\alpha$-closed maps and $^{**}g\alpha$-open maps is introduced in this paper using $^{**}g\alpha$-closed set and $^{**}g\alpha$-open set. This study includes some properties of $^{**}g\alpha$-closed maps and $^{**}g\alpha$-open maps along with its results.

1. Introduction

In 1963, N. Levin introduced semi open sets and semi continuity in topological space [8]. In 1991, a week form of continuous function called generalized continuous maps was introduced and studied by K. Balachandran, Sundaram and H. Maki [1]. Y. Gnanambal and K. Balachandran [7] introduced gpr-continuous function and studied some of its properties in the topological spaces. M. Vigneshwaran and R. Devi [10] studied by introducing $^{*}g\alpha$-continuous function. A. Singaravelan [9] introduced $^{**}g\alpha$-continuous function in the topological space.

In this paper, discussed characteristic of $^{**}g\alpha$-continuous functions in

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topological space and newly introduced and studied about $^{**}g\alpha$ -closed ($^{**}g\alpha$ -open) maps in topological spaces and some of its results.

2. Preliminaries

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a topological space (X, τ) is called

(i) a generalized α-closed set (briefly $g\alpha$-closed) [5] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ).

(ii) a gpr-closed [6] set if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

(iii) a $^{*}g\alpha$ -closed set [10] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha$ -open in (X, τ).

(iv) a $^{**}g\alpha$ -closed set [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $^{*}g\alpha$ -open in (X, τ).

Definition 2.2. A function $f : (X, \tau) \to (Y, \sigma)$ is called

(i) a g-continuous[1] if $f^{-1}(V)$ is a g-closed set of (X, τ) for every closed set V of (Y, σ).

(ii) a gpr-continuous [7] if $f^{-1}(V)$ is a gpr-closed set of (X, τ) for every closed set V of (Y, σ).

(iii) a $^{*}g\alpha$ -continuous [10] if $f^{-1}(V)$ is a $^{*}g\alpha$ -closed set of (X, τ) for every closed set V of (Y, σ).

(iv) a $^{**}g\alpha$ -continuous [9] if $f^{-1}(V)$ is a $^{**}g\alpha$ -closed set of (X, τ) for every closed set V of (Y, σ).

(v) a $^{**}g\alpha$ -irresolute [9] if $f^{-1}(V)$ is a $^{**}g\alpha$ -closed set of (X, τ) for every $^{**}g\alpha$ -closed set V of (Y, σ).

3. Characteristics of $^{**}g\alpha$ -Continuous Functions

Definition 3.01. Let D be a subset of a space (Z, τ).

(i) The set $\cap\{F \subset Z; D \subseteq F, F$ is $^{**}g\alpha$ -closed$\}$ is called the $^{**}g\alpha$ -closure of D and is denoted by $^{**}g\alpha - cl(D)$.

(ii) The set $\cup\{F \subset X; F \subseteq D, F$ is $^{**}g\alpha$ -open$\}$ is called the $^{**}g\alpha$ -interior of D and is denoted by $^{**}g\alpha$ - $\text{int}(D)$.

Theorem 3.02. *Let $h : (Z, \tau) \to (W, \eta)$ be a function. Then the following conditions are equivalent.*

(i) *h is $^{**}g\alpha$ -continuous*

(ii) *The inverse image of every open set in (W, η) is $^{**}g\alpha$ -open in (Z, τ).*

Proof. (i) → (ii) Let G is open subset of (W, η). Then $(W - G)$ is closed in (W, η). Since h is $^{**}g\alpha$ -continuous, $h^{-1}(W - G) = Z - h^{-1}(G)$ is $^{**}g\alpha$ -closed in (Z, τ). Hence $h^{-1}(G)$ is $^{**}g\alpha$ -open in (Z, τ).

(ii) → (i) Let V be a closed subset of (W, η), then $(W - V)$ is open in (W, η) hence by hypothesis (ii) $h^{-1}(W - V) = Z - h^{-1}(V)$ is $^{**}g\alpha$ -open in (Z, τ), hence $h^{-1}(V)$ is $^{**}g\alpha$ -closed in (Z, τ). Therefore, h is $^{**}g\alpha$ -continuous.

Theorem 3.03. *Let $h : (Z, \tau) \to (W, \eta)$ be a function. Then the following conditions are equivalent.*

(1) *For all $z \in Z$ and every open set M containing $h(z)$ there exists a $^{**}g\alpha$ -open set N containing z such that $h(N) \subset M$.*

(2) *$h(^{**}g\alpha - Cl(D)) \subset Cl(h(D))$ for every subset D of (Z, τ).*

Proof. (1) → (2) Let $z \in h(^{**}g\alpha - Cl(D))$ then there exists a $t \in {}^{**}g\alpha - Cl(D)$ such that $z = h(t)$. We claim that $z \in Cl(h(D))$ and let N be any open neighborhood of z. Since $t \in {}^{**}g\alpha - Cl(D)$ there exists an ${}^{**}g\alpha$ open set M such that $t \in M$ and $M \cap D \neq \phi$. Therefore $z = h(t) \in Cl(h(D))$. Hence $h({}^{**}g\alpha - Cl(D)) \subset Cl(h(D))$.

(2) → (1) Let $t \in Z$ and N be any open set containing $h(t)$. Let $D = h^{-1}(W - N)$, since $h({}^{**}g\alpha - Cl(D)) \subset Cl(h(D)) \subset (W - N)$, ${}^{**}g\alpha - Cl(D)$ $\subset h^{-1}(W - N) = D$. Hence ${}^{**}g\alpha - Cl(D) = D$. Since $h(t) \in N$ implies $t \in h^{-1}(N)$ implies $t \notin D$ implies $t \notin {}^{**}g\alpha - Cl(D)$. Thus there exists an open set M containing t such that $M \cap D = \phi$ implies $h(M) \cap h(D) = \phi$. Therefore $h(M) \subset N$.

Theorem 3.04. *If* $h : (W, \tau) \to (Z, \gamma)$ *is continuous function, then* $h({}^{**}g\alpha - Cl(S)) \subset Cl(h(S))$ *for every subset S of* (W, τ).

Proof. Given $S \subset h^{-1}(h(S))$, we have $S \subset h^{-1}(Cl(h(S)))$ now $Cl(h(A))$ is closed set in (Z, γ) and hence $h^{-1}(Cl(h(S)))$ is a ${}^{**}g\alpha$ -closed set containing S. Consequently ${}^{**}g\alpha - Cl(S) \subset h^{-1}(Cl(h(A)))$, therefore $h({}^{**}g\alpha - Cl(S)) \subset$ $h^{-1}(Cl(h(S))) \subset Cl(h(S))$ implies $h({}^{**}g\alpha - Cl(S)) \subset Cl(h(A))$.

Theorem 3.05. *Let* $h : (W, \tau) \to (Z, \gamma)$ *be a function from a topological space* (W, τ) *into topological space* (Z, γ), *then the following conditions are equivalent*

(1) *For every subset G of* (W, τ), $h({}^{**}g\alpha - Cl(G)) \subset Cl(h(G))$.

(2) *For each subset Q of* Y, ${}^{**}g\alpha - Cl(G) \subset h^{-1}(Cl(Q))$.

Proof. (1) → (2) Suppose that (1) holds and let Q be any subset of (Z, γ), replacing G by $h^{-1}(Q)$ we get from (2) $h({}^{**}g\alpha - Cl(h(h^{-1}(Q))) \subset Cl(h(Q))$. Hence ${}^{**}g\alpha - Cl(G) \subset h^{-1}(Cl(Q))$.

(2) → (1) Suppose that (ii) holds. Let $Q = h(G)$, where G is a subset of (W, τ). then we get from (ii) ${}^{**}g\alpha - Cl(G) \subset^{**} g\alpha - Cl(h(h^{-1}(G))) \subset$

$h^{-1}(Cl(h(g)))$. Therefore $h(^{**}g\alpha - Cl(G)) \subset (Cl(G))$.

Theorem 3.06. *Let* $h : (W, \tau) \to (Z, \gamma)$ *be a* $^{**}g\alpha$ *-continuous map and let* J *be a* $^{**}g\alpha$ *-closed subset of* (W, τ). *Then the restriction* $h_J : (J, \tau_J) \to (Z, \gamma)$ *is also* $^{**}g\alpha$ *-continuous.*

Proof. Let N be any closed set in (Z, γ). Since f is $^{**}g\alpha$ -continuous, $h^{-1}(N)$ is $^{**}g\alpha$ -closed in (W, τ). Let $h^{-1}(N) \cap M = M_1$, then M_1 is $^{**}g\alpha$ -closed in (W, τ) by (if G is an $^{**}g\alpha$ -closed set and N is closed set. Hence $G \cap N$ is an $^{**}g\alpha$ -closed set). Since $(h_J)^{-1}(N) = h^{-1}(N) \cap J = J_1$, we need to show that J_1 is $^{**}g\alpha$ -closed in (J, τ_J). Let U be any $^{**}g\alpha$ -open set of (J, τ_J). Such that $J_1 \subseteq U$. Since U is $^{**}g\alpha$ -open set of (J, τ_J), $U = K \cap J$ for some $^{**}g\alpha$ -open set in (W, τ) by (if $G \in {}^{**}g\alpha - O(W_O)$, then $G = L \cap W_O$ for some $L \in {}^{**}g\alpha - O(W)$, where (W, τ) is a topological space and W_O is a sub space of (W, τ). Now $J_1 \subseteq K \cap J$ and so $J_1 \subseteq K$. Since J_1 is $^{**}g\alpha$ -closed in (W, τ). $Cl(J_1) \subseteq K$, we have $Cl_{J1}(J_1) = Cl(J_1) \cap J$ $\subseteq K \cap J = U$ and therefore J_1 is $^{**}g\alpha$ -closed in (J, τ_J) and hence h_J is $^{**}g\alpha$ -continuous.

4. $^{**}g\alpha$ -Closed Maps and $^{**}g\alpha$ -Open Maps in Topological Space

Definition 4.01. A map $k : (W, \gamma) \to (Z, \mu)$ is said to be $^{**}g\alpha$ -closed if the image of every closed set in (W, γ), $^{**}g\alpha$ -closed in (Z, μ).

Definition 4.02. A map $k : (W, \gamma) \to (Z, \mu)$ is said to be $^{**}g\alpha$ -open if the image of every open set in (W, γ), $^{**}g\alpha$ -open in (Z, μ).

Theorem 4.03. *Every closed map is a* $^{**}g\alpha$ *-closed map.*

Proof. Let $k : (W, \gamma) \to (Z, \mu)$ be closed map and N be a closed set in

(W, γ), then $k(N)$ is closed, every closed set is $^{**}g\alpha$-closed hence $k(N)^{**}g\alpha$-closed in (Z, μ), thus k is $^{**}g\alpha$-closed. The reverse implication of the above theorem need not be true from the following example.

Example 4.04. Let $W = \{l, m, n\} = Z$ with topologies $\gamma = \{W, \tau, \{l\}, \{m, n\}\}$ and $\mu = \{Z, \tau, \{m\}, \{l, m\}\}$, define $k : (W, \gamma) \to (Z, \mu)$ by $k(l) = l$, $k(m) = m$, $k(n) = n$.

$^{**}g\alpha$-closed sets are $W, \tau, \{n\}, \{l, n\}, \{m, n\}$. Then k is $^{**}g\alpha$-closed map but not closed as the image of the closed set $\{m, n\}$ in (W, γ) is $\{m, n\}$ is not closed set in (Z, μ).

Theorem 4.05. *A map $k : (W, \gamma) \to (Z, \mu)$ is $^{**}g\alpha$-closed map $\Leftrightarrow$ for each subset G of (Z, μ) and for each open set P containing $k^{-1}(G)$ there is a $^{**}g\alpha$-open set Q of (Z, μ) such that $G \subseteq Q$ and $k^{-1}(Q) \subseteq P$.*

Proof. Suppose k is $^{**}g\alpha$-closed. Let G be a subset of (Z, μ) and P be an open set of W such that $k^{-1}(G) \subseteq P$ then $Q = Z - k(W - P)$ is a $^{**}g\alpha$-open set containing G such that $k^{-1}(G) \subseteq P$. Conversely, suppose that L is a closed of W. Then $k^{-1}(Z - k(L)) \subseteq W - L$ and $W - L$ is open by hypothesis, there is a $^{**}g\alpha$-open set Q of (Z, μ) such that $Z - k(L) \subseteq Q$ and $k^{-1}(Q) \subseteq W - L$. Therefore $L \subseteq W - k^{-1}(Q)$ hence $Z - Q \subseteq k(L \subseteq k$ $(W - k^{-1}(Q)) \subseteq Z - Q$ which implies $k(L) = Z - Q$. Since $Z - Q$ is $^{**}g\alpha$-closed, $k(L)$ is $^{**}g\alpha$-closed and thus k is $^{**}g\alpha$-closed map.

Theorem 4.06. *A map $k : (W, \gamma) \to (Z, \mu)$ is a continuous, $^{**}g\alpha$-closed map from a normal space W onto space Z, then Z is normal.*

Proof. Let K and L are disjoint closed sets of Z, then $k^{-1}(k)$ and $k^{-1}(l)$ are disjoint closed sets of W. Since W is normal there are disjoint open sets U, V in W such that $k^{-1}(K) \subseteq U$ and $k^{-1}(L) \subseteq V$. Since $^{**}g\alpha$-closed by previous theorem there are open sets G and H in Z, such that $K \subseteq G$, $L \subseteq H$ implies $k^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$, since U, V are disjoint $\text{int}(G)$ and $\text{int}(H)$ are disjoint open sets. Since G is $^{**}g\alpha$-open, K is closed and $K \subseteq G$, $K \subseteq \text{int}(G)$, similarly $L \subseteq \text{int}(H)$.

Theorem 4.07. *A map* $k : (W, \gamma) \to (Z, \mu)$ *is an open, continuous,* $^{**}g\alpha$*-closed surjection where W is regular, then Z is regular.*

Proof. Let U be an open set containing a point of W such that $k(m) = p$, since W is regular and k is continuous, there is an open set V, such that $m \in V \subseteq Cl(V) \subseteq k^{-1}(V)$. Here $p \in k(V) \subseteq k(Cl(V)) \subseteq U$. Since k is $^{**}g\alpha$-closed, $k(Cl(V))$ is $^{**}g\alpha$-closed set contained in the open set U. $Cl(k(Cl(V))) \subseteq U$ and hence $p \in k(V) \subseteq Cl(k(V)) \subseteq U$ and $k(V)$ is open, since k is open. Hence Z is regular.

Theorem 4.08. *A map* $k : (W, \gamma) \to (Z, \mu)$ *is continuous, and* $^{**}g\alpha$*-closed and M is a* $^{**}g\alpha$*-closed set of W, then* $k(M)$ *is* $^{**}g\alpha$*-closed.*

Proof. Let $k(M) \subseteq N$ where N is an open set of Z. Since k is continuous, $k^{-1}(N)$ is an open set containing M. Hence $Cl(M) \subseteq k^{-1}(N)$ as M is a $^{**}g\alpha$-closed set. Since k is $^{**}g\alpha$-closed $k(Cl(M))$ is a $^{**}g\alpha$-closed set contained in the open set N, which implies that $Cl(k(Cl(M)))(N)$ and hence $Cl(k(M)) \subseteq N$ so $k(M)$ is a $^{**}g\alpha$-closed.

Theorem 4.09. *A map* $k : (W, \gamma) \to (Z, \mu)$ *is continuous, and* $^{**}g\alpha$*-closed and V is a* $^{**}g\alpha$*-closed set of W, then* $f_V : V \to (Z, \mu)$ *is continuous and* $^{**}g\alpha$*-closed.*

Proof. Let E be closed set of V, then E is a $^{**}g\alpha$-closed set of Z from the above theorem, it follows that $k_V(E) = k(E)$ is $^{**}g\alpha$-closed set of Z. Hence k_V is $^{**}g\alpha$-closed also k_V is continuous.

Theorem 4.10. *A map* $k : (W, \gamma) \to (Z, \mu)$ *is* $^{**}g\alpha$*-closed and* $G = k^{-1}(B)$ *for some closed set B of Z, then* $f_G : G \to Z$ *is a* $^{**}g\alpha$*-closed.*

Proof. Let F be a closed set in G, then there is a closed set H in W. Such that $F = G \cap H$ then $k_G(F) = k(G \cap H) = k(H) \cap k(G) = k(H) \cap B$. Since k is a $^{**}g\alpha$-closed, $k(H)$ is $^{**}g\alpha$-closed in Z, $k(H) \cap B$ is $^{**}g\alpha$-closed in Z. Since the intersection of a $^{**}g\alpha$-closed set and a closed set is $^{**}g\alpha$-closed set. Hence k_G is $^{**}g\alpha$-closed.

Theorem 4.11. *Every open map is a* $^{**}g\alpha$*-open map.*

Proof. Let $k : (W, \gamma) \to (Z, \mu)$ be open map and G be a open set in W, then $k(G)$ is open, every open set is $^{**}g\alpha$-closed hence $k(G)^{**}g\alpha$-open in Z, thus k is $^{**}g\alpha$-open.

The converse of the above theorem need not be true from the following example.

Example 4.12. Let $W = \{l, m, n\} = Z$ with topologies $\gamma = \{W, \tau, \{l, m\}\}$ and $\mu = \{Z, \tau, \{l, n\}\}$, define $k : (W, \gamma) \to (Z, \mu)$ by $k(l) = l$, $k(m) = m$, $k(n) = n$.

$^{**}g\alpha$-open sets are $W, \phi, \{l\}, \{m\}, \{l, m\}$. Then k is $^{**}g\alpha$-open map because the image of $\{l, m\}$ in W is $\{l, m\}$ $^{**}g\alpha$-open in Z, but not open map because the image of $\{l, m\}$ in W is not in open in Z.

Theorem 4.13. *If* $k : (W, \gamma) \to (Z, \mu)$ *is closed map and* $j : (Z, \mu) \to (H, \eta)$ *is* $^{**}g\alpha$*-closed map, then the composition* $j \circ k : (W, \gamma) \to (H, \eta)$ *is* $^{**}g\alpha$*-closed map.*

Proof. Let M be any closed set in (W, γ). Since f is closed map, $k(M)$ is closed set in (Z, μ). Since j is $^{**}g\alpha$-closed map, $j(k(N))$ is $^{**}g\alpha$-closed set in (H, η). That is $j \circ k(N) = j(k(N))$ is $^{**}g\alpha$-closed and hence $j \circ k$ is $^{**}g\alpha$-closed map.

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