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ROMAN DOMINATION NUMBER OF DOUBLE FAN GRAPHS

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Abstract

Ian Stewart writes about a Strategy of Emperor Constantine for defending the Roman Empire in his article in 1999. Cockayne following this in 2004 introduced the concept of Roman domination in Graphs. This paper gives an overview of Roman domination in Graph theory. The paper starts with the introduction of the Roman domination function in graphs followed by its definition and properties. The paper discusses about Roman domination function in Double fan graphs.

1. Motivation

In the 4th century A. D., Constantine the Great issued a decree for the protection of the Roman Empire. Constantine decreed that each city in the empire had its own troops stationed there to defend itself or to lie in ambush near a city with two adjacent armies. In this way, if an unprotected city were to be attacked, the nearest town would be able to send out invitations without leaving the city. The natural problem is deciding how many troops are enough to defend the state. This problem is common in graphs; the minimum number of forces required to defend the graph G is the Roman domination number G, expressed in $\gamma_R(G)$. In such systems, we usually want the

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smallest dominating set on the graph G, its size is the domination number G, indicated by $\gamma(G)$.

2. Roman Domination

Roman domination of graphs was generally described by Cockayne, (University of Victoria, Canada) Dreyer, (RAND Corporation, Santa Monica, USA) Hedetniemi, (Clemson University, USA) and Hedetniemi were encouraged, in part, by an article in Ian Stewart's Scientific American article entitled "Defend" Roman Empire. A Roman dominating function (RDF) in graph G is the function $f(v_0, v_1, v_2)$ which satisfies that each vertex $u \in V_0$ is adjacent to at least one vertex $v \in V_2$. The Roman rule number G, indicated by $\gamma_R(G)$ is the smallest weight among all RDFs in G. To say in formal way, a Roman dominating function (or RDF) of G is a function $f : V(G) \to \{0, 1, 2\}$ such that if $f(v) = 0$, $f(w) = 2$ for some neighbor w of v. The minimum weight of roman dominating function f is $\gamma_R(G)$.

3. Definition

In a graph $G = (V, E)$ a dominating set D is roman dominating set, if S and T are two subsets of D and satisfying the condition that every vertex u in S is adjacent to exactly one to one a vertex v in $V - D$ as well as adjacent to some vertex in T. The roman domination number $\gamma_{rdf}(G)$ of G is the minimum cardinality of a roman dominating set of G. A subset D of V is said to be a dominating set of G if every vertex in $V \setminus D$ is adjacent to a vertex in D. A dominating set with minimum cardinality is said to be a minimum dominating set. The domination number $\gamma(G)$ of the graph G is the minimum cardinality of the dominating set in G.

4. Properties of Roman Dominating Sets

A Roman dominating function on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G.

For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$, and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V \mid f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. Note that there exists a $1-1$ correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is a Roman dominating function (RDF) if $V_2 \succ V_0$, where $\succ$ means that the set V_2 dominates the set V_0, i.e. $V_0 \subseteq N[V_2]$. The weight of f is $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1$. The Roman domination number, denoted $\gamma R(G)$, equals the minimum weight of an RDF of G, and we say that a function $f = (V_0, V_1, V_2)$ is a γ_R-function if it is an Roman dominating function and $f(V) = \gamma_R(G)$.

Preposition 1. *For any graph* G, $\gamma(G) \leqslant \gamma R(G) \leqslant 2\gamma(G)$.

Preposition 2. *For any graph G of order n,* $\gamma(G) = \gamma_R(G)$ *if and only if* $G = K_n$.

Proposition 3. *Let* $f = (V_0, V_1, V_2)$ *be any* γ_R-*function. Then*

- $G[V_1]$, *the subgraph induced by* V_1 *has maximum degree* 1.
- *No edge of G joins* V_1 *and* V_2.
- *Each vertex of* V_0 *is adjacent to at most two vertices of* V_1.
- V_2 *is a* γ-*set of* $G[V_0 \cup V_2]$.
- *Let* $H = G[V_0 \cup V_2]$. *Then each vertex* $v \in V_2$ *has at least two H-pn's* (*i.e. private neighbours relative to* V_2 *in the graph H*).
- *If v is isolated in* $G[V_2]$ *and has precisely one external H-pn, say* $w \in V_0$, *then* $N(w) \cap V_1 = \emptyset$.
- *Let* k_1 *equal the number of non-isolated vertices in* $G[V_2]$, *let* $C = \{v \in V_0 : |N(v) \cap V_2| \geq 2\}$, *and let* $|C| = c$. *Then* $n_0 \geq n_2 + k_1 + c$.

Proposition 4. *Let* $f = (V_0, V_1, V_2)$ *be a* γ_R-*function of an isolate-free* *graph G, such that* n_1 *is a minimum. Then*

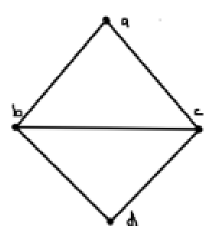
- V_1 *is independent, and* $V_0 \cup V_2$ *is a vertex cover.*
- $V_0 \succ V_1$.
- *Each vertex of* V_0 *is adjacent to at most one vertex of* V_1, *i.e.* V_1 *is a* 2-*packing.*
- *Let* $v \in G[V_2]$ *have exactly two external H-pn's* w_1 *and* w_2 *in* V_0. *Then there do not exist vertices* $y_1, y_2 \in V_1$ *such that* (y_1, w_1, v, w_2, y_2) *is the vertex sequence of a path* P_5.
- $n_0 \geqslant 3n/7$, *and this bound is sharp even for trees.*

Proposition 5. *For any graph G of order n and maximum degree* Δ, $2n\Delta + 1 \leqslant \gamma R(G)$.

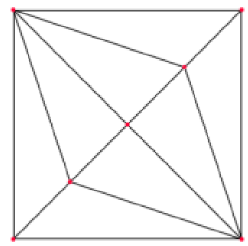
Proposition 6. *For a graph G on n vertices,* $\gamma R(G) \leqslant n2$ $In((1 + \delta(G))/2)1 + \delta(G)$.

5. Fan Graph

A graph which joins the empty graph K_m on m nodes and the path graph P_n on n nodes is called fan graph. If $m = 1$ then it is called fan graph and if $m = 2$ it is called double fan. The $(r, 2)$ graph is isomorphic to the complete tripartite graph $K_{1,1,r}$ and the $(r, 3)$ graph is to $K_{1,2,r}$. The fan graph $F_{4,1}$ is called as gem graph.

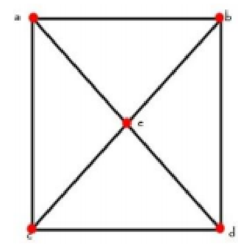


Fan graph $F_{1,3}$.

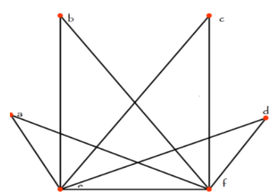


Double Fan graph $F_{(2,\ 5)}$.

Roman domination number of Fan graphs



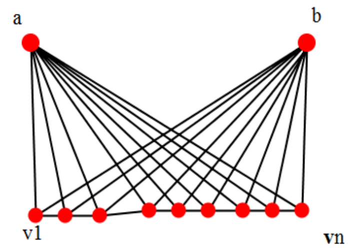
In the given fan graph $F_{2,\,3}$. Let D be the Dominating set. Here a and d dominate b, c, e. Vertex e dominates a, b, c, d. Let S and T be the subsets of D. Let $S = \{e\}$ and $T = \{a, d\}$. Vertex e in S is adjacent to exactly one vertex c in $V - D$ as well as adjacent to vertex e in T. Hence the Roman domination number is 3.



In this graph, let D be the dominating set. Here e dominates a, b, c, d, f. Let $S = \{b\}$ and $T = \{e, c\}$. Here b is adjacent to exactly one vertex f in D as well as to vertex e in T. Hence the Roman domination number is 3.

6. Roman Domination Number of Double Fan

For a graph $G = (V, E)$, let $f : V \to \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V \mid f(v) = i\}$ and $|V_i| = n_i$, for $i = 0, 1, 2$. There exists a $1-1$ correspondence between the functions $f : V \to \{0, 1, 2\}$ and the ordered partitions (V_0, V_1, V_2) of V. Thus, we will write $f = (V_0, V_1, V_2)$. A function $f = (V_0, V_1, V_2)$ is a Roman dominating function (RDF) if $V_2 < V_0$.



Let $G = P_n + 2K_1$. Here, $V(G) = \{a, b, v_i; 1 \leq i < n\}$ and $E(G) = \{av_i, vv_i; 1 \leq i < n\} \cup \{v_i v_{i+1}; 1 \leq i \leq n-1\}$.

Let $V_0 = \{v_i \backslash 1 \leq i \leq n\}$; $V_1 = \{a\}$; $V_2 = \{b\}$. Clearly, The set V_2 dominates the set V_0. (i.e.) $V_0 \subseteq N(V_2)$.

$f = (V_0, V_1, V_2)$ is a Roman domination function. Now, $f(v) = 2_{n2+n1}$ $= 2(1) + 1 = 3$. Obviously, there is no function f such that $f(v) < 3$. Hence Roman domination number of G is $P_n + 2K_1$.

Research Work

Roman Domination Problem could not be easily solved in the general case. Studies have shown the Roman domination function for different classes of graphs:

- Interval graphs, cographs, asteroidal triple-free graphs and graphs with a d-octopus by Liedl off et al. (2005);
- Corona graphs by Yero et al. (2013);
- Grid graphs by Curro (2014);
- Generalized Sierpinski graphs by Ramezani et al. (2016);
- Generalized Petersen Graphs $GP(n; 2)$ by Wang et al. (2011) and $GP(n; 3)$ and $GP(n; 4)$ by Zhiqiang Zhang and Xu (2014);

• Cardinal product of paths and cycles in Klobucar and Pulji *c* (2014, 2015);

• Strongly chordal graphs by Liu and Chang (2013);

• Digraphs by Sheikholeslami and Volkmann (2011);

• Complementary prisms by Al Hashim (2017) and others.

Conclusion

In this paper we have discussed the Roman domination number of double graphs. In the future work the problem of finding these Roman domination numbers for other challenging classes of graphs could be considered. Another direction of future research would be to determine other Roman domination numbers for graphs along with its algorithm.

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