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# t - LACEABILITY IN HIERARCHICAL HYPERCUBE GRAPHS

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## Abstract

The concept of Hamilton- $t$ -laceability is an important consideration in the interconnection network such as hierarchical hypercube network. A connected graph $G$ is said to be Hamilton- t -laceable (Hamilton- $t^*$ -laceable) if there exists a Hamiltonian path between every pair (at least one pair) of distinct vertices $u$ and $v$ in $G$ with the property $d(u, v) = t$ such that $1 \leq t \leq diam(G)$ where $diam(G)$ is the diameter of $G$. The existence of at least a Hamiltonian path is useful whenever we need to label processors with numbers $1, 2, 3, \ldots, p$ so that adjacent processors get successive numbers. In this article we have studied the Hamilton-$t_o$ -laceability of an $n$-dimensional hierarchical hypercube graph for $n \geq 6$. Further we have discussed that an $n$ – dimensional hierarchical hypercube graph $HHC_n$, $n \geq 6$ is Hamilton-$t_e$ -laceability with laceability number $\lambda_{(t)} = 1$.

## 1. Introduction

The graph of $n$-dimensional hypercube denoted by $H_n$ is a popular network and there are $2^n$ nodes present in $H_n$ each is uniquely represented by a binary sequence $a_{n-1}, a_{n-2}, a_{n-3}, \ldots, a_0$ of length $n$. The two nodes in $H_n$ are said to be adjacent if and only if the nodes differ exactly at one bit

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position. If $n$ increases it will be very difficult to design and fabricate the nodes of $H_n$ because of its large fanout. To overcome this, the hierarchical hypercube was proposed in [1] which is a two level structure that takes hypercubes as basic modules and connects them in a hypercube fashion. The hierarchical hypercube graph is suitable for massively parallel systems in which large number of nodes are connected with small diameter and low degree when compared to a hypercube of same size. The hierarchical hypercube graph possesses many appealing properties such as symmetry and logarithmic diameter which results in easy and fast algorithm. Since the topology of hierarchical hypercube graph is closely connected to the topology of the hypercube it inherits some favorable properties from hypercube. An $n$-dimensional hierarchical hypercube denoted by $HHC_n$ where $n = 2^m + m, m \geq 1$ is called perfect hierarchical hypercube graph which is more significant because of its symmetrical structure whereas $HHC_n$ with $n < 2^m + m, m \geq 1$ is called a non-perfect hierarchical hypercube graph.

**Definition 1.** A connected graph $G$ is termed as Hamilton- $t_o(t_e)$-laceable if there exists in $G$, a Hamiltonian path between every pair of its vertices $u$ and $v$ with the property $d(u, v) = t$ for all odd (even) $t$ such that $1 \leq t \leq diam\, G$.

**Definition 2.** Let $u$ and $v$ be any two arbitrary vertices of a connected graph $G$. Let $F_e$ be the set of minimum number of edges which are not in $G$. If $P$ is any path in $G$ then $P \cup F_e$ is a Hamiltonian path in $G$ from $u$ to $v$. Further $|F_e|$ is the $t$ laceability number denoted by $\lambda_{(t)}$ and $F_e$ are the $t$-laceability edges of $G$ with respect to the vertices $(u, v)$.

**Definition 3.** A graph is Hamilton-$t$-connected if it is Hamilton-t-laceable for every $t$ where $1 \leq t \leq diam(G)$.

**Construction of $HHC_n$ from $H_k$ for $k = 2^m, m \geq 1$.**

When $m = 1$, $HHC_n$ where $n = 2^m + m = 3$ is simply a cycle of length 8. That is, in the graph of $H_{2^1}$ every vertex is replaced by $H_1$ and the resulting graph is $HHC_3$.

**Figure 1.** $HHC_3$ constructed from $H_2$.

Similarly to construct $HHC_5$ for $m = 2$ and $n < 2^m + m$, every vertex of $H_3$ is replaced with $H_2$ and connect the edges in a hypercube manner.

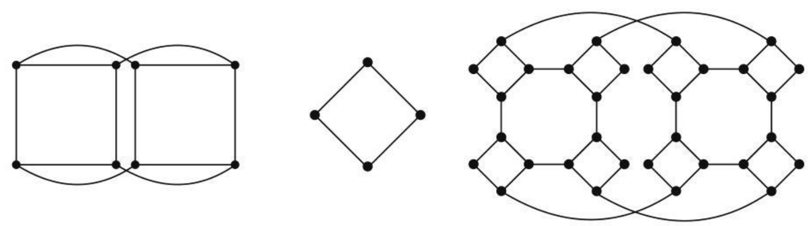

**Figure 2.** $HHC_5$ constructed from $H_3$.

## 2. Results and Discussion

**Theorem 2.1.** *Let $G_h$ be the hypercube graph $H_m$ for $m \geq 2$. If $G^* = HHC_n$, $n \geq 6$ is the graph obtained by embedding $2^k$ copies of $G_h$ for $k = n - m$ then $G^*$ is Hamilton-$t_o$-laceable.*

**Proof of Theorem 2.1.** Let $G^*$ be partitionable to $2^k$ smaller subgraphs of $G_h, 1 \leq h \leq 2^k$ for $k = n - m$ having $2^k$ vertices. Since every $G_h$ is bipartite the vertex set is subdivided into two sets $V_1$ and $V_2$ where $V_1 = \{v_x, 1 \leq x \leq \frac{2^n}{2}\}$ and $V_2 = \{v_y, 1 \leq y \leq \frac{2^n}{2}\}$. We now color the vertices of $G^*$ by using two different colors (red and black) in such a way that adjacent vertices are not colored with the same color. Let the vertices of $V_1$ represents red color and the vertices of $V_2$ represents black color. Let the distance between every pair of vertices of different colors is odd and same color is even in $G^*$. Consider the pair of vertices $(v_r, v_b)$ which are colored different. We assume that $v_r$ is colored in red and $v_b$ is colored in black. Then we trace a Hamiltonian path of $G^*$ as picking red vertex and then walking along an edge adjacent to it to another vertex of black color and continuing until we reach the last vertex of $G^*$.

To establish our result, we have two cases.

**Case (i).** Let the vertices $v_r$ and $v_b$ in the same subgraph say $G_1$ of $G^*$.

Since each of the hypercube graph $G_h$ is bipartite and Hamilton laceable we obtain the path $P_1$ covering all the vertices of $G_1$ except the vertex $v_r$. Let the sequences of vertices in $P_1$ be $\{v_{b_1}, v_{r_1}, v_{b_2}, v_{r_2}, \ldots, v_b\}$. Further, as $G^*$ is bipartite the vertices $v_r$ and $v_b$ are of different colors in $G_1$ and their neighboring vertices $\phi_1(v_b)$ and $\phi_1(v_r)$ are colored different in $G_2$. Also note that the neighbors of $\phi_1(v_b)$ and $\phi_1(v_r)$ are $\phi_2(v_r)$ and $\phi_2(v_b)$ in $G_3$ which are colored different and so on. We now trace Hamiltonian paths $P_2$ from $\phi_1(v_b)$ and $\phi_1(v_r)$ in $G_2$, $P_3$ from $\phi_2(v_r)$ and $\phi_2(v_b)$ in $G_3$ etc. Finally we obtain the path $P_{2^k}$ from $\phi_{2^k-1}(v_b)$ and $\phi_{2^k-1}(v_r)$ in $G_{2^k}$. The concatenation of these paths $P_1, P_2, P_3, \ldots, P_{2^k}$ along with $2^k$ edges $(v_r, \phi_1(v_b))$, $(\phi_1(v_r), \phi_2(v_b))$, $(\phi_2(v_r), \phi_3(v_b))$, $\ldots$, $(\phi_{2^k-1}(v_r), \phi_{2^k-1}(v_r), v_{b_1})$ forms the Hamiltonian path of $G^*$ that joins vertices $v_r$ and $v_b$.

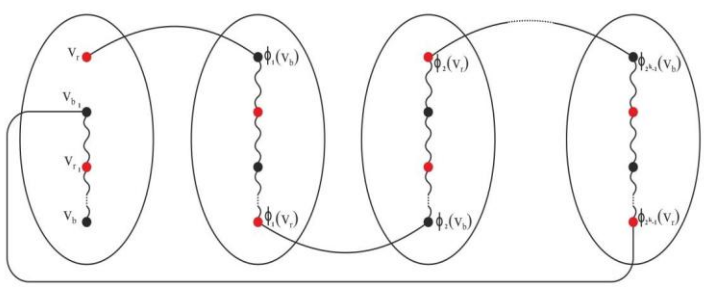

**Figure 3.** Hamiltonian path between $v_r$ and $v_b$ in $G_1$ of $G^*$.

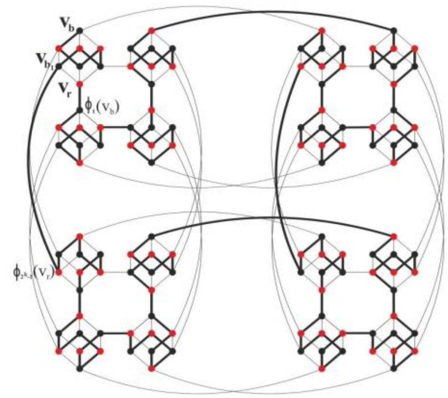

**Figure 4.** Hamiltonian path between $v_r$ and $v_b$ in and $HHC_7$.

**Case (ii).** Let $v_r$ and $v_b$ be in two different subgraphs of $G^*$ say $G_1$ and $G_{2^k}$. Pick the red vertex $v_r$ in $G_1$ and $v_b$, the black vertex in $G_{2^k}$. Also let us consider any black vertex $v_{b_i}$ in $G_1$. Obviously its neighboring vertex $\phi_1(v_{r_i})$ in $G_2$ will be colored red and its neighbor $\phi_1(v_{b_i})$ in $G_3$ is a black vertex and so on. Since each of $G_h$ is bipartite and Hamilton laceable we obtain paths $P_1$ joining $(v_r, v_{b_i})$ in $G_1$, $P_2$ joining $\phi_1(v_{r_i})$ to $\phi_1(v_{b_i})$, $P_3$ joining $\phi_2(v_{r_i})$ to $\phi_2(v_{b_i})$ etc. Finally the path $P_{2^k}$ is obtained by joining vertices $\phi_{2^k-1}(v_{r_i})$ and $v_b$ in $G_{2^k}$. The concatenation of these paths $P_1, P_2, P_3, \ldots, P_{2^k}$ along with $(2^k - 1)$ edges $(v_{b_i}, \phi_1(v_{r_i}))$, $(\phi_1(v_{b_i}), \phi_2(v_{r_i}))$, $(\phi_2(v_{b_i}), \phi_3(v_{r_i}))$, $\ldots$, $(\phi_{2^k-2}(v_{b_i}), \phi_{2^k-1}(v_{r_i}))$, $(\phi_{2^k-1}(v_{r_i}), v_b)$ forms the Hamiltonian path of $G^*$.

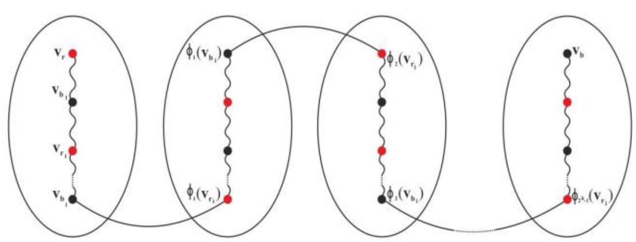

**Figure 5.** Hamiltonian path between $v_r$ and $v_b$, $v_r \in G_1$, $v_b \in G_{2^k}$ of $G^*$.

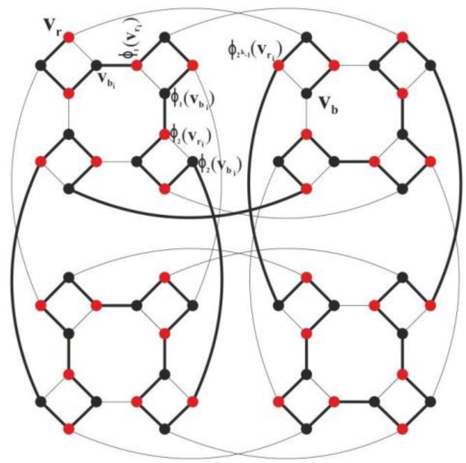

**Figure 6.** Hamiltonian path between $v_r$ and $v_b$ in $HHC_6$.

This completes the proof.

**Theorem 2.2.** *Let* $G_h$ *be the hypercube graph* $H_m$ *for* $m \geq 2$. *If* $G^* = HHC_n$, $n \geq 6$ *is the graph obtained by embedding* $2^k$ *copies of* $G_h$ *for* $k = n - m$ *then* $G^*$ *is Hamilton-* $t_e$ *-laceable with* $\lambda_{(t)} = 1$.

**Proof of Theorem 2.2.** Let $G^* = HHC_n$, $n \geq 6$ be the graph of $2^n$ vertices, partition able to two smaller subgraphs $G'_h$ for $1 \leq h \leq 2^k - 1$ and $G''_h$ for $h = 1$. It is obvious that the graph $G'_h$ contains $(2^k - 1)$ number of $H_m$ and $G''_h$ contains only one $H_m$ for $m \geq 2$. Since $G^*$ is known to be bipartite, we color the vertices in such a way that no two adjacent vertices of $G^*$ can have the same color. Let the distance between two vertices which are of same color (different color) be even (odd). Let $F_e = \{(v_i, v_j) = 1\}$ be the faulty edge linked to the two vertices $v_i$ and $v_j$ of $G''_h$ which are colored same. This makes the subgraph $G''_h$ non bipartite. Further we note that the vertices $v_i$ and $v_j$ are adjacent with same color. Let $S$ and $E$ be the pair of vertices of same color. Now we can trace a Hamiltonian path of $G^*$ as picking the vertex $S$, walking along an edge adjacent to it and then to another vertex of different color. We have to continue this until we get the last vertex $E$ of $G^*$.

To establish our result, we have four cases to discuss.

**Case (i).** $S$, $E$ belongs to the same $H_m$ of $G'_h$.

**Sub case (i).** Let $S$ and $E$ be two red vertices in the same $H_m$ of $G'_h$. Choose two other black vertices $A$ and $B$ in $G'_h$. Since $G'_h$ is bipartite and properly colored we obtain a path $P_1$ from $S$ to $A$ and a path $P_2$ from $E$ to $B$ such that the two paths are disjoint and covers all the vertices of $G'_h$.

Also by considering two vertices $\phi(A)$ and $\phi(B)$ in $G''_h$ which are neighboring vertices of $A$ and $B$ colored in red, we obtain a path $P_3$ from $\phi(A)$ to $\phi(B)$ including $F_e = \{(v_i, v_j) = 1\}$. The concatenation of these paths $P_1, P_2, P_3$ along with the edges $(A, \phi(A))$ and $(B, \phi(B))$ is the Hamiltonian path joining $S$ and $E$ in $G^*$.

**Sub case (ii).** Let $S$ and $E$ be two black vertices in the same $H_m$ of $G'_h$. Choose two other red vertices $A$ and $B$ in $G'_h$. Since $G'_h$ is bipartite and properly colored we obtain a path $P_1$ from $S$ to $A$ and a path $P_2$ from $E$ to $B$ such that the two paths are disjoint and covers all the vertices of $G'_h$. Also by considering two vertices $\phi(A)$ and $\phi(B)$ in $G''_h$ which are neighboring vertices of $A$ and $B$ colored in black, we obtain a path $P_3$ from $\phi(A)$ to $\phi(B)$ including $F_e = \{(v_i, v_j) = 1\}$. The concatenation of these paths $P_1, P_2, P_3$ along with the edges $(A, \phi(A))$ and $(B, \phi(B))$ is the Hamiltonian path joining $S$ and $E$ in $G^*$.

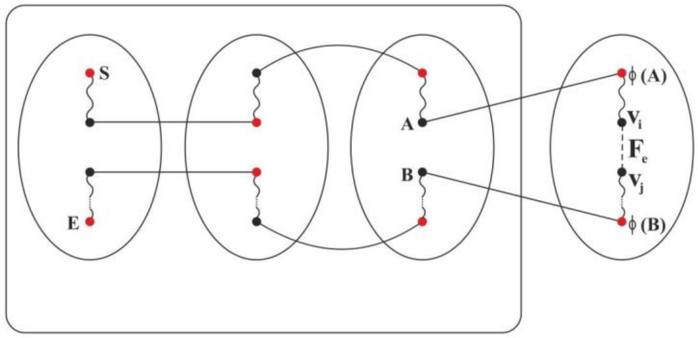


**Figure 7.** Hamiltonian path between $S$ and $E$ of same color belongs to the same $H_m$ of $G'_h$.

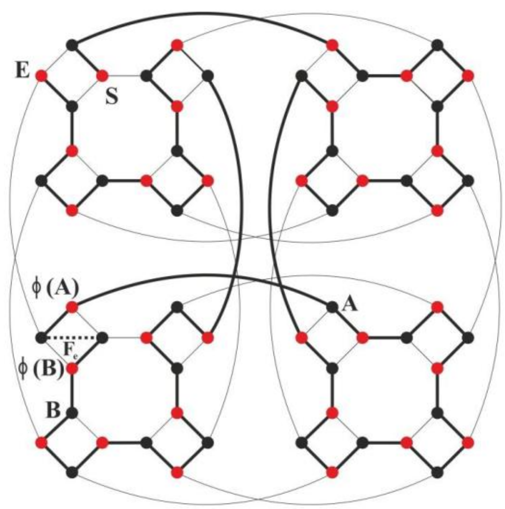


**Figure 8.** Hamiltonian path between $S$ and $E$ where $d(S, E) = 2$ in $HHC_6$.

**Case (ii).** $S$, $E$ belongs to different $H_m$ of $G'_h$.

**Sub case (i).** Let $S$ be the red vertex in one $H_m$ and $E$ be any other red vertex in the neighboring $H_m$ of $G'_h$. Also choose two black vertices $A$ and $B$ in $G'_h$. Then we trace a path $P_1$ from $S$ to $A$ and a path $P_2$ from $E$ to $B$. Further choose two vertices $\phi(A)$ and $\phi(B)$ in $G''_h$ such that $\phi(A) \neq A$ and $\phi(B) \neq B$. Also we obtain a path $P_3$ from $\phi(A)$ to $\phi(B)$ including the faulty edge $F_e = \{(v_i, v_j) = 1\}$. The concatenation of these paths $P_1, P_2, P_3$ along with the edges $(A, \phi(A))$ and $(B, \phi(B))$ is the Hamiltonian path joining $S$ and $E$ in $G^*$.

**Sub case (ii).** Let $S$ be the black vertex in one $H_m$ and $E$ be any other black vertex in the neighboring $H_m$ of $G''_h$. Also choose two red vertices $A$ and $B$ in $G'_h$. Then we trace a path $P_1$ from $S$ to $A$ and a path $P_2$ from $E$ to $B$. Further choose two vertices $\phi(A)$ and $\phi(B)$ in $G''_h$ such that $\phi(A) \neq A$ and $\phi(B) \neq B$. Also we obtain a path $P_3$ from $\phi(A)$ to $\phi(B)$ including the faulty edge $F_e = \{(v_i, v_j) = 1\}$. The concatenation of these paths $P_1, P_2, P_3$ along with the edges $(A, \phi(A))$ and $(B, \phi(B))$ is the Hamiltonian path joining $S$ and $E$ in $G^*$.

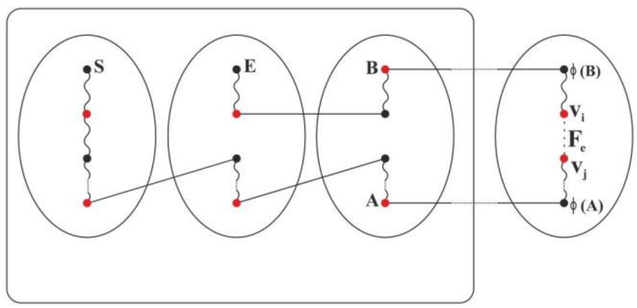


**Figure 9.** Hamiltonian path between $S$ and $E$ of different color belongs to different $H_m$ of $G'_h$.

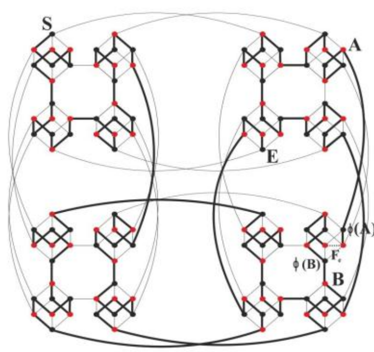


**Figure 10.** Hamiltonian path between $S$ and $E$ where $d(S, E) = 8$ in $HHC_7$.

**Case (iii)** $S, E \in G''_h$ where $S$ and $E$ are red or black vertices.

Let $S$ and $E$ be any red or black vertices of the non-bipartite graph $G''_h$. Choose a vertex $A$ in $G''_h$ such that $S \neq A$. Since $G'_h$ is bipartite, properly colored and Hamiltonian laceable, we obtain a path $P_1$ from $S$ to $A$ including the faulty edge $F_e = \{(v_i, v_j) = 1\}$. Also we choose a vertex $B \in G'_h$ such that $B \neq A$. Then we obtain a path $P_2$ from $B$ to $\phi(A)$ where $\phi(A)$ is the adjacent vertex of A such that $A \neq \phi(A)$. The concatenation of these paths $P_1$ and $P_2$ along with the edges $(A, \phi(A))$ and $(E, B)$ is the Hamiltonian path joining $S$ and $E$ in $G^*$.

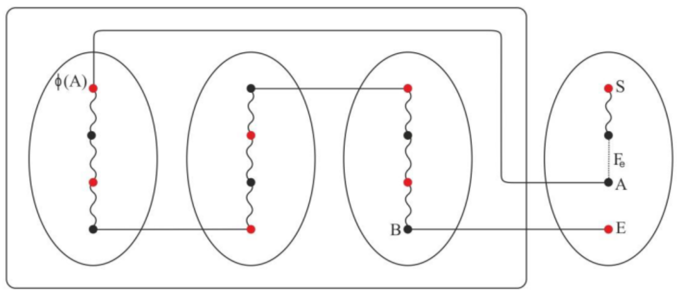


**Figure 11.** Hamiltonian path between $S$ and $E$ of same color belongs to $G''_h$.

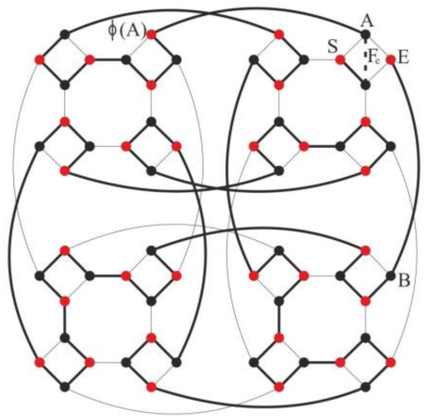


**Figure 12.** Hamiltonian path between $S$ and $E$ of $G''_h$ where $d(S, E) = 2$ in $HHC_6$.

**Case (iv).** $S \in G'_h$, $E \in G''_h$ where $S$ and $E$ are red or black vertices.

We choose a vertex $A$ of $G'_h$ such that $S$ and $A$ are of different colors and the vertex $E = \phi(A) \in G''_h$. Note that $A \neq \phi(A)$ where $\phi(A)$ is the adjacent vertex of A. We trace a path $P_1$ from $S$ to $A$ that covers all the vertices of $G'_h$ and $P_2$ from $E$ to $\phi(A)$ in $G''_h$ including the faulty edge $F_e = \{(v_i, v_j) = 1\}$. The concatenation of these paths $P_1$ and $P_2$ along with the edge $(A, \phi(A))$ is the Hamiltonian path that joins $S$ and $E$.

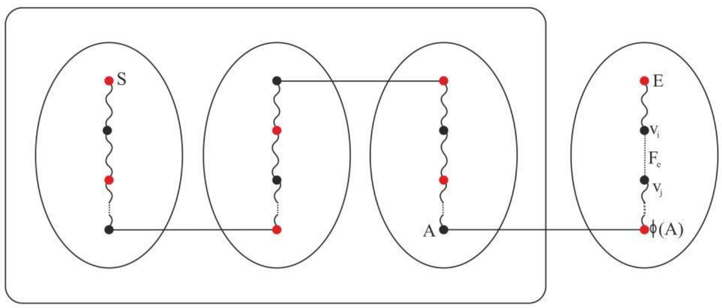


**Figure 13.** Hamiltonian path between $S$ and $E$ of same color where $S \in G'_h$, $E \in G''_h$.

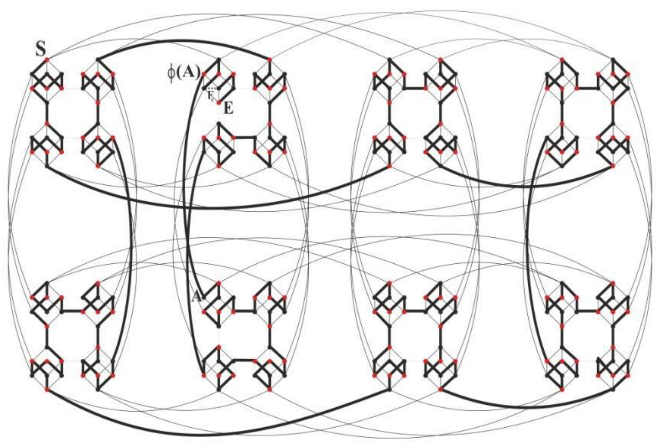


**Figure 14.** Hamiltonian path between $S$ and $E$ of $G''_h$ where $d(S, E) = 4$ in $HHC_8$.

This completes the proof.

## Conclusion

In this article we have studied the Hamilton- $t$ - laceability property of $HHC_n$ for $n \geq 6$. Precisely we have shown that the graph of $HHC_n$ for $n \geq 6$ is Hamilton-$t_o$-laceable and Hamilton-$t_e$-laceable with laceability number $\lambda_{(t)} = 1$ such that $1 \leq t \leq diam(G)$. Further we conclude that $HHC_n$ for $n \geq 6$ is Hamilton- $t$ -connected as it is Hamilton- $t$ -laceable for every $t$ where $1 \leq t \leq diam(G)$.


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