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Graphs with large vertex and edge-reconstruction number

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Abstract The reconstruction problem is considered to be one of the most important unsolved problems in graph theory. Proposed in 1942, the Graph Reconstruction Conjecture states that every simple, finite and undirected graph G with three or more vertices can be reconstructed up to isomorphism to the original graph from the collection of all unlabelled vertex-deleted subgraphs of G.

Related to this conjecture, Harary and Plantholt came up with the idea of the reconstruction number – the minimum number of vertex-deleted subgraphs required in order to identify a graph up to isomorphism. The edge-reconstruction number of a graph is analogously defined.

Myrvold and independently Bollobás proved that almost all graphs have reconstruction number equal to 3. An analogous result for edges showed that almost every graph has an edge-reconstruction number of 2.

We shall look into some results obtained in recent years which dealt with graphs that have large vertex and edge-reconstruction numbers.

Keywords Reconstruction conjectures; vertex-reconstruction number; edge-reconstruction number
AMS subject classifications 05C60

1 Introduction

The Reconstruction Conjecture states that a simple, finite and undirected graph with at least three vertices is uniquely determined, up to isomorphism, from its collection of vertex-deleted subgraphs [10, 23]. Until now, there are no known counter-examples to this conjecture and it is widely believed to be true.

In 1964, Harary formulated the related topic of edge-reconstruction where one can reconstruct a graph from its edge-deleted subgraphs and it states that all simple finite graphs with at least four edges are edge-reconstructible [8].

A vertex-deleted subgraph of a graph G is a subgraph $G-v$ obtained by deleting from G a vertex v and all the edges incident to it. Similarly an edge-deleted subgraph of G is a subgraph $G-e$ obtained by deleting the edge e. The *deck* of a graph G, denoted by $\mathcal{D}(G)$, is the collection of all unlabelled vertex-deleted subgraphs of G, and the elements of $\mathcal{D}(G)$ are referred to as *cards*.

Note that all graphs in $\mathcal{D}(G)$ are unlabelled and we do not identify isomorphic graphs in $\mathcal{D}(G)$. Therefore $\mathcal{D}(G)$ is a multiset rather than a set of representations of isomorphism classes of graphs. The *edge-deck* of G, denoted by $\mathcal{ED}(G)$, is similarly defined to be the collection of all edge-deleted subgraphs of G.

A *reconstruction* (*edge-reconstruction*) of G is a graph H with $\mathcal{D}(G)=\mathcal{D}(H)$ ($\mathcal{ED}(G)=\mathcal{ED}(H)$). A graph G is *reconstructible* (*edge-reconstructible*) if every reconstruction of G is isomorphic to G. This means that G is reconstructible (edge-reconstructible) if it can be obtained uniquely, up to isomorphism, from its deck (edge-deck).

A graph G is reconstructible from a subdeck $S \subseteq \mathcal{D}(G)$ if it is uniquely determined from S, up to isomorphism. That is every graph which has S as a subdeck must be isomorphic to G. In most cases where a class of graphs has been shown to be reconstructible, it has turned out that only a few of the graphs in the deck were needed.

When research on reconstruction problems seemed to have come to a standstill, Harary and Plantholt [9] wrote a short paper which introduced the notion of reconstruction numbers of graphs. While the Reconstruction Conjecture is concerned with the possibility of reconstructing graphs from a deck, reconstruction numbers indicate how easy or difficult it is to reconstruct a graph from its deck.

Thus, the following two questions can be put forward [19].

(i) What is the minimum k required so that a graph G can be reconstructed from *some* subdeck of size k?

(ii) What is the minimum k required so that a graph G can be reconstructed from *any* subdeck of size k?

The minimum value of k in the first case is called the *existential* or *ally reconstruction number* of a graph G, denoted by $\exists rn(G)$ or $rn(G)$. Most often the existential number is just called the *reconstruction number* of G.

On the other hand the minimum value of k in the second case is called the *universal* or *adversary reconstruction number* of G, denoted by $\forall rn(G)$ or $adv\text{-}rn(G)$.

The *edge-reconstruction number*, denoted by $ern(G)$ and the *adversary edge-reconstruction number*, denoted by $adv\text{-}ern(G)$ are analogously defined.

In this paper we shall consider (vertex-) reconstruction and edge-reconstruction numbers. Any graph-theoretic notation and definitions not explicitly defined can be found in [5] or [13]. For a recent survey on reconstruction, see [12].

2 Reconstruction numbers

The simplest observation that we can make about reconstruction numbers is that no two cards alone from the deck of a graph G are sufficient to reconstruct G uniquely.

Observation 2.1 *For any graph G,* $rn(G) \geq 3$.

Proof. Suppose that $rn(G) = 2$. Then there are two cards $G-u$ and $G-v$ which uniquely identify G. Construct graph H as follows: if $uv \in E(G)$, then $H = G - uv$; if $uv \notin E(G)$, then $H = G + uv$. This means that $H \not\simeq G$ but $G-u$ and $G-v \in \mathcal{D}(H)$. □

Observation 2.2 *For any graph G,* $rn(G) = rn(\overline{G})$ *where* $\overline{G}$ *is the complement of G.*

By means of a computer search amongst graphs of order at most 7, Harary and Plantholt showed that only five graphs and their complements have reconstruction number greater than 3. In fact they conjectured that almost all graphs have reconstruction numbers equal to 3. This was later proved by Myrvold [19] and independently by Bollobás [4].

Theorem 2.1 [4, 19] *Almost every graph has reconstruction number equal to* 3.

An analogous result for edges can be found in [13].

Theorem 2.2 [13] *Almost every graph has an edge-reconstruction number equal to* 2.

2.1 Empirical evidence

The data in Table 1, obtained by Rivshin and Radziszowski [21] gives a very good idea of how strong Theorem 2.1 is.

Theorem 2.1 (Theorem 2.2) follows from the fact that almost all graphs have the property that deleting any three different vertices (two different edges) will give non-isomorphic subgraphs which means that almost every graph is highly asymmetric.

We can also see that all graphs with high reconstruction numbers require graphs of non-prime order which supports Harary and Plantholt's conjecture that if G is a graph of odd prime order, then $rn(G) = 3$ [9]. Table 1 shows that it can be difficult to find graphs with large reconstruction number, even if this number is equal to 4.

Order										
Rec. no.	3	4	5	6	7	8	9	10	11
3	4	8	34	150	1044	12334	274666	12005156	1018997864
4		3		4		8		6	
5				2		2	2	4	
6						2			
7								2	

Table 1. Number of graphs with given order and given reconstruction number

2.2 Graphs with large reconstruction numbers

The first graphs for which reconstruction numbers were studied, were disconnected and regular graphs.

The following theorem was first proved by Mryvold [20], and then some problems with the proof were identified and corrected by Molina [17].

Theorem 2.3 [20, 17] *Given a disconnected graph G where not all components are isomorphic,* $rn(G) = 3$.

The next theorem by Myrvold shows that disconnected graphs whose components are all isomorphic have an upper bound.

Theorem 2.4 [20] *Whenever the components of a disconnected graph are all isomorphic and each component is of order c,* $rn(G) \leq c+2$.

Myrvold also showed that the upper bound can be attained when the graph consists of p copies of the complete graph K_c. This is because every card of pK_c is isomorphic to $K_{c-1} \cup (p-1)K_c$ but any $c+1$ such cards are blocked by $K_{c-1} \cup K_{c+1} \cup (p-2)K_c$.

In the next result, Asciak and Lauri [2] showed that in fact these make up the only class of disconnected graphs that attains the upper bound $c+2$ and that there are no disconnected graphs, where $rn(G) = c+1$.

Theorem 2.5 [2] *Given a disconnected graph G of the form pC where C has order c and is not a complete graph,* $rn(G) \leq c$.

In addition, we can say that the complements of disjoint unions of complete graphs are the complete multi-partite graphs which have reconstruction number greater than 3.

The computer searches of McMullen and Radziszowski [16] for all graphs of order less than 11, and Rivshin and Radziszowski [21] for all graphs of order up to 11, confirmed the theoretical results obtained on disconnected graphs. In their studies they unearthed two examples of disconnected graphs of the form pH where H is of order c and not complete, and $rn(pH) > 3$. These are the graphs $2C_4$ and $2P_4$, both of which have $rn = 4$.

The two graphs shown in Figure 1 are the complements of $2C_4$ and $2P_4$, and both have $rn = 4$.

The only graph with reconstruction number of 4 and which does not fit any of the above mentioned categories is the self-complementary graph P_4. Harary and Plantholt remarked that P_4 is an exception due to its low order.

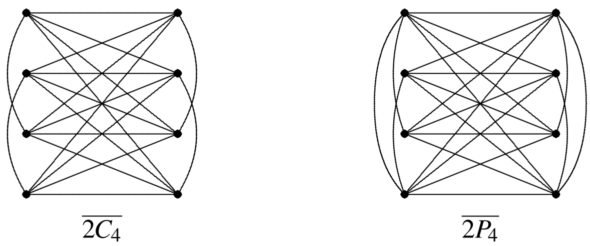


Figure 1. The complement of $2C_4$ and $2P_4$

This raises the question of investigating the gap between 4 and c for the reconstruction number of disconnected graphs.

The situation with regular graphs is quite similar because the reconstructibility of such graphs is ery easy to show but the determination of their reconstruction number proves to be quite difficult. Myrvold [19] has shown that r-regular graphs have reconstruction number at most $r+3$ and that this bound is attained by pK_{r+1}. The following by Asciak [1] further shows that in fact pK_{r+1} is the only r-regular graph with reconstruction number $r+3$.

Theorem 2.6 [1] *Let G be an r-regular graph. Suppose that $rn(G) = r+3$, then $G = pK_{r+1}$, for some $p > 1$.*

Here, as in the case of disconnected graphs, the gap between reconstruction numbers 3 and $r+3$ is waiting to be explored.

McMullen and Radziszowski [16] used *nauty* [14] and Condor [7] to construct two new classes of graphs with high vertex reconstruction numbers.

The first class of graphs is called *redundantly connected cycles*. These graphs defined as $RCC_{n,j}$ are regular graphs with n redundantly connected cycles, each of length j. These can be obtained as follows. Take n ($n > 2$) disjoint copies of cycles each of length $j \geq 3$. Let $v_{c,i}$, $i \in \{0, 1, \ldots, j-1\}$ denote the i^{th} vertex of the c^{th} cycle. Then for each $c \neq d$, join the vertices $v_{c,i}$ and $v_{d,i+1}$, where addition is modulo j.

From the following result by McMullen and Radziszowski [16], we see that the reconstruction number of every graph in such class is greater than 3.

Theorem 2.7 [16] *For any graph $G = RCC_{n,j}$ where $n \geq 2$ and $j \geq 3$, $rn(G) > n+1$.*

Figure 2 shows an example of $RCC_{2,5}$ whose reconstruction number is 4.

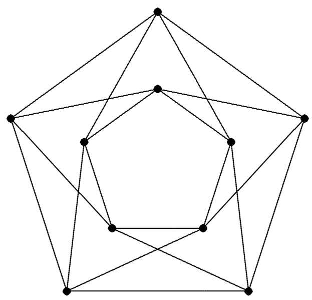


Figure 2. The regular graph $RCC_{2,5}$

The second class is of the form $K_{c_1} \longleftrightarrow^b K_{c_2}$, which is a graph formed by disjoint graphs K_{c_1} and K_{c_2} with b additional edges forming a partial matching between K_{c_1} and K_{c_2}. Theorem 2.8 specifies a range of values of c_1, c_2 and b for the graph $K_{c_1} \longleftrightarrow^b K_{c_2}$ for which $rn(G) > 3$.

Theorem 2.8 [16] *For every graph $G = K_c \longleftrightarrow^b K_c$ where $c \geq 3$ and $2 \leq b \leq c-1$, $rn(G) > 3$.*

Such graphs consist of two K_c components, connected with b additional edges which share no common vertex. This class was previously identified by Harary and Plantholt who showed that such graphs have $rn \geq \min\{b+1, c-b+2\}$. Moreover in his project, McMullen [15] showed that for a graph G of the form $K_c \longleftrightarrow^{c-1} K_c$, $rn(G) \geq c$.

The graph $K_4 \longleftrightarrow^3 K_4$ is presented in Figure 3 whose reconstruction number is 5.

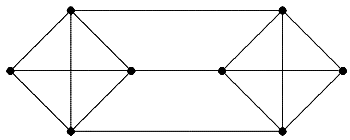


Figure 3. The graph $K_4 \longleftrightarrow^3 K_4$

We can also add the class of the complement of $K_c \longleftrightarrow^b K_c$ which is $K_{c,c}$ less b non-adjacent edges whose reconstruction number is greater than 3, and a particular example is shown in Figure 4.

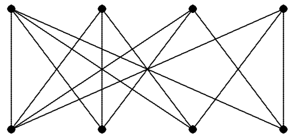


Figure 4. The graph $K_{4,4}$ less 3 non-adjacent edges

For more information on such constructions of graphs the reader is invited to read [16].

Brewster et al. [6] used the *lexicographic product* to construct infinite families of graphs with high reconstruction numbers. The best way to think of the lexicographic product of graphs G and H, denoted by $G[H]$, is by replacing each vertex u of G by a copy H_u, of H and adding all edges between two copies that replace adjacent vertices. They showed that every member of $\{G[P_4] : G \text{ is vertex transitive}\}$ has reconstruction number at least 4.

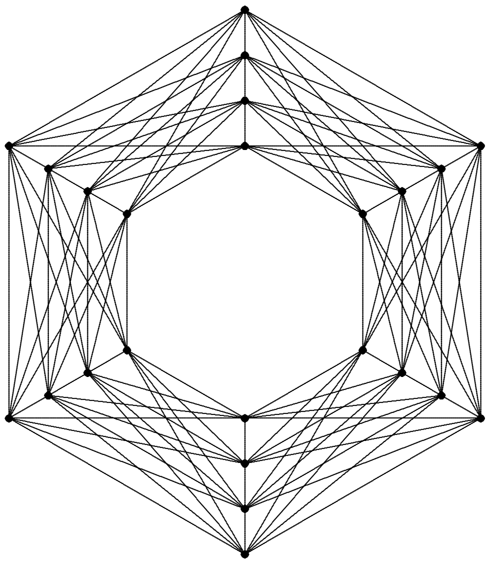


Figure 5. The lexicographic product $C_6[P_4]$ where $rn > 4$

This is neither an RCC (defined also by Brewster et al. as $C_j[\overline{K}_n]$) nor its complement.

2.3 Graphs with large edge-reconstruction numbers

Table 2 shows the number of graphs having at least one edge, with a given edge-reconstruction number for all orders between 3 and 11, both inclusive.

Order									
Edge rec. no.	3	4	5	6	7	8	9	10	11
1	3	5	9	18	23	35	46	64	71
2			14	115	980	12242	274523	12004951	1018997596
3		1	6	16	31	57	81	130	167
4				2	5	4	9	10	15
5						3	3	5	6
6							1	2	2
7								1	1
8									1

Table 2. Number of graphs with given order and given edge-reconstruction number

The results concur with the theoretical result obtained by Lauri [11] that $ern(G)$ is almost always 2.

In [18], Molina started to tackle the edge-reconstruction number of disconnected graphs. He showed that the edge-reconstruction results are similar to the vertex reconstruction results stated by Myrvold [20], but a significant difference is that whereas the vertex reconstruction number of a graph is always 3 or more, the edge-reconstruction number of a disconnected graph is often 2. The following theorem gives Molina's main results:

Theorem 2.9 [18] *Let G be a disconnected graph with at least four edges and at least two non-trivial components (that is, components that have more than one vertex).*

(1) If not all components are isomorphic, then $ern(G) \leq 3$.

(2) If all components are isomorphic, then $ern(G) \leq t+2$ *where t is the number of edges in a component.*

(3) If there exists a pair of non-isomorphic components in which one component has a cycle and G does not have any components isomorphic to either K_3 *or* $K_{1,3}$*, then* $ern(G) \leq 2$.

He also observed that the value of $t+2$ is attained, giving as an example the graph consisting of p copies of $K_{1,t}$.

In [3], Asciak and Lauri used line graphs in order to prove and extend Molina's results. In fact they proved the following results:

Theorem 2.10 [3] *Let G be a disconnected graph with at least four edges and the property that all components are isomorphic to a graph H.*

(1) If H is isomorphic to K_3, then $ern(G) = 2$.

(2) If H is isomorphic to $K_{1,3}$, then $ern(G) = 5$.

(3) If H is not isomorphic to K_3 or $K_{1,3}$, then $ern(G) \leq t+2$, where t is the number of edges in H. Moreover, if $ern(G) \geq t+1$ then $H \simeq K_{1,t}$.

Theorem 2.11 [3] *Let G be a disconnected graph consisting of exactly two types of non-trivial components, namely those isomorphic to K_3 and those isomorphic to $K_{1,3}$. Then $ern(G) = 3$.*

They also tried to investigate conditions which force or do not allow $ern(G)$ to be equal to 2, and also showed that in general, there is no straightforward relationship between the edge-reconstruction number of G and that of its components.

Rivshin performed a computer search [22] which showed that, out of more than a billion graphs on at most eleven vertices, only fifty-six disconnected graphs have edge-reconstruction number greater than 3. Out of these fifty-six disconnected graphs, only four graphs do not have isolated vertices as components, namely $2K_{1,2}, 2K_{1,3}, 2K_{1,4}$ and $3K_{1,2}$. All of these results led Asciak and Lauri [3] to make the following conjecture.

Conjecture 2.1 [3] *Suppose that $ern(G) > 3$ for a disconnected graph all of whose components are isomorphic to H. Then H is isomorphic to the star $K_{1,r}$ where r is the number of edges.*

2.3.1 Examples of some classes of graphs with large reconstruction number

The following are examples of disconnected graphs having high edge-reconstruction number:

(1) Graphs of the form $pP_3 \cup kK_1$, $p \geq 2$, $k \geq 0$ have $ern = 4$.

(2) Graphs of the form $pK_3 \cup kK_1$, $p \geq 1$, $k \geq 1$ have $ern = 4$.

(3) Graphs of the form pP_5, $p \geq 1$ have $ern = 3$.

(4) Graphs of the form $P_5 \cup kK_1$, $k \geq 1$ have $ern = 4$.

We also noticed that many of the graphs with high edge-reconstruction number are spanning subgraphs of $K_{r,2}$.

The following result identifies a class of graphs that have high edge-reconstruction number.

Theorem 2.12 *A graph of the form $K_{r,2}$ less two non-adjacent edges, where $r \geq 4$, has $ern(G) = |V(G)| - 3$ or $r - 1$.*

Proof. Let G be a graph in the form of $K_{r,2}$ less 2 non-adjacent edges as shown in Figure 6.

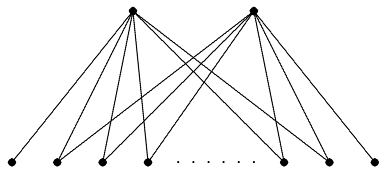
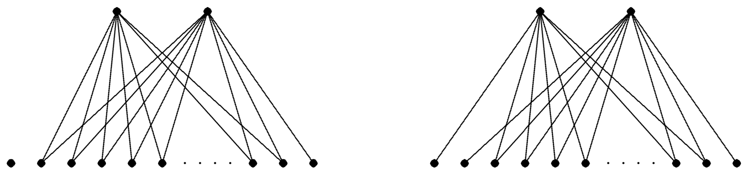


Figure 6. The graph G

Then the $\mathcal{ED}(G)$ is made up of 2 edge-cards of the form C_1 and $2(r-2)$ edge-cards of the form C_2, where C_1 consists of $K_{r-1,2}$ less an edge and an isolated vertex, while C_2 consists of $K_{r,2}$ less 2 adjacent edges and a non-adjacent edge, as shown in Figure 7.



Edge-card C_1 Edge-card C_2

Figure 7. The two edge-cards C_1 and C_2

Now consider the graph G' which is of the form of $K_{r,2}$ less 2 adjacent edges. Its edge-deck is made up of 3 different types of edge-cards, namely types A, B and C. Altogether there will be $(r-2)$ edge-cards of type A which consists of $K_{r,2}$ less 2 adjacent edges, and a non-adjacent edge, $(r-2)$ edge-cards of type B consisting of $K_{r,2}$ less 3 adjacent edges and 2 edge-cards of type C which consists of $K_{r-1,2}$ less an edge and an isolated vertex.

It is easy to check that whether we are given

(i) $(r-2)$ copies of edge-card C_2, or

(ii) $(r-3)$ copies of C_2 and a copy of C_1, or

(iii) two copies of C_1 and $(r-4)$ copies of C_2,

then the graph G' has $(r-2)$ edge-cards in its edge-deck in common with the edge-cards of G. Therefore $(r-2)$ edge-cards are not sufficient to reconstruct G uniquely, so that $ern(G) > (r-2)$.

We now need to show that $(r-1)$ edge-cards are sufficient to reconstruct G uniquely. We choose $(r-1)$ copies of C_2 from the edge-cards of G, and let H be a graph on $(r+2)$ vertices with edge-cards $H-e_i \simeq C_2$, for some edges $e_i \in E(H)$, $i=1,2,\ldots,(r-1)$. We show that $H \simeq G$.

Suppose that an edge e_i is connected to two vertices of degree $(r-1)$ and $(r-2)$ in C_2. Then there can only be one edge-card in common with the edge-cards of H. Any other edge deletion other than the mentioned edge e_i in C_2, results in an edge-card not isomorphic to any $(r-1)$ copies of H. This excludes the possibility of having an edge e_i in H connecting the two vertices of degree $(r-1)$ and $(r-2)$ in C_2.

The same argument follows if an edge e_i is connected either to two vertices of degree 2 or to two vertices of degree 1 in C_2.

Now if the replacing edge joins a vertex of degree 1 to the vertex of degree $(r-1)$ in C_2, there will only be $(r-2)$ edge-cards in common with the edge-cards $H-e_i$, which contradicts the above assumption. Therefore the replacing edge must join a vertex of degree 1 to the vertex of degree $(r-2)$ in C_2, that is, $G \simeq H$.

Hence $ern(G) = r-1$. □

3 Conclusion

We hereby present different tables of graphs of order less than 11 whose reconstruction number are considered to be large. Each table shows graphs and their complements of different orders having the same reconstruction number.

Reconstruction number $= 4$		
Order	Graph G	Graph $\overline{G}$
4	$2K_2$	C_4
	P_4	P_4
6	$3K_2$	$RCC_{2,3} \simeq C_3[\overline{K_2}]$
	$K_3 \longleftrightarrow^2 K_3$	$K_{3,3}$ less 2 non-adjacent edges
8	$2P_4$	$\overline{2P_4}$
	$4K_2$	$K_{2,2,2,2}$
	$2C_4$	$\overline{2C_4}$
	$K_4 \longleftrightarrow^2 K_4$	$K_{4,4}$ less 2 non-adjacent edges
10	$5K_2$	$K_{2,2,2,2,2}$
	$K_5 \longleftrightarrow^2 K_5$	$K_{5,5}$ less 2 non-adjacent edges
	$RCC_{2,5}$	$\overline{RCC_{2,5}}$

Table 3. Graphs of order less than 11 whose $rn = 4$.

Reconstruction number $= 5$		
Order	Graph G	Graph $\overline{G}$
6	$2K_3$	$K_{3,3}$
8	$K_4 \longleftrightarrow^3 K_4$	$K_{4,4}$ less 3 non-adjacent edges
9	$3K_3$	$RCC_{3,3} \simeq K_{3,3,3}$
10	$K_5 \longleftrightarrow^3 K_5$ $K_5 \longleftrightarrow^4 K_5$	$K_{5,5}$ less 3 non-adjacent edges $K_{5,5}$ less 4 non-adjacent edges

Table 4. Graphs of order less than 11 whose $rn = 5$.

Reconstruction number $= 6$		
Order	Graph G	Graph $\overline{G}$
8	$2K_4$	$K_{4,4} \simeq RCC_{2,4}$

Table 5. Graphs of order 8 whose $rn = 6$.

Reconstruction number $= 7$		
Order	Graph G	Graph $\overline{G}$
10	$2K_5$	$K_{5,5}$

Table 6. Graphs of order 10 whose $rn = 7$.

Table 7 gives a list of predicted graphs of order 12 with different reconstruction numbers.

Graphs of order 12		
Reconstruction number	Graph G	Graph $\overline{G}$
4	$6K_2$ $3C_4$ $\{K_6 \longleftrightarrow^b K_6 : 2 \leq b \leq 4\}$ $RCC_{2,6}$	$K_{2,2,2,2,2,2}$ $\overline{3C_4}$ $K_{6,6}$ less b non-adjacent edges $\overline{RCC_{2,6}}$
5	$4K_3$	$K_{3,3,3,3}$
6	$3K_4$ $K_6 \longleftrightarrow^5 K_6$	$K_{4,4,4}$ $\overline{K_6 \longleftrightarrow^5 K_6}$
8	$2K_6$	$K_{6,6} \simeq RCC_{3,4}$

Table 7. Predicted graphs of order 12 whose $rn \geq 4$.

In this survey, we have pointed out that in the case of the edge-reconstruction number, certain dis connected graphs and spanning subgraphs of $K_{r,2}$ seem to be some of the prerequites needed for a graph to have large edge-reconstruction number. We have also identified a class of graphs in the form of K_{r-2} less two non-adjacent edges to have large edge-reconstruction number.

We also offer the following open problems:

- Can we find an *upper bound* for the *rn* values for the graphs of the form $K_c \stackrel{b}{\longleftrightarrow} K_c$ and $RCC_{n,j}$?
- Is there a *recursive construction* (along the lines of the lexicographic product) which increases the reconstruction number?
- Can Harary's conjecture that if G is a graph of odd prime order, then $rn(G) = 3$ be proved? Or is it possible to find a counterexample?

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