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FIXED POINT THEOREM FOR A PAIR OF *s*-α-CONTRACTIONS IN *b*-DISLOCATED METRIC SPACES

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Abstract

The purpose of this work is to prove common fixed point theorem in complete b-dislocated metric space. We extend the existing results of self mapping to a pair of s-α-contraction in b-Dislocated metric space satisfying several more conditions.

1. Introduction

Fixed point theory (FPT) finds applications in various fields like economics, engineering, physics, mathematics etc. The most familiar Banach contraction principle is the main concept to study in FPT. In 1989, Bakhtin [1] (and Czerwik [5]), developed the extension of BCP in metric space. Recently there are a number of extensions of metric space and various results are proved using these. The distance of a point with the self may not be zero, which are studied in dislocated metric space. Dislocated metric space and b-Dislocated metric space are given and studied by Hitzler and Seda [9] and Nawab Hussain et al. [10] respectively. For single valued, set-valued mappings and different types of contractions authors applied the conditions for existence and uniqueness of a fixed point in these spaces. Quasicontractions and g-quasicontractions first studied by L. B. Ciric [4], [16]

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Consider a non empty set A, a metric space (B, d) and $g, h : A \to B$ such that $d(gp, gp) \leqslant kd(hp, hq)$, for all $p, q \in A$, $0 \leqslant k < 1$ for some k.

In 1968, using the Banach contraction principle, Goebel [16] obtained a coincidence theorem for g and h on an set A using values in a complete metric space B satisfying $g(A) \subseteq h(A)$. The given condition was first observed by Machuca [15] in 1967 held down by heavy topological constraints. However, considering given condition when $A = B$ and introducing a new sequence of iterates, Jungck [12] used commutativity of g and h, shown the existence of a common fixed point. Elegancy of this result with its proof fascinated several scholars and subsequently a multitude of coincidence for maps having the full force of commutativity or restricted commutativity, were obtained. Several new applications have been suggested in various papers. ([2] [7] [21] [3]). In 1976, Jungck [12] concluded the Banach contraction principle by using commuting mappings, entrenched the perception of mappings that are weakly commuting due to Sessa [20] and produced the pair of mappings which are compatible and commutes on the set of coincidence points. Jungck and Rhoades [13] and Dhage [6] described if self-mappings pair commute at their coincidence points will be weakly compatible. Afterwards, this approach of weak commutativity is weakened and used by Singh [23], Jungck [12], Pathak [19], Mishra [17], Gairola and Whitfield [8], Pant [18], Tivari and Singh [24] and others. Here our motive is to introduce FPT for $s - \alpha$- quasicontractions with the use of double self mappings in b-dislocated metric space which satisfies different set of constraints. The following known definitions are necessary for our discussion.

2. Preliminaries (Concepts and Methods)

Definition 2.1 [25]. Consider a non-empty set B with mapping $d_l : B \times B \to [0, \infty)$ is known to be dislocated metric (d_l- metric) if it satisfies these constraints for any $p, q, r \in B$:

1. if $d_l(p, q) = 0$, then $p = q$;

2. $d_l(p, q) = d_l(q, p)$;

3. $d_l(p, q) \leqslant d_l(p, r) + d_l(r, q)$. The pair (B, d_l) is known as a dislocated metric space. But when $p = q$, $d_l(p, q)$ may not be 0.

Definition 2.2 [22]**.** Consider a sequence $\{p_n\}$ in d_l-metric space (B, d_l)

(1) if only, for $\epsilon > 0, \exists\, n_0$ belongs N s.t (such that) for all $m, n \geqslant n_0$, we get $d_l(p_m, p_n) < \epsilon$ or $\lim_{n,m\to\infty} d_l(p_n, p_m) = 0$ then it is named as Cauchy sequence,

(2) it is convergent relative to d_l $\exists$, $p \in B$ so that $d_l(p_n, p) \to 0$ as $n \to \infty$. By these circumstances, the limit of $\{p_n\}$ is called p and $p_n \to p$.

Definition 2.3 [14]**.** Consider B is a non empty set with mapping $b_d : B \to B \to [0, \infty)$ is named as b-dislocated metric provided that it satisfies the properties for any $p, q, r \in B$ and $s \geqslant 1$:

1. if $b_d(p, q) = 0$, then $p = q$;

2. $b_d(p, q) = b_d(q, p)$;

3. $b_d(p, q) \leqslant s[b_d(p, r) + b_d(r, q)]$.

The space (B, b_d) is known as b-dislocated metric space.

Definition 2.4 [25]**.** Consider (B, b_d) is a b_d- metric space, and a sequence of points in B is denoted by $\{p_n\}$. Some point $p \in B$ is known to be the limit of $\{p_n\}$ if $\lim_{n\to\infty} b_d(p_n, p) = 0$ then we assert $\{p_n\}$ is b_d- convergent to p and indicate it by $p_n \to p$ as $n \to \infty$. The limit in a b_d- metric space is unique for a b_d- convergent sequence.

Definition 2.5 [25]**.** In a b_d- metric space (B, b_d) let a sequence $\{p_n\}$ is named as b_d- Cauchy sequence iff, for $\epsilon > 0$, $\exists$, n_0 belongs N s.t for all $n, m > n_0$, we are having $b_d(p_n, p_m) < \epsilon$ or $\lim_{n,m\to\infty} b_d(p_n, p_m) = 0$. Every b_d- convergent sequence is a b_d- Cauchy sequence in a b_d- metric space.

Definition 2.6 [11]**.** Consider the self mappings pair (R, P) described on a metric space (B, d) is known as weakly-compatible : provided that the mappings which commute at their coincidence dence points i.e, $R_p = P_p$ for some $p \in B$ implies $RP_p = PR_p$.

3. Results

By the denition of quasi-contraction given by Ciric [4] we introduce the following definition for pair of self mappings on b-dislocated metric space.

Definition 3.1. Consider a complete b-dislocated metric space (B, b_d) along the parameter $s \geqslant 1$. Provided that $R, P : B \to B$ are self mappings which satisfy: $s^2 b_d(Rp, Rq) \leqslant \alpha \max\{b_d(Rp, Rq), b_d(Rp, Pp), b_d(Rq, Pq), b_d(Rp, Pq), b_d(Rq, Pp)\}$ for all p, q belongs B, $\alpha \in \left[0, \frac{1}{2}\right]$.

Then R and P are called a $s - \alpha$ quasi-contraction. Further, we show the existence of common fixed point theorem for $s - \alpha$ quasi-contraction for two mappings in a class of space which is larger than metric and b-metric spaces.

Theorem 3.2. *Consider the pair* (R, P) *of self-mappings on a complete b-dislocated metric space* (B, b_d) *s.t*

(a) $\overline{R(B)} \subseteq P(B)$

(b) $p, q \in B$ *and some* $\alpha \in \left[0, \frac{1}{2}\right)$

$$s^2 b_d(Rp, Rq) \leqslant \alpha \max\{b_d(Rp, Rq), b_d(Rp, Pp), b_d(Rq, Pq), b_d(Rp, pq), b_d(Rq, Pp)\} \tag{1}$$

(c) $\overline{R(B)}$ *is a complete subspace of B.*

Then R and P have a coincidence point. If (R, P) is weakly compatible pair, the mappings R and P has a common fixed point in (B, b_d).

Proof. Let p_0 be an arbitrary element in B. By (a). $\overline{R(B)} \subseteq P(B)$. This implies $R(B) \subseteq \overline{R(B)} \subseteq P(B)$. Therefore, we can define a sequence,

$$\{Rp_0, Rp_1, Rp_2, \ldots, Rp_n, Rp_{n+1}, \ldots\} \tag{2}$$

s.t. $Rp_n = Pp_{n+1}$ for n is non-negative integers.

Now, we present that sequence is characterized by (2) is a Cauchy sequence. Using (1) with $p = p_n$ and $q = p_{n+1}$, we have $s^2 b_d(Rp_n, Rp_{n+1}) \leqslant \alpha \max\{b_d(Rp_n, Rp_{n+1}), b_d(Rp_n, Pp_n), b_d(Rp_{n+1}, Pp_n), b_d(Rp_n, Pp_n), b_d(Rp_{n+1}, Pp_n)\}$. As $Rp_n = Pp_{n+1}$ for n is non-negative integers

We have,

$$s^2 b_d(Rp_n, Rp_{n+1}) \leqslant \alpha \max\{b_d(Rp_n, Rp_{n+1}), b_d(Rp_n, Pp_n), b_d(Rp_{n+1}, Rp_n), b_d(Rp_n, Rp_n)\ b_d(Rp_{n+1}, Rp_n)\}$$

$$s^2 b_d(Rp_n, Rp_{n+1}) \leqslant \alpha \max\{b_d(Rp_n, Rp_{n+1}), b_d(Rp_n, Pp_n), 0, b_d(Rp_{n+1}, Pp_n)\}$$

$$s^2 b_d(Rp_n, Rp_{n+1}) \leqslant \alpha \max\{b_d(Rp_n, Rp_{n+1}), b_d(Rp_n, Pp_n), b_d(Rp_{n+1}, Pp_n)\}$$

$$\leqslant \alpha \max\{b_d(Rp_n, Rp_{n+1}), b_d(Rp_n, Rp_{n-1}), b_d(Rp_{n+1}, Rp_{n-1})\}.$$

If $b_d(Rp_{n-1}, Rp_n) \leqslant b_d(Rp_n, Rp_{n+1})$ for some $n \in N$, then from the above inequality we have three cases:

case 1. if $b_d(Rp_n, Rp_{n+1})$, is maximum value on R.H.S on solving $b_d(Rp_n, Rp_{n+1}) \leqslant \frac{\alpha}{s^2} b_d(Rp_n, Rp_{n+1})$ which is a contradiction since, $\frac{\alpha}{s^2} < 1$.

case 2. if $b_d(Rp_n, Rp_{n-1})$ is maximum value on R.H.S on solving we get, $b_d(Rp_n, Rp_{n+1}) \leqslant \frac{\alpha}{s^2}(Rp_{n-1}, Rp_n)$ since, $\frac{\alpha}{s^2} < 1$. Hence, for all $n \in N$, $b_d(Rp_n, Rp_{n+1}) \leqslant b_d(Rp_{n-1}, Rp_n)$. And also by case 2, we get,

$$b_d(Rp_n, Rp_{n+1}) \leqslant \frac{\alpha}{s^2} b_d(Rp_{n-1}, Rp_n) \tag{3}$$

similarly,

$$b_d(Rp_{n-1}, Rp_n) \leqslant \frac{\alpha}{s^2} b_d(Rp_{n-2}, Rp_{n+1}) \tag{4}$$

from (3) and (4) we have , for all $n \geqslant 0$

$$b_d(Rp_n, Rp_{n+1}) \leqslant k b_d(Rp_{n-1}, Rp_n) \leqslant \ldots \leqslant k^n b_d(Rp_0, Rp_1) \tag{5}$$

$$k = \frac{\alpha}{s^2}; \ 0 \leqslant k < 1.$$

Taking $\lim_{n\to\infty}$ in (5) we get $b_d(Rp_n, Rp_{n+1}) \to 0$. Further, we show that $\{Rp_n\}$ is b_d-Cauchy sequence. Let $m > 0, n > 0$ with $m > n$, using definition (1.3)

$$b_d(Rp_n, Rp_m) \leqslant s[b_d(Rp_n, Rp_{n+1}) + b_d(Rp_{n+1}, Rp_m)]$$

$$\leqslant s b_d(Rp_n, Rp_{n+1}) + s^2 b_d(Rp_{n+1}, Rp_{n+2}) + s^3 b_d(Rp_{n+2}, Rp_{n+3}) \ldots$$

$$\leqslant s k^n b_d(Rp_0, Rp_1) + s^2 k^{n+1} b_d(Rp_0, Rp_1) + s^3 k^{n+2} b_d(Rp_0, Rp_1) + \ldots$$

$$= s k^n b_d(Rp_0, Rp_1)[1 + sk + (sk)^2 + (sk)^3 + \ldots]$$

$$\leqslant \frac{s}{1 - sk} k^n b_d(Rp_0, Rp_1).$$

Taking $\lim_{(n,m)\to\infty}$, we have, $b_d(Rp_n, Rp_m) \to 0$. As $sk < 1$. Therefore, $\{Rp_n\}$ is a b_d-Cauchy sequence in complete b-dislocated metric space (B, b_d). Since, $\overline{R(B)}$ is a complete subspace of B, we have $\lim_{n\to\infty} Rp_n = \lim_{n\to\infty} Pp_{n+1} = h \in P(B)$ therefore, $\exists l \in B$ s.t. $P(l) = h$ we claim that $Pl = Rl$. If not then using (1) with $p = l, q = p_n$, we have $s^2 b_d(Rl, Rp_n) \leqslant \alpha \max\{b_d(Rl, Rp_n), b_d(Rl, Pl), b_d(Rp_n, Pp_n), b_d(Rl, Pp_n), b_d(Rp_n, Pl)\}$. Taking $\lim_{n\to\infty}$ we get,

$$s^2 b_d(Rl, h) \leqslant \alpha \max\{b_d(Rl, h), b_d(Rl, Pl), b_d(h, h), b_d(Rl, h), b_d(h, Pl)\}$$

$$s^2 b_d(Rl, Pl) \leqslant \alpha \max\{b_d(Rl, h), b_d(Rl, Pl), 0, b_d(h, h), b_d(h, h)\}$$

$$s^2 b_d(Rl, Pl) \leqslant \alpha \max\{b_d(Rl, Pl), b_d(Rl, Pl), 0, 0, 0\}$$

$$s^2 b_d(Rl, Pl) \leqslant \alpha \ \{b_d(Rl, Pl)\}$$

$$b_d(Rl, Pl) \leqslant \frac{\alpha}{s^2} \{b_d(Rl, Pl)\}$$

which is a contradiction since $\frac{\alpha}{s^2} < 1$. Hence, $Rl = Pl$ that proves 'l' is a coincidence point of pair (R, P) we are given the pair (R, P) is weakly compatible. So we have, $Ph = PRl = RPl = Sh$. Further we present that the pair (R, P) has a common fixed point h. We claim that, $Rh = h$ again using (1) with $p = h, q = l$

$$s^2 b_d(Rh, Rl) \leqslant \alpha \max\{b_d(Rh, Rl), b_d(Rh, Ph), b_d(Rl, Pl), b_d(Rh, Pl), b_d(Rl, Ph)\}$$

$$s^2 b_d(Rh, h) \leqslant \alpha \max\{b_d(Rh, h), b_d(Rh, Ph), b_d(h, h), b_d(Sh, h), b_d(h, Th)\}$$

$$s^2 b_d(Rh, h) \leqslant \alpha \max\{b_d(Rh, h), 0, 0, b_d(Rh, h), b_d(h, Rh)\}$$

$$b_d(Rh, h) \leqslant \frac{\alpha}{s^2}\{b_d(Rh, h)\}$$

which is not possible since $\frac{\alpha}{s^2} < 1$.

Hence $Rh = h$ or $Th = h$.

$\therefore$ h is common fixed point of (R, P).

4. Conclusion

Using the definition of quasi-contraction we have introduced the pair of self mapping (R, P) in $s - \alpha$ quasi-contraction that commutes at the point of coincidence with the given pair of mapping is weakly compatible. By using given definitions the existence of a common fixed point has shown.

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