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# ON STABILITY OF A CUBIC FUNCTIONAL EQUATION IN NEUTROSOPHIC NORMED SPACES

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## Abstract

In this paper, we determine some stability results concerning the cubic functional equation $f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$ in Neutrosphic Normed Spaces.

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## 1. Introduction

Fuzzy set theory is a powerful hand set for modeling uncertainly and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, computer programming, nonlinear dynamical systems, nonlinear operators, statistical convergence etc. The fuzzy topology proved to be a very useful tool to deal with such situations where the use of classical theories breaks down. The most fascinating application of fuzzy topology in quantum particle physics arises in string and $e^{(\infty)}$-theory.

Stability problem of a functional equation was first posed by Ulam [23] which was answered by Hyers [6] and then generalized by Rassias [14] for addictive mappings and linear mappings respectively. Since then several stability problems for various functional equations have been investigated in [14] and various fuzzy stability results concerning Cauchy, Jensen and quadratic functional equations were discussed.

After a while, Smarandache [18] introduced the notion of Neutrosophic Sets [NS], which is the different kind of the notation of the classical set theory by adding an intermediate membership function. This set is a formal setting trying to measure the truth, indeterminacy and falsehood. Later on, the concepts of statistical convergence of double sequences have been analyzed in IFNS by Mursaleen and Mohiuddin [9]. Quite recently, Kirisci and Simsek [7] introduced the notion of Neutrosophic normed space and statistical convergence. Since Neutrosophic Normed Space [NNS] is a natural generalization of IFNS and statistical convergence.

In this paper, we determine some stability results concerning the cubic functional equation $f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$ in NNS. We define the Neutrosophic continuity of the cubic mappings of the function and prove that the existence of a solution for any approximately cubic mappings implies the completeness of NNS.

**Definition 1.1.** The 6-tuple $(X, \mu, \vartheta, \omega, *, \diamond)$ is said to be a Neutrosophic Normed Space (NNS) if $X$ is a vector space, $*$ and $\diamond$ are the CTN and CTC, respectively and $\mu, \vartheta, \omega$ are Normed spaces on $X \times (0, \infty)$ fulfilling the conditions below: For each $x, y \in X$ and for each $s, t > 0, \emptyset \neq 0$,

(NNS-1) $0 \leq \mu(x, t) \leq 1, 0 \leq \upsilon(x, t) \leq 1, 0 \leq \omega(x, t) \leq 1$, for all $t \in (0, \infty)$;

(NNS-2) $\mu(x, t) + \upsilon(x, t) + \omega(x, t) \leq 3$;

(NNS-3) $\mu(x, t) > 0$;

(NNS-4) $\mu(x, t) > 0$; if and only if $x = 0$;

(NNS-5) $\mu(\emptyset x, t) = \mu\left(x, \frac{1}{|\emptyset|}\right)$, for each $\emptyset \neq 0$;

(NNS-6) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$;

(NNS-7) $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous and increasing;

(NNS-8) $\lim_{t \to \infty} \mu(x, t) = 1$ and $\lim_{t \to 0} \mu(x, t) = 0$;

(NNS-9) $\upsilon(x, t) < 1$;

(NNS-10) $\upsilon(x, t) = 0$ if and only if $\vartheta = 0$;

(NNS-11) $\upsilon(\emptyset x, t) = \lambda\left(x, \frac{t}{|\emptyset|}\right)$ for each $\emptyset \neq 0$;

(NNS-12) $\upsilon(x, t) \diamond \upsilon(y, s) \geq \upsilon(x + y, t + s)$;

(NNS-13) $\upsilon(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous and increasing;

(NNS-14) $\lim_{t \to \infty} \upsilon(x, t) = 0$ and $\lim_{t \to 0} \upsilon(x, t) = 1$;

(NNS-15) $\omega(x, t) < 1$;

(NNS-16) $\omega(x, t) = 0$ if and only if $\omega = 0$;

(NNS-17) $\omega(\emptyset x, t) = \omega\left(x, \frac{t}{|\emptyset|}\right)$, for each $\emptyset \neq 0$;

(NNS-18) $\omega(x, t) \diamond \omega(y, s) \geq \omega(x + y, t + s)$;

(NNS-19) $\omega(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous and increasing;

(NNS-20) $\lim_{t \to \infty} \omega(x, t) = 0$ and $\lim_{t \to 0} \omega(x, t) = 1$;

Then $(\mu, \vartheta, \omega)$ is called Neutrosophic Norm (NN).

**Example 1.2.** Let $(X, \|\cdot\|)$ be a normed space, $a * b = ab$ and $a \diamond b = \min\{a + b, 1\}$ for all $a, b \in 0, 1$. For all $x \in X$ and every $t < 0$, consider

$$\mu(x, t) = \begin{cases} \dfrac{t}{t + \| x \|} \text{ if } t > 0 \\ 0 \text{ if } t \leq 0 \end{cases}, \; \upsilon(x, t) = \begin{cases} \dfrac{\| x \|}{t + \| x \|} \text{ if } t > 0 \\ 0 \text{ if } t \leq 0 \end{cases} \text{ and}$$

$$\omega(x, t) = \begin{cases} \dfrac{\| x \|}{t} \text{ if } t > 0 \\ 0 \text{ if } t \leq 0 \end{cases}$$

Then $(X, \mu, \vartheta, \omega, *, \diamond)$ is an NNS.

The concepts of convergence and Cauchy sequences in an NNS are studied.

Let $(X, \mu, \vartheta, \omega, *, \diamond)$ be an NNS. Then, a sequence $x = (x_k)$ is said to be Neutrosophic convergent to $L \in X$ if $\lim \mu(x_k - L, t) = 1, \lim \upsilon(x_k - L, t) = 0$ and $\omega(x_k - L, t) = 0$, for all $t > 0$. In this case we write $x_k \to L$ as $k \to \infty$.

Let $(X, \mu, \vartheta, \omega, *, \diamond)$ be an NNS. Then $x = (x_k)$ is said to be Neutrosophic Cauchy sequences if $\lim \mu(x_{k+p} - x_k, t) = 1, \lim \upsilon(x_{k+p} - x_k, t) = 0$ and $\omega(x_{k+p} - x_k, t) = 0$ for all $t > 0$ and $p = 1, 2, \ldots$.

Let $(X, \mu, \vartheta, \omega, *, \diamond)$ be an NNS. Then $(X, \mu, \vartheta, \omega, *, \diamond)$ is said to be complete if every neutrosophic Cauchy sequences if $(X, \mu, \vartheta, \omega, *, \diamond)$ neutrosophic convergent in $(X, \mu, \vartheta, \omega, *, \diamond)$.

## 2. Neutrosophic Stability

The functional equation $f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$ (2.1) is called the cubic functional equation, since the function $f(x) = cx^3$ is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. We begin with a generalized Hyers-Ulam-Rassias type theorem in NNs for the cubic functional equation.

**Theorem 2.1.** *Let X be a linear space and let* $(Z, \mu', \upsilon', \omega')$ *be an NNS. Let* $\varphi : X \times X \to Z$ *be a function such that for some* $0 < \alpha < 8$, $\mu'(\varphi(2x, 0), t) \geq \mu'(\alpha\varphi(x, 0), t)$, $\upsilon'(\varphi(2x, 0), t) \leq \upsilon'(\alpha\varphi(x, 0), t)$ *and*

$$\omega'(\varphi(2x, 0), t) \leq \omega'(\alpha\varphi(x, 0), t) \tag{2.1.1}$$

*and* $\lim_{n\to\infty} \mu'(\varphi(2^n x, 2^n y), 8^n t) = 1$, $\lim_{n\to\infty} \upsilon'(\varphi(2^n x, 2^n y), 8^n t) = 0$ *and* $\lim_{n\to\infty} \omega'(\varphi(2^n x, 2^n y), 8^n t) = 0$, *for all* $x, y$ *in* $X$ *and* $t > 0$. *Let* $(Y, \mu, \vartheta, \omega)$ *be an neutrosophic Banach space and let* $f : X \to Y$ *be a* $\emptyset$*-approximately cubic mapping in the sense that*

$$\mu(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), t) \geq \mu'(\emptyset(x, y), t),$$

$$\upsilon(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), t) \leq \upsilon'(\emptyset(x, y), t) \text{ and}$$

$$\omega(f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), t) \leq \omega'(\emptyset(x, y), t)$$

(2.1.2)

*for all* $t > 0$ *and all* $x, y \in X$. *Then there exists a unique cubic mapping* $C : X \to Y$ *such that* $\mu(C(x) - f(x), t) \geq \mu'(\varphi(x, 0), (8 - \alpha)t)$, $\upsilon(C(x) - f(x), t) \leq \upsilon'(\varphi(x, 0), (8 - \alpha)t)$ *and* $\omega(C(x) - f(x), t) \geq \omega'(\varphi(x, 0), (8 - \alpha)t)$ *for all* $x \in X$ *and all* $t > 0$. (2.1.3)

**Proof.** Put $y = 0$ in (2.1.2). Then for all $x \in X$ and $t > 0$

$\mu\left(\frac{f(2x)}{8} - f(x), \frac{t}{16}\right) \geq \mu'(\varphi(x, 0), t)$, $\upsilon\left(\frac{f(2x)}{8} - f(x), \frac{t}{16}\right) \leq \upsilon'(\varphi(x, 0), t)$ and $\omega\left(\frac{f(2x)}{8} - f(x), \frac{t}{16}\right) \leq \omega'(\varphi(x, 0), t)$ (2.1.4) Replacing $x$ by $2^n x$ in (2.1.4) and using (2.1.1) we obtain

$$\mu\left(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}, \frac{t}{16(8^n)}\right) \geq \mu'(\varphi(2^n x, 0), t) \geq \mu'\left(\varphi(x, 0), \frac{t}{\alpha^n}\right),$$

$$v\left(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}, \frac{t}{16(8^n)}\right) \leq v'(\varphi(2^n x, 0), t) \leq v'\left(\varphi(x, 0), \frac{t}{\alpha^n}\right) \quad \text{and}$$

$$\omega\left(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}, \frac{t}{16(8^n)}\right) \leq \omega'(\varphi(2^n x, 0), t) \leq \omega'\left(\varphi(x, 0), \frac{t}{\alpha^n}\right), \quad \text{for all}$$

$x \in X$, $t > 0$ and $n \geq 0$. By replacing $t$ by $\alpha^n t$, we get

$$\mu\left(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}, \frac{\alpha^n t}{16(8^n)}\right) \geq \mu'(\varphi(x, 0), t),$$

$$v\left(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}, \frac{\alpha^n t}{16(8^n)}\right) \leq v'(\varphi(x, 0), t) \text{ and}$$

$$\omega\left(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}, \frac{\alpha^n t}{16(8^n)}\right) \leq \omega'(\varphi(x, 0), t).$$

It follows from $\frac{f(2^n x)}{8^n} - f(x) = \sum_{k=0}^{n-1}\left(\frac{f(2^{k+1}x)}{8^{k+1}} - \frac{f(2^k x)}{8^k}\right)$ and (2.1.5) that

$$\mu\left(\frac{f(2^n x)}{8^n} - f(x), \sum_{k=0}^{n-1}\frac{\alpha^k t}{16(8^k)}\right) \geq \prod_{k=0}^{n-1}\mu\left(\frac{f(2^{k+1}x)}{8^{k+1}} - \frac{f(2^k x)}{8^k} \cdot \frac{\alpha^k t}{16(8^k)}\right)$$

$$\geq \mu'(\varphi(x, 0), t),$$

$$v\left(\frac{f(2^n x)}{8^n} - f(x), \sum_{k=0}^{n-1}\frac{\alpha^k t}{16(8^k)}\right) \leq \coprod_{k=0}^{n-1} v\left(\frac{f(2^{k+1}x)}{8^{k+1}} - \frac{f(2^k x)}{8^k} \cdot \frac{\alpha^k t}{16(8^k)}\right)$$

$v'(\varphi(x, 0), t)$ and

$$\omega\left(\frac{f(2^n x)}{8^n} - f(x), \sum_{k=0}^{n-1}\frac{\alpha^k t}{16(8^k)}\right) \leq \coprod_{k=0}^{n-1}\omega\left(\frac{f(2^{k+1}x)}{8^{k+1}} - \frac{f(2^k x)}{8^k} \cdot \frac{\alpha^k t}{16(8^k)}\right)$$

$$\geq \omega'(\varphi(x, 0), t) \tag{2.1.6}$$

for all $x \in X$, $t > 0$ and $n > 0$, where $\prod_{j=1}^{n} a_j = a_1 * a_2 \ldots * a_n$, $\coprod_{j=1}^{n} a_j = a_1 \diamond a_2 \diamond \ldots \diamond a_n$. By replacing $x$ with

$2^m x$ in (2.1.6) we have,

$$\mu\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, \sum_{k=0}^{n-1} \frac{\alpha^k t}{16(8)^{k+m}}\right) \geq \mu'(\varphi(2^m x, 0), t) \geq \mu'\left(\varphi(x, 0), \frac{t}{\alpha^m}\right),$$

$$v\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, \sum_{k=0}^{n-1} \frac{\alpha^k t}{16(8)^{k+m}}\right) \leq v'(\varphi(2^m x, 0), t) \leq v'\left(\varphi(x, 0), \frac{t}{\alpha^m}\right)$$

and

$$\omega\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, \sum_{k=0}^{n-1} \frac{\alpha^k t}{16(8)^{k+m}}\right) \leq \omega'(\varphi(2^m x, 0), t) \leq \omega'\left(\varphi(x, 0), \frac{t}{\alpha^m}\right).$$

Thus $\mu\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, \sum_{k=m}^{n+m-1} \frac{\alpha^k t}{16(8)^k}\right) \geq \mu'(\varphi(x, 0), t),$

$v\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, \sum_{k=m}^{n+m-1} \frac{\alpha^k t}{16(8)^k}\right) \leq v'(\varphi(x, 0), t)$ and

$\omega\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, \sum_{k=m}^{n+m-1} \frac{\alpha^k t}{16(8)^k}\right) \leq \omega'(\varphi(x, 0), t),$ for all $x \in X, t > 0,$ $m > 0$ and $n > 0$. Hence

$$\mu\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, t\right) \geq \mu'\left(\varphi(x, 0), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k t}{16(8)^k}}\right),$$

$$v\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, t\right) \leq v'\left(\varphi(x, 0), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k t}{16(8)^k}}\right) \quad \text{and}$$

$$\omega\left(\frac{f(2^{n+m}x)}{8^{n+m}} - \frac{f(2^m x)}{8^m}, t\right) \leq \omega'\left(\varphi(x, 0), \frac{t}{\sum_{k=m}^{n+m-1} \frac{\alpha^k t}{16(8)^k}}\right) \tag{2.1.7}$$

for all $x \in X, t > 0, m \geq 0$ and $n \geq 0$, since $0 < \alpha < 8$ and $\sum_{k=0}^{\infty} \left(\frac{\alpha}{8}\right)^k < \infty$ the Cauchy criterion for convergence in NNS shows that $\left(\frac{f(2^n x)}{8}\right)$ is a

Cauchy sequence in $(Y, \mu, v, \omega)$. Since $(Y, \mu, v, \omega)$ is complete, this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and $m = 0$ in (2.1.7) to obtain

$$\mu\left(\frac{f(2^n x)}{8^n} - f(x), t\right) \geq \mu'(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{16(8)^k}}),$$

$$v\left(\frac{f(2^n x)}{8^n} - f(x), t\right) \leq v'(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{16(8)^k}}) \quad \text{and}$$

$$\omega\left(\frac{f(2^n x)}{8^n} - f(x), t\right) \leq \omega'(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{16(8)^k}}),$$

for all $t > 0$ and $n > 0$. Thus, we obtain

$$\mu(C(x) - f(x), t) = \mu\left(C(x) - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right) * \mu\left(\frac{f(2^n x)}{8^n} - f(x), \frac{t}{2}\right)$$

$$\geq \mu'(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{8(8)^k}}),$$

$$v(C(x) - D(x), t) = v\left(C(x) - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right) \diamond v\left(\frac{f(2^n x)}{8^n} - f(x), \frac{t}{2}\right)$$

$$\leq v'(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{8(8)^k}}) \text{ and}$$

$$\omega(C(x) - D(x), t) = \omega\left(C(x) - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right) \diamond \omega\left(\frac{f(2^n x)}{8^n} - f(x), \frac{t}{2}\right)$$

$$\leq \omega'(\varphi(x, 0), \frac{t}{\sum_{k=0}^{n-1} \frac{\alpha^k}{8(8)^k}})$$

for large $n$. Taking the limit as $n \to \infty$ and using the definition of NNS, we get

$$\mu(C(X) - f(x), t) \geq \mu'(\varphi(x, 0), (8 - \alpha)t),$$

$v(C(X) - f(x), t) \le v'(\varphi(x, 0), (8 - \alpha)t)$ and

$\omega(C(X) - f(x), t) \le \omega'(\varphi(x, 0), (8 - \alpha)t)$

Replacing $x$ and $y$ by $2^n x$ and $2^n y$ respectively in (2.1.2) we have

$$\mu(C(x) - D(x), t) = \mu\left(\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, t\right) \ge \mu\left(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right)$$

$$* \mu\left(\frac{f(2^n x)}{8^n} - f(x), \frac{t}{2}\right) \ge \mu'\left(\varphi(2^n x, 0), \frac{8^n(8-\alpha)t}{2}\right) \ge \mu'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right)$$

$$v(C(x) - D(x), t) = v\left(\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, t\right) \le v\left(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right)$$

$$\diamond\, v\left(\frac{f(2^n x)}{8^n} - f(x), \frac{t}{2}\right) \le v'\left(\varphi(2^n x, 0), \frac{8^n(8-\alpha)t}{2}\right) \le v'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right)$$

$$\omega(C(x) - D(x), t) = \omega\left(\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, t\right) \le \omega\left(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right)$$

$$\diamond\, \omega\left(\frac{f(2^n x)}{8^n} - f(x), \frac{t}{2}\right) \le \omega'\left(\varphi(2^n x, 0), \frac{8^n(8-\alpha)t}{2}\right) \le \omega'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right)$$

Since $\lim_{n\to\infty} \frac{8^n(8-\alpha)t}{2\alpha^n} = \infty$, we get

$$\lim_{n\to\infty} \mu'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right) = 1, \lim_{n\to\infty} \vartheta'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right) = 0 \quad \text{and}$$

$$\lim_{n\to\infty} \omega'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right) = 0.$$

Therefore, $\mu(C(x) - D(x), t) = 1$, $\vartheta(C(x) - D(x), t) = 0$ and

$\omega(C(x) - D(x), t) = 0$, for all $t > 0$.

Hence $C(x) = D(x)$.

$$\mu\left(\frac{f(2^n(2x+y))}{8^n} + \frac{f(2^n(2x-y))}{8^n} - \frac{2f(2^n(x+y))}{8^n} - \frac{2f(2^n(x-y))}{8^n} - \frac{12f(2^n x)}{8^n}, t\right)$$

$\geq \mu'(\varphi(2^n x \cdot 2^n y), 8^n t),$

$$v\left(\frac{f(2^n(2x+y))}{8^n} + \frac{f(2^n(2x-y))}{8^n} - \frac{2f(2^n(x+y))}{8^n} - \frac{2f(2^n(x-y))}{8^n} - \frac{12f(2^n x)}{8^n}, t\right)$$

$\leq v'(\varphi(2^n x \cdot 2^n y), 8^n t)$ and

$$\omega\left(\frac{f(2^n(2x+y))}{8^n} + \frac{f(2^n(2x-y))}{8^n} - \frac{2f(2^n(x+y))}{8^n} - \frac{2f(2^n(x-y))}{8^n} - \frac{12f(2^n x)}{8^n}, t\right)$$

$\leq \omega'(\varphi(2^n x \cdot 2^n y), 8^n t),$

for all $x, y \in X$ and for all $t > 0$. Since $\lim_{n\to\infty} \mu'(\varphi(2^n x \cdot 2^n y), 8^n t) = 1$, $\lim_{n\to\infty} v'(\varphi(2^n x \cdot 2^n y), 8^n t) = 0$ and $\lim_{n\to\infty} \omega'(\varphi(2^n x \cdot 2^n y), 8^n t) = 0$. We observe that $C$ fulfills (2.1). To prove the uniqueness of the cubic function $C$, assume that there exists a cubic function $D : X \to Y$ which satisfies (2.1.3). For fix $x \in X$, clearly $C(2^n x) = 8^n C(x)$ and $D(2^n x) = 8^n D(x)$ for all $n \in \mathbb{N}$. It follows from (2.1.3) that

$$\mu(C(x) - D(x), t) = \mu\left(\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, t\right) \geq \mu\left(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right)$$

$$* \mu\left(\frac{f(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, \frac{t}{2}\right) \geq \mu'\left(\varphi(2^n x, 0), \frac{8^n(8-\alpha)t}{2}\right) \geq \mu'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right),$$

$$v(C(x) - D(x), t) = v\left(\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, t\right) \leq v\left(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right)$$

$$\diamond\, v\left(\frac{f(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, \frac{t}{2}\right) \leq v'\left(\varphi(2^n x, 0), \frac{8^n(8-\alpha)t}{2}\right) \leq v'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right)$$

$$\omega(C(x) - D(x), t) = \omega\left(\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, t\right) \leq \omega\left(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{t}{2}\right)$$

$$\diamond\, \omega\left(\frac{f(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, \frac{t}{2}\right) \leq \omega'\left(\varphi(2^n x, 0), \frac{8^n(8-\alpha)t}{2}\right) \leq \omega'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right)$$

Since $\lim_{n\to\infty} \frac{8^n(8-\alpha)t}{2\alpha^n} = \infty$, we get

$\lim_{n\to\infty} \mu'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right) = 1$, $\lim_{n\to\infty} \vartheta'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right) = 0$ and $\lim_{n\to\infty} \omega'\left(\varphi(x, 0), \frac{8^n(8-\alpha)t}{2\alpha^n}\right) = 0$.

Therefore, $\mu(C(x) - D(x), t) = 1$, $\vartheta(C(x) - D(x), t) = 0$ and $\omega(C(x) - D(x), t) = 0$, for all $t > 0$.

Hence $C(x) = D(x)$.

**Example 2.2.** Let $X$ be a Hilbert Space and $Z$ be a normed space. Denote by $(\mu, \vartheta, \omega)$ and $(\mu', \vartheta', \omega')$ the Neutrosophic norms given as in Example (1.2) on $X$ and $Z$, respectively. Let $\varphi : X \times X \to Z$ be defined by $(x, y) = 8(\| x \|^2 + \| y \|^2)z_0$, where $z_0$ is a fixed unit vector in $Z$. Define $f : X \to X$ by $f(x) = \| x \|^2 x + \| x \|^2 x_0$ for some unit vector $x_0 \in X$. Then $\mu(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t) \geq \frac{t}{t + 8\| x \|^2 + 8\| y \|^2}$ $= \mu'(\varphi(x, y), t)$,

$\vartheta(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t) \leq \frac{8\| x \|^2 + 8\| y \|^2}{t + 8\| x \|^2 + 8\| y \|^2}$ $\leq \vartheta'(\varphi(x, y), t)$ and

$\omega(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t) \leq \frac{8\| x \|^2 + 8\| y \|^2}{t}$ $\leq \omega'(\varphi(x, y), t)$.

Also, $\mu'(\varphi(2x, 0), t) = \frac{t}{t + 32\| x \|^2} = \mu'(4\varphi(x, 0), t)$,

$\vartheta'(\varphi(2x, 0), t) = \frac{32\| x \|^2}{t + 32\| x \|^2} = \vartheta'(4\varphi(x, 0), t)$ and

$\omega'(\varphi(2x, 0), t) = \frac{32\| x \|^2}{t} = \omega'(4\varphi(x, 0), t)$. Hence, conditions of Theorem (2.1), for $\alpha = 4$ are fulfilled. Therefore, there is a unique cubic mapping $C : X \to X$ such that $\mu(C(x) - f(x), t) \geq \mu'(\varphi(x, 0), 4t)$, $\vartheta(C(x) - f(x), t) \leq \vartheta'(\varphi(x, 0), 4t)$ and $\omega(C(x) - f(x), t) \geq \omega'(\varphi(x, 0), 4t)$ in the following theorem we consider the case $\alpha > 8$.

**Theorem 2.3.** *Let X be a linear space and Let* $(Z, \mu', \vartheta', \omega')$ *be an NNS. Let* $\varphi : X \times X \to Z$ *be a function such that for some* $\alpha > 8$,

$\mu'(\varphi(x/2, 0), t) \geq \mu'(\varphi(x, 0), \alpha t)$, $\vartheta'(\varphi(x/2, 0), t) \leq \vartheta'(\varphi(x, 0), \alpha t)$, $\omega'(\varphi(x/2), 0, t) \leq \omega'(\varphi(x, 0), \alpha t)$ *and*

$\lim_{n\to\infty} \mu'(8^n \varphi(2^{-n}x, 2^{-n}y), t) = 1$, $\lim_{n\to\infty} \vartheta'(8^n \varphi(2^{-n}x, 2^{-n}y), t) = 0$, $\lim_{n\to\infty} \omega'(8^n \varphi(2^{-n}x, 2^{-n}y), t) = 0$, *for all* $x, y \in X$ *and all* $t > 0$. *Let* $(Y, \mu, \vartheta, \omega)$ *be an Neutrosophic Banach Spaceand let* $f : X \to Y$ *be a* $\varphi$*-approximately cubic mapping in the sense of* (2.1.2). *Then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$\mu(C(x) - f(x), t) \geq \mu'(\varphi(x, 0), (\alpha - 8)t)$, $\vartheta(C(x) - f(x), t) \leq \vartheta'(\varphi(x, 0), (\alpha - 8)t)$ $\omega(C(x) - f(x), t) \leq \omega'(\varphi(x, 0), (\alpha - 8)t)$ *for all* $x \in X$ *and for all* $t > 0$.

**Proof.** The techniques are similar to that of Theorem (2.1). Hence we present a sketch of proof. Put $y = 0$ in (2.1.1), we get,

$\mu(2f(2x) - 16f(x), t) \geq \mu'(\varphi(x, 0), t)$, $\vartheta(2f(2x) - 16f(x), t) \leq \vartheta'(\varphi(x, 0), t)$ and

$\omega(2f(2x) - 16f(x), t) \leq \omega'(\varphi(x, 0), t)$, for all $x \in X$ and $t > 0$. Therefore,

$\mu\left(f(x) - 8f\left(\frac{x}{2}\right), t\right) \geq \mu'(\varphi(x, 0), 2\alpha t)$, $\vartheta\left(f(x) - 8f\left(\frac{x}{2}\right), t\right) \leq \vartheta'(\varphi(x, 0), 2\alpha t)$ and

$\omega\left(f(x) - 8f\left(\frac{x}{2}\right), t\right) \leq \omega'(\varphi(x, 0), 2\alpha t)$, for all $x \in X$ and $t > 0$. For each $x \in X$, $n \geq 0$, $m \geq 0$ and $t > 0$, we can deduce

$$\mu(8^{n+m} f(2^{-(n+m)}x) - 8^m f(2^{-m}x), t) \geq \mu'(\varphi(x, 0), \frac{t}{\sum_{k=m+1}^{n+m} \frac{\alpha^k}{16(8)^k}}),$$

$$\vartheta(8^{n+m} f(2^{-(n+m)}x) - 8^m f(2^{-m}x), t) \leq \vartheta'(\varphi(x, 0), \frac{t}{\sum_{k=m+1}^{n+m} \frac{\alpha^k}{16(8)^k}}) \text{ and}$$

$$\omega(8^{n+m} f(2^{-(n+m)}x) - 8^m f(2^{-m}x), t) \leq \omega'(\varphi(x, 0), \frac{t}{\sum_{k=m+1}^{n+m} \frac{\alpha^k}{16(8)^k}}). \qquad (2.3.1)$$

Thus, $(8^m f(2^{-m}x))$ is a Cauchy sequence in Neutrosophic Banach Space. Therefore, there is a function $C : X \to Y$ defined by $C(x) = \lim_{n\to\infty} 8^n f(2^{-n}x)$, (2.3.1) with $m = 0$ implies $\mu(C(x) - f(x), t) \geq \mu'(\varphi(x, 0), (\alpha - 8)t)$, $\vartheta(C(x) - f(x), t) \leq \vartheta'(\varphi(x, 0), (\alpha - 8)t)$ and

$\omega(c(x) - f(x), t) \leq \omega'(\varphi(x, 0), (\alpha - 8)t)$, for all $x \in X$ and for all $t > 0$.

## 3. Conclusion

We linked here two different disciplines, namely, the fuzzy spaces and functional equations. We established Hyers-Ulam-Rassias stability of a cubic functional equation $f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$ in the set of NNS.

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