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SPARSE GRAPH MATCHING OF AZTEC DIAMOND GRAPH

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Abstract

One of the important concepts of Graph Theory is Matching Theory. Several Concepts on Matching Theory has been dealt in [1, 2, and 3]. The technique of maximum matching on directed graphs have been studied in [1]. In this paper, a new approach of forming Aztec diamond using Domino tiling is studied and the Aztec Diamond graph is formed by dimer covering which will be a sparse graph. The matching of sparse graph is found based on geometric multiplicity of Eigen values using exact controllability network.

2020 Mathematics Subject Classification: 05C70

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1. Introduction

One of the important concepts of Graph Theory is Matching Theory. Under this topic we discuss the method to find maximum matching of an undirected Aztec diamond graph using Largest Geometric Multiplicity of Eigen values. Maximum matching nodes can be obtained using the largest geometric multiplicity through the transpose of adjacency matrix. In this paper an Aztec diamond graph of order N is the dual graph of Aztec diamond of order N which is obtained from Dominoes by Domino tilling. Basic idea of this graph is obtained from [1]. Maximum Matching based on largest geometric multiplicity of an undirected graph is obtained from [3].

Notation: $G_z = (V_z, E_z)$, where G_z is the Undirected Aztec diamond graph, V_z be the vertices (Corners of the unit squares) and E_z is the Edges (lines joining the vertices).

2. Preliminaries

Definition 2.1. (Size): The size of undirected Aztec graph is the number of nodes in the graph G_z and it is denoted by $|G_z| = N$.

Definition 2.2. (Geometric Multiplicity (Aztec graph)): The largest number of linearly independent Eigen vectors corresponding to an Eigen value is known as Geometric Multiplicity.

Definition 2.3. (Matching of an undirected graph): The independent set of edges where no two share a node is matching M of an undirected graph.

Definition 2.4. (Matching Node): If a node is incident to an edge then it is matched, otherwise it is unmatched.

Definition 2.5. (Sparse graph): A graph is called Sparse if the number of edges is much less than the possible number of edges.

Note: An Undirected graph can have at most $\frac{n(n-1)}{2}$ edges.

Definition 2.6. (Maximum Matching of an undirected graph): The Matching of Maximum cardinality among all matching is known as Maximum Matching M^* if all the nodes are matched then the Maximum Matching M^*

is a perfect matching. The edges corresponding to the matched nodes are the matched edges.

Definition 2.7. (Domino): A Domino is the union of two adjacent units square [4].

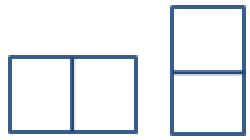


Figure 1. Domino.

Definition 2.8. (Domino Tiling): A Domino Tiling is a set of dominoes whose interior is disjoint and the union is the whole Aztec Diamond [4].

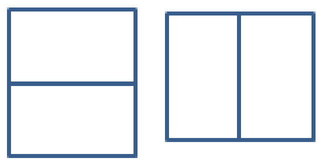
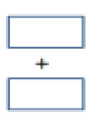
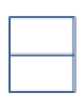
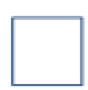
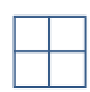


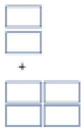
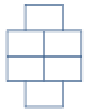
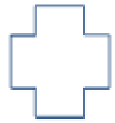
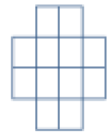
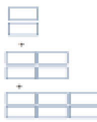
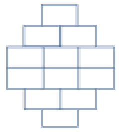
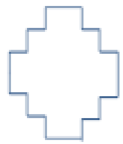
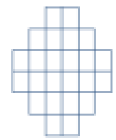
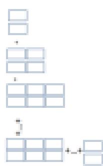
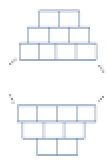
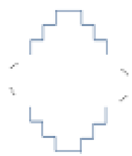
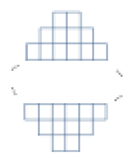
Figure 2. Domino Tiling.

3. Aztec Diamond Using Domino Tiling's

The Domino tiling of Domino the union of two adjacent squares forms a perfect Dominoes covering of an Aztec diamond of order N if the interior is disjoint without any overlap. Then the interior of the covering is filled by adjacent unit squares forms an undirected Aztec Diamond of order N

The Aztec diamond of order N is illustrated below:

No. of Domino		Domino tiling	Dominoes covering of Aztec Diamond of order		Aztec Diamond of order	
	2			1		1

	6			2		2
	12			3		3
				N		N

Definition 3.1. (Diamond graph): The Aztec diamond graph dual of the Aztec diamond is formed by dimer covering of each domino tiling of the dual graph of the Aztec diamond.

Definition 3.2. (Dimer covering): A Dimer covering is a subset of edges so that each vertex is incident to only one edge.

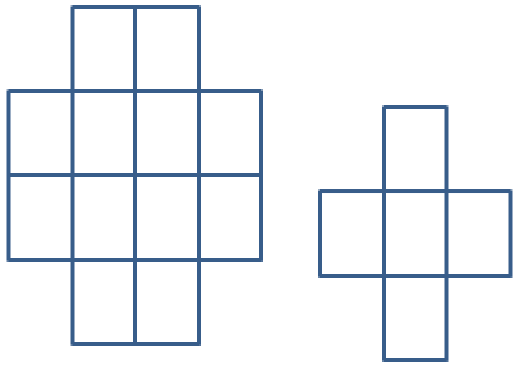


Figure 3. Aztec diamond graph of the Aztec diamond of order 2.

4. Largest Geometric Multiplicity

The minimum number of driver's node N_D of an undirected graph is obtained by the largest geometric multiplicity $\mu(\lambda_i)$ of the Eigen value λ_i of A_i

$$\mu(\lambda_i) = \dim V_{\lambda_j} = N - rank\{\lambda_j I_N - A\} \tag{4.1}$$

Where $\lambda_j = 1, 2, 3, \ldots, N$ represent the distinct Eigen values of A and I_N is the unit matrix with the same as A.

$$N_D = \max\{\mu(\lambda_j)\} \tag{4.2}$$

4.1. Maximum matching of an Undirected Graph based on Largest Geometric Multiplicity

From the condition explained in [2] for any undirected graph in general $\mu(\lambda_j)$ is the largest geometric multiplicity of the Eigen values λ_j of A. Let matrix A' be the column canonical form of matrix $\lambda_j I_N - A$. Then Linearly independent rows in A' are matched nodes and linearly dependent rows are unmatched nodes.

5. Theorem. *The Aztec diamond graph of order N dual to the Aztec diamond of order N is perfectly matched if the graph is sparse and the rows of its adjacency matrix are linearly independent*.

Proof. Consider an undirected Aztec diamond graph of order $N = n_i, n_i \geq 2$ which is formed by adjacent squares. The vertices of the graph is the squares and the edges that are incident on the vertices without any self-loops. In this graph the number edges will be much less than the possible number of edges (Sparse graph) and in the adjacency matrix all rows will be linearly independent then the expected Eigen values $\lambda_j (j = 1, 2, 3 \ldots N)$ of the adjacency matrix A will be zero.

$$E(\lambda) = \frac{1}{N} \sum_{j=1}^{N} \lambda_j = \frac{1}{N} \sum_{j=1}^{N} a_{jj} = 0 \tag{5.1}$$

If the rows are linearly independent then by equation (4.1) the Geometric Multiplicity $\mu(\lambda_j)$ corresponding to the Eigen values $\lambda_j = 0$ is also zero.

$$rank\{\lambda_j I_N - A\} = rank(-A) = rank(A) = N \tag{5.2}$$

(N number of non-zero rows)

Therefore, $\mu(\lambda_j) = N - rank\{\lambda_j I_N - A\} = N - N = 0$ (5.3)

Thus, all the nodes (N) are matched nodes, since according to maximum matching of an undirected graph the largest Geometric multiplicity $\mu(\lambda_j)$ is related to unmatched nodes and if it is zero means all the nodes are matched.

In general, the maximum matching of an undirected Aztec diamond graph of order N with V vertices formed by the Recurrence Relation as,

$$F_n = F_{n-1} + 4n, \; F_2 = 12, \; n \geq 3 \tag{5.4}$$

Edges $E = 4n_i^2, n_i \geq 2$ (5.5)

Then the graph without self-loops based on its Largest Geometric multiplicity will have a perfect matching with maximum number of matched edges $\frac{N}{2}$ and exactly two .perfectly matched graph for each graph of order N if all the rows of its adjacency matrix A is linearly independent.

The above proof is illustrated by an undirected Aztec diamond graph by choosing the order of graph as N.

Consider an undirected Aztec diamond graph of order N.

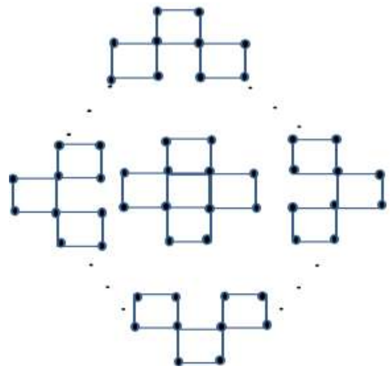


Figure 4. Aztec diamond graph of order N.

5.1 Matrix Form. The Adjacency matrix is, $A \in R_{N \times N}$ of a graph (V, E) with vertices.

$V\{v_1, v_2, \ldots, v_N\}$ is given by

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 0 & 1 & 0 & . & . & . & 0 \\ 1 & 0 & 0 & 1 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & 1 & 1 & 0 & 0 & 1 \\ 0 & . & . & . & 0 & 1 & 0 & 0 & 1 \\ 0 & . & . & . & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

It is a sparse graph without self-loops and from the Theorem 5 it is known that the expected mean Eigen value is zero (i. e) $\lambda_j = 0$.

5.2. Largest Multiplicity of a Sparse Graph. The Eigen value zero corresponds to the largest geometric multiplicity. Each Eigen value of A has geometric multiplicity exactly 0 and each row in A is linearly independent. The nodes corresponding to the linearly independent rows are matched nodes with numerical.

$$rank(\lambda_m I_N - A) = rank(-A) = rank(A) = F_n \tag{5.6}$$

Therefore, all nodes are perfectly matched.

5.3. Perfectly Matched Edges: The Perfectly Matched Edges for the F_n matched nodes are $\frac{F_n}{2}$ given in red lines. It is given in a way that no two edges share a node. Figure 3 shows that there are 2 different perfectly matched graphs for an undirected Aztec diamond graph with F_n nodes and $E = 4n_i^2$ edges.

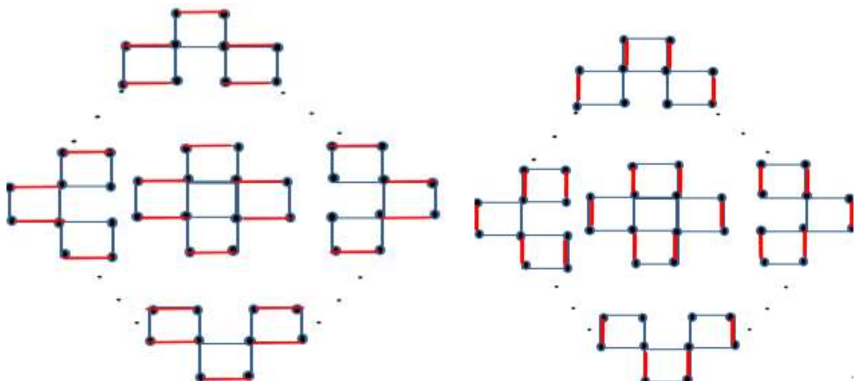
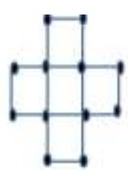
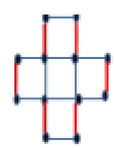
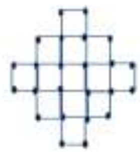
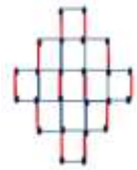
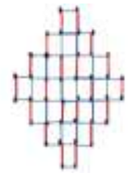
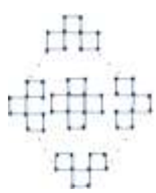
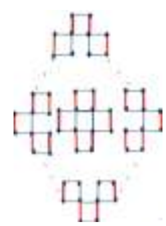


Figure 5. Perfectly matched edges for order.

The Aztec diamond graph for $N = 2, 3, 4, 5, \ldots$ is illustrated below:

Order	Aztec diamond of order $N = n_i, n_i \geq 2$	No. of vertices	No. of edges	Perfectly matched edges	Order
2		12	16		2

3		24	36		3
4		40	64		4
5		60	100		5
⋮	⋮	⋮	⋮	⋮	⋮
N		$F_n = F_{n-1}$ $+4n$ for $F_n = 12,\ n \geq 3$	$E = 4n_i^2,$ $n_i \geq 2$		N

4. Conclusion

In this topic the new method for finding maximum matched nodes of an undirected Aztec diamond graph based on largest geometric multiplicity of Eigen values. The result proved in this paper is for an undirected Aztec diamond (sparse) graph if the rows of its adjacent matrix are linearly independent then based on geometric multiplicity it gives a perfectly matched nodes and the corresponding matched edges. Perfect Matching of an undirected Aztec diamond graph for order $N = n_i$, $n_i \geq 2$ and the number of vertices formed as a Recurrence relation $F_n = F_{n-1} + 4n$ for $F_n = 12$, $n \geq 3$, the corresponding edges $E = 4n_i^2$ and the number perfectly matched graph is 2 discussed and calculated in this paper.

References

[1] Yunyun Yang and Gang Xie, Maximum Matchings of a Digraph Based on the Largest Geometric Multiplicity, Hindawi Publishing Corporation, Vol. 2016, Article ID 4702387.

[2] Y.-Y. Liu, J.-J. Slotine and A.-L. Barabási, Controllability of complex networks, Nature 473(7346) (2011), 167-173.

[3] Z. Z. Yuan, C. Zhao, Z. R. Di, W.-X. Wang and Y.-C. Lai, Exact controllability of complex networks, Nature Communications Article 2447 4 (2013), 1-12.

[4] Juanjo Rue, Domino tiling's of the Aztec Diamond, Snapshots of modern Mathematics from Oberwolfach 16 (2015), 1-10.

[5] M. Antony, S. Kalaiarasi, P. Tamizharasi, Cordial Labeling of n chain Aztec diamond graphs, UGC Sero, MRP-5873/15.

[6] Eric W. Weisstein, Domino, from Mathworld Resource.

[7] Stephen Andrilli and David Hecker, Elementary Linear Algebra (Fifth Edition), 2016, ISBN978-0-12- 00853-9.

[8] J. Jini and S. Hemalatha, Perfect Matching of Bridge Graph based on Geometric and Algebraic Multiplicity, Journal of Xidian University, ISSN NO: 1001-2400, Volume 14, ISSUE 11, 2020.

[9] Lowell W. Beineke and Robin J. Wilson, Topics in algebraic graph theory - edited by academic Consultant Peter J. Cameron.