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TOPP-LEONE POWER RAYLEIGH DISTRIBUTION WITH PROPERTIES AND APPLICATION IN ENGINEERING SCIENCE

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Abstract

Standard distributions do not essentially have a reasonable fit to all kinds of data sets, implying the need to make generalisations of standard distributions in order to improve their usefulness in data modelling. In this paper, we developed a new generalization of power Rayleigh distribution by employing the Topp-Leone generated family of distributions. The distribution which has been developed is known as the Topp-Leone power distribution (TLPRD). The distinct structural properties of the formulated distribution including moments, moment generating function, incomplete moments, order statistics, different measure of entropies, mean deviations have been discussed. In addition expressions for survival function, hazard rate function, reverse hazard rate function and mean residual function are obtained explicitly. The behaviour of probability density function (pdf) and cumulative distribution function (cdf) are illustrated through different graphs. The estimation of the established distribution parameters are performed by maximum likelihood estimation method. A simulation analysis has been carried out to evaluate and compare the effectiveness of estimators in terms of their bias, variance and mean square error (MSE). Eventually the versatility of the established distribution is examined through real life data set related to engineering science.

1. Introduction

Probability distributions are useful for inferring statistical inferences and interpreting data. These findings may be used to make some well-informed

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decisions. As a result, having a distribution that accurately represents the data is important. Various classical distributions have been widely used for the interpretation of real-world data observations and data modelling in a range of contexts, such as biology, architecture, economics, medicine, and actuarial science. However, most of these generic distributions have drawbacks in correctly fitting any actual data. As a result, it is critical to determine which distribution should be used to model the results. As a result, researchers are constantly attempting to expand existing classical distributions in order to enhance their goodness-of-fit and achieve more versatility in modelling data.

The Rayleigh distribution, named after the Lord Rayleigh, is a continuous probability distribution. Due to its wide range of applications, researchers have extended Rayleigh distribution for instance exponentiated Rayleigh distribution by Voda [15], transmuted Rayleigh distribution by Faton Merovci [8], transmuted generalized Rayleigh distribution by Faton Merovci [10], Weibull-Rayleigh distribution by Faton Merovci [10], transmuted inverse Rayleigh distribution by Afaq et al. [4], compound Poisson Rayleigh Burr-XII distribution by Ibrahim M [12] weighted Rayleigh distribution by odd Lindley- Rayleigh distribution by Terna Godfrey Ieren [14], odd generalized exponential Rayleigh distribution by Albert Luguterah [2], Topp-Leone Rayleigh distribution by Fatoki olayode [11], an extension of Rayleigh distribution by Ateeq K. et al. [5], new generalisation of Rayleigh distribution by A. A. Bhat et al. [7]. The probability density function (pdf) of Rayleigh distribution with scale parameter α is defined by

$$h(y, \alpha) = \alpha y e^{-\frac{\alpha}{2} y^2}; y > 0, \alpha > 0 \tag{1.1}$$

In this paper, we attempt to formulate an extension of power Rayleigh distribution which was proposed by A. A. Bhat et al. [7], by employing Topp-Leone generated family of distributions. The cumulative distribution function (cdf) of power Rayleigh distribution is given as

$$G(y, \alpha, \beta) = 1 - e^{-\frac{\alpha}{2} y^{2\beta}}; y > 0, \alpha, \beta > 0 \tag{1.2}$$

The associated probability density function (pdf) of power Rayleigh distribution is given as

$$g(y, \alpha, \beta) = \alpha\beta y^{2\beta-1} e^{-\frac{\alpha}{2} y^{2\beta}}; y > 0, \alpha, \beta > 0 \quad (1.3)$$

The paper is onwards outlined as; in section 2, we formulate the proposed distribution and illustrate the behaviour of pdf and cdf through different graphs. In section 3, several basic statistical properties have been studied. In section 4, explicit expression of ageing properties have been derived and through various graphical illustrations, the behaviour of these expressions has been shown. In section 5, the mean deviations about mean and median have been derived. In section 6 and 7, two different entropies of the distribution have been studied. In section 8, we provide order statistics of the distribution. In section 9, the estimation of the parameters has been performed by applied maximum likelihood estimation. In section 10, simulation procedure has been carried out. In section 11, real data set is presented to illustrate the versatility of the formulated distribution. Section 12, ends with concluding remarks.

2. Topp-Leone Power Rayleigh Distribution (TLPRD)

Let us suppose $G(y, \xi)$ denotes the distribution function of baseline distribution, which depends on parameter vector ξ. Then the Topp-Leone generated family of distribution proposed by Ali et al. (2016), is given as

$$F(y, \xi) = \int_0^{G(y, \xi)} 2\theta(1-y)(2y-y^2)^{\theta-1} dy$$

$$= [G(y, \xi)]^{\theta}[2 - G(y, \xi)]^{\theta}$$

$$= [1 - (\overline{G}(y, \xi))^2]^{\theta}; y > 0, \xi, \theta > 0 \quad (2.1)$$

The associated probability density function of TLPRD is given by

$$f(y, \xi) = 2\theta g(y, \xi)\overline{G}(y, \xi)[1 - (\overline{G}(y, \xi))^2]^{\theta-1}; y > 0, \xi, \theta > 0, \quad (2.2)$$

where

$$g(y, \xi) = G'(y, \xi) \text{ and } \overline{G}(y, \xi) = 1 - G(y, \xi)$$

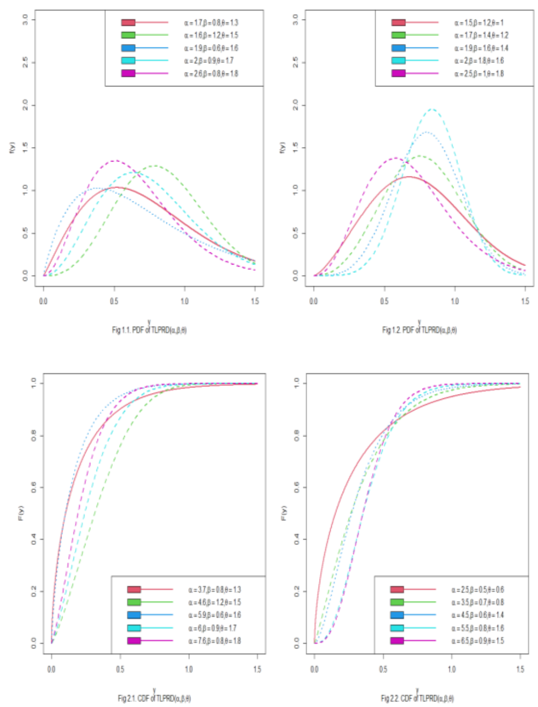
Using equations (1.2) and (1.3) in equation (2.1), we obtain the cumulative distribution function (cdf) of Topp-Leone power Rayleigh distribution (TLPRD).

$$F(y, \alpha, \beta, \theta) = (1 - e^{-\alpha y^{2\beta}}); y > 0, \alpha, \beta, \theta > 0 \tag{2.3}$$

The associated probability density function (pdf) of TLPRD is given by

$$f(y, \alpha, \beta, \theta) = 2\alpha\beta\theta y^{2\beta-1} e^{-\alpha y^{2\beta}} (1 - e^{-\alpha y^{2\beta}})^{\theta-1}; y > 0, \alpha, \beta, \theta, > 0 \tag{2.4}$$

Figure (1.1), (1.2), (2.1) and (2.2) expounds different shapes of pdf and cdf of TLPRD for various values of parameters respectively.



2.1. Linear Representation of TLPRD

The linear representation of TLPRD can be obtained by applying following Taylor's expansion

$$e^{-m\theta} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (m\,\theta)^i$$

and

$$(1-x)^{\alpha-1} = \sum_{j=0}^{\infty}(-1)^j\binom{a-1}{j}x^j$$

Now equation (2.4) can be written as

$$f(y, \alpha, \beta, \theta) = 2\alpha\beta\theta y^{2\beta-1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}(\alpha y^{2\beta})^i\sum_{j=0}^{\infty}(-1)^{-1}\binom{\theta-1}{j}(e^{-\alpha y^{2\beta}})^j$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2(-1)^{i+j}}{i!}\binom{\theta-1}{j}\alpha^{i+1}\beta\theta y^{2\beta(i-1)-1}e^{-j\alpha y^{2\beta}}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}v_{i,j}2\alpha\beta y^{2\beta(i+1)}e - j\alpha y^{2\beta}, \tag{2.5}$$

where $v_{i,j} = \frac{(-1)^{i+j}}{i!}\binom{\theta-1}{j}\alpha^j\theta$

3. Statistical Properties of TLPRD

This section is devoted to study and derive various statistical properties of formulated distribution. They are as follows

3.1. Asymptotic behaviour

This section explores the limiting behaviour of TLPRD. That is by assigning the limit to pdf and cdf of TLPRD. This is described in the following way

For the pdf

$$\lim_{y\to\infty} f(y, \alpha, \beta, \theta) = \lim_{y\to\infty} 2\alpha\beta\theta y^{2\beta-1}e^{-\alpha y^{2\beta}}(1-e^{-\alpha y^{2\beta}})^{\theta-1}$$

$$= 2\alpha\beta\theta(\infty)^{2\beta-1}e^{-\alpha(\infty)^{2\beta}}(1-e^{-\alpha(\infty)^{2\beta}})^{\theta-1} = 0$$

$$\lim_{y\to 0} f(y, \alpha, \beta, \theta) = \lim_{y\to 0} 2\alpha\beta\theta y^{2\beta-1}e^{-\alpha y^{2\beta}}(1-e^{-\alpha y^{2\beta}})^{\theta-1}$$

$$= 2\alpha\beta\theta(0)^{2\beta-1}e^{-\alpha(0)^{2\beta}}(1 - e^{-\alpha(0)^{2\beta}})^{\theta-1} = 0$$

For the cdf

$$\lim_{y\to\infty} F(y, \alpha, \beta, \theta) = \lim_{y\to\infty}(1 - e^{-\alpha y^{2\beta}})^{\theta}$$

$$= (1 - e^{-\alpha(\infty)^{2\beta}})^{\theta} = 1$$

$$\lim_{y\to 0} F(y, \alpha, \beta, \theta) = \lim_{y\to 0}(1 - e^{-\alpha y^{2\beta}})^{\theta}$$

$$= (1 - e^{-\alpha(0)^{2\beta}})^{\theta} = 0$$

According to the aforementioned derivations, the formulated distribution is unimodal and a true probability distribution.

3.2. Moments of TLPRD

Let Y be a random variable follows TLPRD. Then r^{th} moment denoted by μ'_r is given as

$$\mu'_r = E(Y^r) = \int_0^{\infty} y^r f(y, \alpha, \beta, \theta)dy$$

Using equation (2.5), we have

$$\mu'_r = \int_0^{\infty} y^r \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v_{i,j} 2\alpha\beta y^{2\beta(i+1)-1} e^{-j\alpha y^{2\beta}} dy$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v_{i,j} 2\alpha\beta g \int_0^{\infty} y^{2\beta(i+1)+r-1} e^{-j\alpha y^{2\beta}} dy$$

Making substitution $j\alpha y^{2\beta} = z$ and $0 < z < \infty$, we have

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{v_{i,j}}{j}\int_0^{\infty} \left(\left(\frac{z}{j\alpha}\right)^{\frac{1}{2\beta}}\right)^{2\beta i + r} e^{-z}dz$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{v_{i,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+r}{2\beta}}} \int_0^{\infty} z^{\frac{2\beta i+r}{2\beta}} e^{-z} dz$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{v_{i,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+r}{2\beta}}} \int_0^{\infty} z^{\frac{2\beta(i+r)+r}{2\beta}} e^{-z} dz$$

After solving the integral, we get

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{v_{i,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+r}{2\beta}}} \Gamma\left(\frac{2\beta(i+1)+r}{\beta}\right)$$

Substituting $r = 1, 2, 3, 4$ we obtain first four moments of the distribution about origin. The variance (σ^2), skewness $(\sqrt{\beta_1})$, kurtosis (β_2), coefficient of variation (C.V) and index of dispersion (γ) of TLPRD can be obtained by using following formulae

$$\sigma^2 = \mu'_2 - (\mu'_1)^2, \ \sqrt{\beta_1} = \frac{\mu'_3 - 3\mu'_2\mu'_1 + (\mu'_1)^3}{\sigma^3}$$

$$\beta_2 = \frac{\mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4}{\sigma^4}, \ C.V = \frac{\sqrt{\mu'_2 - (\mu'_1)^2}}{\mu'_1}, \ \gamma = \frac{\mu'_2 - (\mu'_1)^2}{\mu'_1}$$

3.3. Moment generating function of TLPRD

Let Y be a random variable follows TLPRD. Then the moment generating function of the distribution denoted by $M_\gamma(t)$ is given

$$M_Y(t) = E(e^{ty}) = \int_0^{\infty} e^{ty} f(y, \alpha, \beta, \theta) dy$$

Using Taylor's series

$$= \int_0^{\infty} \left(1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \dots\right) f(y, \alpha, \beta, \theta) dy$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} y^r f(y, \alpha, \beta, \theta) dy$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(Y^r)$$

$$M_Y(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r}{r!} \frac{v_{i,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+r}{2\beta}}} \Gamma\left(\frac{2\beta(i+1)+r}{2\beta}\right)$$

The characteristics function of the TLPRD denoted as $\phi_Y(t)$ can be obtained by replacing $t = lt$, $l = \sqrt{-1}$ is given by

$$\phi_Y(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(lt)^r}{r!} \frac{v_{i,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+r}{2\beta}}} \Gamma\left(\frac{2\beta(i+1)+r}{2\beta}\right)$$

3.4. Incomplete moments of TLPRD

The q^{th} incomplete moment of TLPRD about origin is given by

$$T_q(s) = \int_0^{s} y^s f(y, \alpha, \beta, \theta) dy$$

Using equation (2.5), we have

$$T_q(s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} v_{i,j} 2\alpha\beta \int_0^{s} y^{2\beta(i+1)+s-1} e^{-j\alpha y^{2\beta}} dy$$

Making substitution $j\alpha y^{2\beta} = z$ and $0 < z < j\alpha s^{2\beta}$, we have

$$T_q(s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{v_{i,j}}{j} \int_0^{j\alpha s^{2\beta}} \left(\left(\frac{z}{j\alpha}\right)^{\frac{1}{2\beta}}\right)^{2\beta i+s} e^{-z} dy$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{v_{i,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+s}{2\beta}}} \int_0^{j\alpha s^{2\beta}} z^{\left(\frac{2\beta i+s}{2\beta}+1\right)-1} e^{-z} dy$$

After solving the integral, we get

$$T_q(s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{v_{i,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+s}{2\beta}}} \gamma\left(\frac{2\beta(i+1)+s}{2\beta}, j\alpha s^{2\beta}\right)$$

3.5. Quantile function of TLPRD

The quantile function of any distribution can be put in the following form

$$Q(u) = Y_q = F^{-1}(u)$$

where $Q(u)$ denotes the quantile function of $F(y)$ for $u \in (0, 1)$.

Let us suppose

$$F(y) = \left(1 - e^{-\alpha y^{2\beta}}\right)^{\dot{\theta}} = u \tag{3.1}$$

After simplifying equation (3.1), we obtain quantile function of TLPRD as

$$Q(u) = Y_q\left[\frac{-1}{\alpha}\log\left(1 - u^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2\beta}}$$

4. Reliability Analysis of TLPRD

Suppose Y be a continuous random variable with cdf $F(y)$, $y \geq 0$. Then its reliability function which is also called survival function is defined as

$$S(y) = p_r(Y > y) = \int_y^{\infty} f(y)dy = 1 - F(y)$$

The survival function of TLPRD is given as

$$S(y, \alpha, \beta, \theta) = 1 - F(y, \alpha, \beta, \theta) \tag{4.1}$$

Using equation (2.3) in equation (4.1), we get

$$S(y, \alpha, \beta, \theta) = 1 - \left(1 - e^{-\alpha y^{2\beta}}\right)^{\theta} \tag{4.2}$$

The hazard rate function of a random variable y is given as

$$h(y, \alpha, \beta, \theta) = \frac{f(y, \alpha, \beta, \theta)}{S(y, \alpha, \beta, \theta)} \tag{4.3}$$

Using equation (2.4) and (4.2) in equation (4.3), we obtain hazard rate function of TLPRD

$$h(y, \alpha, \beta, \theta) = \frac{2\alpha\beta\theta\ y^{2\beta-1}e^{-\alpha y^{2\beta}}\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta-1}}{1-\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}}$$

Reverse hazard rate function of random variable y is given by

$$h_r(y, \alpha, \beta, \theta) = \frac{f(y, \alpha, \beta, \theta)}{F(y, \alpha, \beta, \theta)} \tag{4.4}$$

Using equations (2.3) and (2.4) in equation (4.4), we obtain reverse hazard rate function of TLPRD

$$h_r(y, \alpha, \beta, \theta) = \frac{2\alpha\beta\theta\ y^{2\beta-1}e^{-\alpha y^{2\beta}}}{1-e^{-\alpha y^{2\beta}}}$$

The mean residual function of random variable y denoted by $m(y)$, can be obtained as

$$m(y, \alpha, \beta, \theta) = \frac{1}{S(y, \alpha, \beta, \theta)}\int_y^\infty tf(t, \alpha, \beta, \theta)\,dt - y$$

Using equation (2.5) in above integral, we obtain the mean residual function of TPLPRD

$$m(y, \alpha, \beta, \theta) = \frac{1}{1-\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}}\int_y^\infty t\sum_{i=0}^\infty\sum_{j=0}^\infty v_{i,\,j}2\alpha\beta t^{2\beta(i+1)-1}e^{-j\alpha y^{2\beta}}dt - y$$

$$= \sum_{i=0}^\infty\sum_{j=0}^\infty \frac{v_{ij}2\alpha\beta}{1-\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}}\int_y^\infty t^{2\beta(i+1)}e^{-j\alpha y^{2\beta}}dt - y$$

Making substitution $j\alpha y^{2\beta} = z$ and $j\alpha y^{2\beta} < z < \infty$, we have

$$m(y, \alpha, \beta, \theta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{v_{i,\,j}\alpha}{\left(1-\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}\right)(j\alpha)^{\frac{2\beta(i+1)+1}{2\beta}}} \int_{j\alpha y^{2\beta}}^{\infty} z^{\frac{2\beta(i+1)+1}{2\beta}} e^{-z} dz - y$$

After solving the integral, we obtain

$$m(y, \alpha, \beta, \theta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{v_{i,\,j}\alpha}{\left(1-\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}\right)(j\alpha)^{\frac{2\beta(i+1)+1}{2\beta}}} \Gamma\left(\frac{2\beta(i+1)+1}{2\beta},\ j\alpha y^{2\beta}\right) - y$$

The following figures (3.1), (3.2), (4.1), (4.2), (5.1) and (5.2) expounds the shapes of survival function, hazard rate function and reverse hazard function respectively for different values of parameters

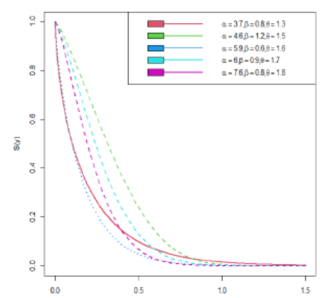


Fig 3.1. SF of TLPRD(α,β,θ)

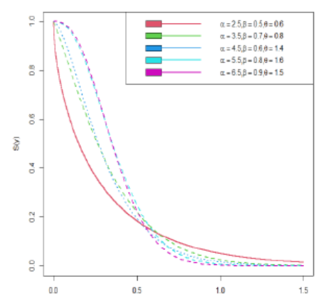


Fig 3.2. SF of TLPRD(α,β,θ)

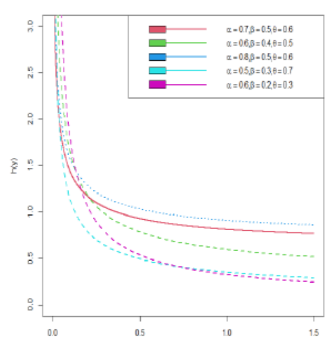


Fig 4.1. HF of TLPRD(α,β,θ)

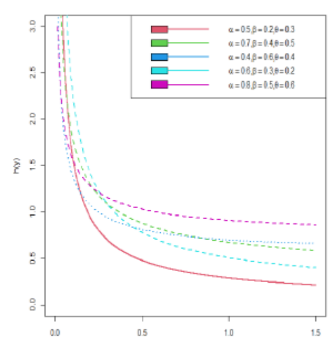


Fig 4.2. HF of TLPRD(α,β,θ)

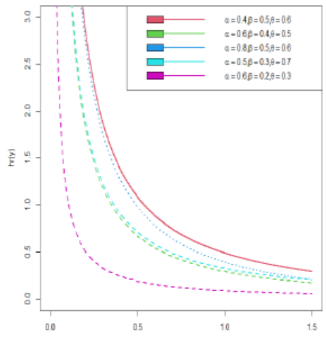


Fig 5.1. RHF of TLPRD(α, β, θ)

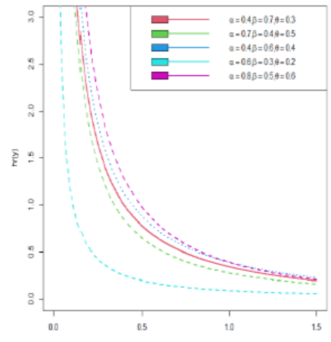


Fig 5.2. RHF of TLPRD(α, β, θ)

5. Mean Deviation from Mean and Median of TLPRD

The quantity of scattering in a population is evidently measured to some extent by the totality of the deviations.

Let Y be a random variable from TLPRD with mean μ. Then the mean deviation from mean is defined as.

$$D(\mu) = E(|Y - \mu|)$$

$$= \int_0^\infty |Y - \mu| f(y)\, dy$$

$$= \int_0^\mu (\mu - y) f(y)\, dy + \int_\mu^\infty (y - \mu) f(y)\, dy$$

$$= 2\mu F(\mu) - 2\int_0^\mu y f(y)\, dy \tag{5.1}$$

Now

$$\int_0^\mu y f(y)\, dy = \sum_{i=0}^\infty \sum_{j=0}^\infty v_{i,\, j} 2\alpha\beta \int_0^\mu y^{2\beta(i+1)} e^{-j\alpha y^{2\beta}} dy$$

Making substitution $j\alpha y^{2\beta} = z$ and $0 < z < j\alpha\mu^{2\beta}$, we have

$$\int_0^{\mu} yf(y)\,dy = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{v_{i,\,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+1}{2\beta}}}\int_0^{j\alpha\mu^{2\beta}} z^{\frac{2\beta(i+1)+1}{2\beta}} e^{-z}dy$$

After solving the above integral, we get

$$\int_0^{\mu} yf(y)\,dy = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{v_{i,\,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+1}{2\beta}}}\gamma\left(\frac{2\beta(i+1)+1}{2\beta},\ j\alpha\mu^{2\beta}\right) \tag{5.2}$$

Substituting equation (5.2) in equation (5.1), we get

$$D(\mu) = 2\mu\left(1 - e^{-\alpha\mu^{2\beta}}\right)^{\theta} - 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{v_{i,\,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+1}{2\beta}}}\gamma\left(\frac{2\beta(i+1)+1}{2\beta},\ j\alpha\mu^{2\beta}\right)$$

Let Y be a random variable from TLPRD with median M. Then the mean deviation from median is defined as.

$$\begin{aligned} D(M) &= E(|\,Y - M\,|) \\ &= \int_0^{\infty} |\,Y - M\,| f(y)dy \\ &= \int_0^{M}(M - y)f(y)\,dy + \int_M^{\infty}(y - M)f(y)\,dy \\ &= \mu - 2\int_0^{M} yf(y)\,dy \end{aligned} \tag{5.3}$$

Now

$$\int_0^{M} yf(y)\,dy = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} v_{i,\,j}2\alpha\beta\int_0^{M} y^{2\beta(i+1)}e^{-j\alpha y^{2\beta}}dy$$

Making substitution $j\alpha y^{2\beta} = z$ and $0 < z < j\alpha M^{2\beta}$, we have

$$\int_0^{M} yf(y)\,dy = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{v_{i,\,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+1}{2\beta}}}\int_0^{j\alpha M^{2\beta}} z^{\frac{2\beta(i+1)+1}{2\beta}} e^{-z}dy$$

After solving the above integral, we get

$$\int_0^M yf(y)\,dy = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{v_{i,\,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+1}{2\beta}}}\gamma\left(\frac{2\beta(i+1)+1}{2\beta},\ j\alpha M^{2\beta}\right) \tag{5.4}$$

Substituting equation (5.4) in equation (5.3), we get

$$D(M) = \mu - 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{v_{i,\,j}\alpha}{(j\alpha)^{\frac{2\beta(i+1)+1}{2\beta}}}\gamma\left(\frac{2\beta(i+1)+1}{2\beta},\ j\alpha\mu^{2\beta}\right)$$

6. Renyi Entropy of TLPRD

Let Y be a continuous random variable with probability density function $f(y,\ \alpha,\ \beta, \theta)$. Then Renyi entropy is defined as

$$T_R(\delta) = \frac{1}{1-\delta}\log\left\{\int_0^{\infty} f^{\delta}(y)\,dy\right\},\ \text{where } \delta > 0 \text{ and } \delta \neq 1.$$

Thus, the Renyi entropy for TLPRD is given as

$$T_R(\delta) = \frac{1}{1-\delta}\log\left\{\int_0^{\infty}\left[2\alpha\beta\theta\ y^{2\beta-1}e^{-\alpha y^{2\beta}}\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta-1}\right]^{\delta} dy\right\}$$

$$= \frac{1}{1-\delta}\log\left\{\int_0^{\infty}(2\alpha\beta\theta)^{\delta}\int_0^{\infty} y^{(2\beta-1)\delta}e^{-\alpha\delta y^{2\beta}}\left(1-e^{-\alpha y^{2\beta}}\right)^{\delta(\theta-1)} dy\right\}$$

We know the following Taylor's and binomial expansion.

$$e^{-m\theta} = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}(m\theta)^i$$

and

$$(1-x)^{-a} = \sum_{j=0}^{\infty}\binom{a+j-1}{j}x^j;\ |\,x\,| \leq 1$$

After applying the above expansions, we obtain

$$T_R(\delta) = \frac{1}{1-\delta}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(2\alpha\beta\theta)^{\delta}(\alpha\delta)^{i}\binom{\delta(1-\theta)j-1}{j}\int_0^{\infty} y^{\delta(2\beta-1)+2\beta i}e^{-\alpha j y^{2\beta}}dy\right\}$$

$$= \frac{1}{1-\delta}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(2\alpha\beta\theta)^{\delta}(\alpha\delta)^{i}\binom{\delta(1-\theta)j-1}{j}\int_0^{\infty} y^{2\delta(\delta-i)+\delta}e^{-\alpha j y^{2\beta}}dy\right\}$$

Making substitution $j\alpha y^{2\beta} = z$ and $0 < z < \infty$, we have

$T_R(\delta)=$

$$\frac{1}{1-\delta}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(2\alpha\beta\theta)^{\delta-1}(\alpha\delta)^{i}\binom{\delta(1-\theta)j-1}{j}\frac{\alpha\theta}{(\alpha j)^{\frac{2\beta(\delta+i)-\delta+1}{2\beta}}}\int_0^{\infty} z^{\left(\frac{2\beta(\delta+i)-\delta+1}{2\phi}\right)-1}e^{-z}dy\right\}$$

Solving the above integral, we get

$T_R(\delta)=$

$$\frac{1}{1-\delta}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(2\alpha\beta\theta)^{\delta-1}(\alpha\delta)^{i}\binom{\delta(1-\theta)j-1}{j}\frac{\alpha\theta}{(\alpha j)^{\frac{2\beta(\delta+i)-\delta+1}{2\beta}}}\Gamma\left(\frac{2\beta(\delta+i)-\delta+1}{2\beta}\right)\right\}$$

7. Tsallis Entropy of TLPRD

Tsallis entropy of order δ for TLPRD is given as

$$S_{\delta} = \frac{1}{\delta-1}\left\{1-\int_0^{\infty} f^{\delta}(y)\,dy\right\}, \text{ where } \delta > 0 \text{ and } \delta \neq 1$$

$$= \frac{1}{\delta-1}\left\{1-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(2\alpha\beta\theta)^{\delta}(\alpha\delta)^{i}\binom{\delta(1-\theta)j-1}{j}\int_0^{\infty} y^{2\beta(\delta+i)-\delta}e^{-\alpha j y^{2\beta}}dy\right\}$$

Solving the above integral, we get

$$S_{\delta} = \frac{1}{\delta-1}\left\{1-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(2\alpha\beta\theta)^{\delta}(\alpha\delta)^{i}\binom{\delta(1-\theta)j-1}{j}\frac{\alpha\theta}{(\alpha j)^{\frac{2\beta(\delta+i)-\delta+1}{2\beta}}}\Gamma\left(\frac{2\beta(\delta+i)-\delta+1}{2\beta}\right)\right\}$$

8. Order Statistics of TLPRD

Let us suppose $Y_1, Y_2, \ldots, Y_n$ be random samples of size n from TLPRD with pdf $f(y)$ and cdf $F(y)$. The probability density function of order statistics is then calculated as follows:

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!} f(y)[F(y)]^{k-1}[1-F(y)]^{n-k} \tag{8.1}$$

Now using the equations (2 .3) and (2 .4) in (8 .1). The probability of k^{th} order statistics of TLPRD is given by

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!} 2\alpha\beta\theta y^{2\beta-1} e^{-\alpha y^{2\beta}}\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta-1}\left[\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}\right]^{k-1}\left[1-\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}\right]^{n-k} \tag{8.2}$$

Then, the pdf of first order statistics 1 Y TLPRD is given as

$$f_{Y_{(l)}}(y) = n2\alpha\beta\theta\ y^{2\beta-1} e^{-\alpha y^{2\beta}}\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta-1}\left[1-\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}\right]^{n-1}$$

Also, the pdf of nth order statistics n Y TLPRD is given as

$$f_{Y_{(n)}}(y) = n2\alpha\beta\theta\ y^{2\beta-1} e^{-\alpha y^{2\beta}}\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta-1}\left[\left(1-e^{-\alpha y^{2\beta}}\right)^{\theta}\right]^{n-1}$$

9. Maximum Likelihood Estimation of TLPRD

Let $y_1, y_2, \ldots, y_n$ be random samples from TLPRD. Then its Likelihood function is given by

$$l = \prod_{i=1}^{n} f(y, \alpha, \beta, \theta) \tag{9.1}$$

Using equation (2.3) in equation (9.1), we have

$$l = \prod_{i=1}^{n} 2\alpha\beta\theta\ y_i^{2\beta-1} e^{-\alpha y_i^{2\beta}}\left(1-e^{-\alpha y_i^{2\beta}}\right)^{\theta-1}$$

$$= (2\alpha\beta\theta)^n \prod_{i=1}^{n} y_i^{2\beta-1}(1 - e^{-\alpha y_i^{2\beta}})^{\theta-1} e^{-\alpha \sum_{i=1}^{n} y_i^{2\beta}} \tag{9.2}$$

Then the log-likelihood function becomes

$$\log\ l = n\ \log\ 2 + n\ \log\ \alpha + n\ \log\ \beta + n\ \log\ \theta + (2\beta - 1)\sum_{i=1}^{n} \log\ y_i$$

$$+ (\theta - 1)\sum_{i=1}^{n} \log\left(1 - e^{-\alpha y_i^{2\beta}}\right) - \alpha \sum_{i=1}^{n} y_i^{2\beta} \tag{9.3}$$

Differentiate equation (9.3), partially w.r.t parameters, we get

$$\frac{\partial \log\ l}{\partial \alpha} = \frac{n}{\alpha} + (\theta - 1)\sum_{i=1}^{n} \frac{y_i^{2\beta} e^{-\alpha y_i^{2\beta}}}{\left(1 - e^{-\alpha y_i^{2\beta}}\right)} - \sum_{i=1}^{n} y_i^{2\beta} \tag{9.4}$$

$$\frac{\partial \log l}{\partial \alpha} = \frac{n}{\beta} + n\sum_{i=1}^{n} \log y_i + 2\alpha(\theta - 1)\sum_{i=1}^{n} \frac{y_i^{2\beta} e^{-\alpha y_i^{2\beta}} \log y_i}{\left(1 - e^{-\alpha y_i^{2\beta}}\right)} - 2\alpha \sum_{i=1}^{n} y_i^{2\beta} \log y_i \tag{9.5}$$

$$\frac{\partial \log\ l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log\left(1 - e^{-\alpha y_i^{2\beta}}\right) \tag{9.6}$$

The equations (9.4), (9.5) and (9.6) are non-linear equations which cannot be expressed in compact form and it is difficult to solve these equations explicitly for α, β and θ. Iterative methods are needed to solve these equations, such as Newton-Rapson method, secant method, Regula-Falsi method etc. The MLE of the parameters denoted as $\hat{\omega}(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ of $\omega(\alpha, \beta, \theta)$ can be obtained by using the above methods.

Since the MLE of $\hat{\omega}$ follows asymptotically normal distribution which is given as

$$\sqrt{n}(\hat{\omega} - \omega) \rightarrow N(0,\ I^{-1}(\omega))$$

where $I^{-1}(\omega)$ is the limiting variance-covariance matrix $\hat{\omega}$ and $I^{-1}(\omega)$ is a

3×3 Fisher information matrix

i.e.

$$I^{-1}(\omega) = -\frac{1}{n}\begin{bmatrix} E\left(\frac{\partial^2 \log l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 \log l}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 \log l}{\partial\alpha\partial\theta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial\beta\partial\alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial\beta^2}\right) & E\left(\frac{\partial^2 \log l}{\partial\beta\partial\theta}\right) \\ E\left(\frac{\partial^2 \log l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2 \log l}{\partial\theta\partial\beta}\right) & E\left(\frac{\partial^2 \log l}{\partial\theta^2}\right) \end{bmatrix},$$

where

$$\frac{\partial \log l}{\partial\alpha^2} = \frac{-n}{\alpha^2} + (1-\theta)\sum_{i=1}^{n}\frac{y_i^{4\beta}e^{-\alpha y_i^{2\beta}}}{\left(1-e^{-\alpha y_i^{2\beta}}\right)^2}$$

$$\frac{\partial^2 \log l}{\partial\beta^2} = \frac{-n}{\beta} + 2\alpha(\theta-1)\sum_{i=1}^{n}\frac{\left(\left(1-\alpha y_i^{2\beta}\right)-e^{-\alpha y_i^{2\beta}}\right)-e^{-\alpha y_i^{2\beta}}y_i^{2\beta}(\log\ y_i)^2}{\left(1-e^{-\alpha y_i^{2\beta}}\right)^2}$$

$$-4\alpha\sum_{i=1}^{n}y_i^{2\beta}(\log\ y_i)^2$$

$$\frac{\partial^2 \log l}{\partial\theta^2} = \frac{-n}{\theta^2}$$

$$\frac{\partial^2 \log l}{\partial\alpha\partial\beta} = \frac{\partial^2 \log l}{\partial\theta\partial\alpha} = 2\alpha(\theta-1)\sum_{i=1}^{n}\frac{\left(\left(1-\alpha y_i^{2\beta}\right)-e^{-\alpha y_i^{2\beta}}\right)-e^{-\alpha y_i^{2\beta}}y_i^{2\beta}\log\ y_i{}^2}{\left(1-e^{-\alpha y_i^{2\beta}}\right)^2}$$

$$-2\sum_{i=1}^{n}y_i^{2\beta}\log\ y_i$$

$$\frac{\partial^2 \log l}{\partial\alpha\partial\theta} = \frac{\partial^2 \log l}{\partial\theta\partial\alpha} = 2\alpha\sum_{i=1}^{n}\frac{e^{-\alpha y_i^{2\beta}}y_i^{2\beta}\log\ y_i}{1-e^{-\alpha y_i^{2\beta}}}$$

$$\frac{\partial^2 \log l}{\partial\beta\partial\theta} = \frac{\partial^2 \log l}{\partial\theta\partial\beta} = \sum_{i=1}^{n} \frac{e^{-\alpha y_i^{2\beta}} y_i^{2\beta}}{1 - e^{-\alpha y_i^{2\beta}}}$$

Hence the approximate $100(1 - \psi)\%$ confidence interval for α, β and θ are respectively given by

$$\hat{\alpha} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\alpha\alpha}^{-1}(\hat{\omega})},\ \hat{\beta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\beta\beta}^{-1}(\hat{\omega})} \text{ and } \hat{\theta} \pm Z_{\frac{\psi}{2}} \sqrt{I_{\theta\theta}^{-1}(\hat{\omega})}$$

where $Z_{\frac{\psi}{2}}$ denotes the ψ^{th} percentile of the standard normal distribution.

10. Simulation Study of TLPRD

In this section, Monte Carlo simulation has been used to validate the maximum likelihood estimators. This is accomplished by employing the TLPRD estimators. The quantile function of the TLPRD has been used to create random numbers from the TLPRD, as discussed in section 3. For sample sizes n=50, 150, 300, 600, 800 and1000, we set three initial parameter values, as well as the simulations are performed 1000 times for each sample for calculation of bias, variance and Mean square error (MSE). It is observed that bias, variance and MSE decrease significantly, when we increase sample size. The efficiency of maximum likelihood estimators is therefore relatively strong, consistent in case of TLPRD.

Table 10.1. The Mean values, Average bias and MSEs of 1,000 simulations of TLPRD for parameter values $\alpha = 0.5$, $\beta = 0.4$, $\theta = 0.6$.

Sample size	parameters	Mean value	Bias	variance	MSE
50	α	0.5713	0.0713	0.2574	0.2625
	β	3.2166	2.8166	3.2900	11.223
	θ	0.7761	0.1761	1.1845	1.2156
150	α	0.5199	0.0199	0.0630	0.0634
	β	2.6710	2.2710	0.4936	5.6513

	θ	0.6309	0.0309	0.0581	0.0590
300	α	0.5087	0.0087	0.0324	0.0324
	β	2.5817	2.1817	0.2170	4.9769
	θ	0.6139	0.0139	0.0262	0.0264
600	α	0.5038	0.0038	0.0139	0.0139
	β	2.5379	2.1379	0.0900	4.6608
	θ	0.6062	0.0062	0.0107	0.0107
800	α	0.4990	-0.0009	0.0109	0.0109
	β	2.5385	2.1385	0.0698	4.6433
	θ	0.6020	0.0020	0.0083	0.0083
1000	α	0.5053	0.0053	0.0083	0.0083
	β	2.5124	2.1124	0.0505	4.5129
	θ	0.6067	0.0067	0.0062	0.0063

Table 10.2. The Mean values, Average bias and MSEs of 1,000 simulations of TLPRD for parameter values $\alpha = 0.8$, $\beta = 0.6$, $\theta = 0.8$.

Sample size	parameters	Mean value	Bias	variance	MSE
50	α	0.5713	0.0713	0.2574	0.2625
	β	2.1065	1.5065	1.5854	3.8552
	θ	1.0579	0.2579	1.8949	1.9614
150	α	0.8444	0.0444	0.1070	0.1090
	β	1.7309	1.1309	0.1775	1.4566
	θ	0.8826	0.0826	0.1485	0.1553
300	α	0.8071	0.0071	0.0522	0.0523
	β	1.7182	1.1182	0.0904	1.3409
	θ	0.8255	0.0255	0.0528	0.0535
600	α	0.8043	0.0043	0.0247	0.0247

	β	1.6892	1.0892	0.0402	1.2267
	θ	0.8132	0.0132	0.0233	0.0235
800	α	0.8033	0.0033	0.0159	0.0159
	β	1.6831	1.0831	0.0261	1.1993
	θ	0.8080	0.0080	0.0146	0.0146
1000	α	0.8070	0.0070	0.0133	0.0134
	β	1.6730	1.0730	0.0203	1.1718
	θ	0.8103	0.0103	0.0120	0.0121

Table 10.3. The Mean values, Average bias and MSEs of 1,000 simulations of TLPRD for parameter values $\alpha = 1.2$, $\beta = 1.6$, $\theta = 0.6$.

Sample size	parameters	Mean value	Bias	variance	MSE
50	α	1.2354	0.0354	0.4791	0.4804
	β	0.8301	-0.7698	0.2493	0.8420
	θ	0.7369	0.1369	0.5479	0.5667
150	α	1.1937	-0.0062	0.1338	0.1338
	β	0.6990	-0.9009	0.0541	0.8659
	θ	0.6172	0.0172	0.0519	0.0522
300	α	1.1991	-0.0008	0.0580	0.0580
	β	0.6994	-0.9005	0.0571	0.8680
	θ	0.6059	0.00592	0.0222	0.0223
600	α	1.2058	0.0058	0.0302	0.0302
	β	0.7148	-0.8851	0.0765	0.8600
	θ	0.6091	0.00916	0.0117	0.0118
800	α	1.2069	0.0069	0.0213	0.0213
	β	0.7304	-0.8695	0.0907	0.8469

	θ	0.6072	0.0072	0.0077	0.0077
1000	α	1.2030	0.0030	0.0160	0.0160
	β	0.7888	-0.8111	0.1320	0.7899
	θ	0.6047	0.0047	0.0056	0.0056

11. Applications

In this section we demonstrate the potentiality of the formulated distribution (TLPRD) by means of real life data set. The formulated distribution is compared with some generalized and existing distribution, which are Topp-Leone Burr distribution (TLBD), Lomax distribution (LXD), power Rayleigh distribution (PRD), Rayleigh distribution (RD), Nadrajah-Haghighi distribution (NHD) and Lindley distribution (LD). The data follows as

Data Set. The data set consists of 63 observations of the gauge lengths of 10mm from Kundu and Raqab [13]. The data was also used by Afify et al. [1]. The data follows

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396,2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624,2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125,3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852,3.871, 3.886, 3.971, 4.024, 4.027,4.225, 4.395, 5.020

The fitted models are compared using empirical goodness of fit measures such as the AIC (Akaike information criterion), CAIC (Consistent Akaike information criterion), BIC (Bayesian information criterion), HQIC (Hannan-Quinn information criteria), and KS (Kolmogorov-Smirov). Each model's p-value is also displayed. A distribution with a lower AIC, CAIC, BIC, and HQIC together with a higher p value is rated as the top distribution.

$$AIC = 2k - 2 \text{ In } l \quad CAIC = \frac{2kn}{n-k-1} - 2 \text{ In } l$$

$$BIC = k \text{ In } n - 2 \text{ In } l \quad HQIC = 2k \; In \; (In(n)) - 2 \text{ In } l$$

Table 1 shows the descriptive statistics for the data set. Table 2 displays the parameter estimates for the data set. Table 3 displays the log-likelihood; Akaike information criteria (AIC) details and some other statistics for the data set.

Table 1. The descriptive statistics of data set.

Min	Q_1	Median	Mean	Q_3	Skew.	Kurt.	Max
1.901	2.554	2.996	3.059	3.421	0.6328	3.2863	5.020

Table 2. The ML Estimates and standard error of the unknown parameters.

Model		TLPRD	TLBD	LXD	PRD	NHD	RD	LD
$\hat{\alpha}$		0.8172	0.0221	0.0061	14.569	0.0047	2.2066	0.5392
$\hat{\beta}$		0.7268	112.53	53.41	2.5246	54.430	……	……
$\hat{\theta}$		37.182	147.30	……	…..	……	……	……
S.E	$\hat{\alpha}$	1.1176	0.0072	0.0032	4.4014	0.0027	0.1390	0.0495
	$\hat{\beta}$	0.3787	37.108	28.59	0.2278	32.465	……	……
	$\hat{\theta}$	79.207	57.081	……	……	……	……	……

Table 3. Performance of distributions.

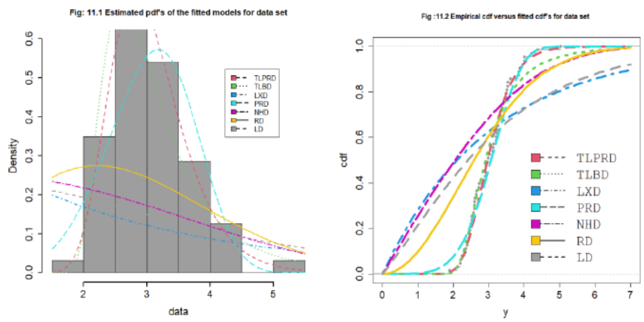
Model	TLPRD	TLBD	LXD	PRD	NHD	RD	LD
$-\log l$	56.310	59.345	134.01	61.95	113.55	93.519	121.35
AIC	118.62	124.69	272.02	127.91	231.11	189.03	244.71
CAIC	119.02	125.09	272.22	128.11	231.31	189.10	244.78
BIC	125.05	131.12	276.30	132.20	235.40	191.18	246.85
HQIC	121.15	127.21	273.70	129.59	232.80	189.88	245.55
K-S Value	0.0812	0.0863	0.4868	0.0875	0.4996	0.3607	0.4308
P Value	0.7994	0.7352	2.13e-13	0.7192	4.36e-14	1.51e-07	1.39e-10

The asymptotic variance-covariance matrix of maximum likelihood estimates under TLPRD for data set is computed as

$$I^{-1}(\omega)=\begin{pmatrix} 1.24903 & -0.42201 & 88.0395 \\ -0.42201 & 0.14342 & -29.5492 \\ 88.0395 & -29.5492 & 6273.84 \end{pmatrix}$$

Therefore, the 95 % confidence interval for α, β and θ are given as $(-1.3731,\ 3.0077)(-0.0154,\ 1.4691)$ and $(-118.061,\ 192.426)$ respectively.

TLPRD outranks the other models as it has the lowest loglikehood and other goodness of fit measures and offers a relatively accurate fit, as shown in table 3.



12. Conclusion

This paper deals with a new generalisation of power Rayleigh distribution called Topp-Leone power Rayleigh distribution. We have added an extra parameter to the power Rayleigh distribution by Topp-Leone-G generator, the main purpose for such modification is that the formulated distribution become more richer and flexible in modelling datasets. Several distinct properties of formulated distribution has been studied and discussed. The model parameters of the distribution are estimated by the known method of maximum likelihood estimation. Eventually, the efficiency of the explored distribution is examined through real data set which reveals that the formulated distribution provides an adequate model fit than competing ones.

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