Advances and Applications in Mathematical Sciences
Volume 21, Issue 1, November 2021, Pages 111-116

DISJOINT UNION OF CYCLE AND TREES OF DIAMETER FOUR IS ODD GRACEFUL

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Abstract

The first graph labeling method is the graceful labeling of graphs which was introduced by Alex Rosa [5] in 1967. In 1991 Gnanajothi [2] introduced a labeling method called odd graceful labeling. An odd graceful labeling is an injection *f* from $V(G)$ to $\{0, 1, 2, \ldots, (2q-1)\}$ such that when each edge *xy* is assigned the label or weight $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, (2q-1)\}$. A graph which admits an odd graceful labeling is called an odd graceful graph. The real life application of odd graceful labeling are the traditional social network, online communities, security control, radar tracking and remote control.

In this paper we prove the odd gracefulness of disjoint union of cycle and tree of diameter four.

1. Introduction

Graph labeling is one of the important areas in Graph Theory. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition. Graph labeling was first introduced by Rosa [5] in 1967. The graceful labeling of a graph G with q edges is an injection from the

2020 Mathematics Subject Classification: 05C69, 05C85, 05C90, 05C20.
Keywords: Odd Graceful Labeling, Cycle, Diameter four trees, Disjoint union.
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Received June 2, 2020; Accepted January 19, 2021

vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

In 1991 Gnanajothi [2] introduced a labeling method called odd graceful labeling. An odd graceful labeling is an injection f from $V(G)$ to $\{0, 1, 2, \ldots, (2q-1)\}$ such that when each edge xy is assigned the label or weight $|f(x) - f(y)|$, the resulting edge labels are $\{1, 3, 5, \ldots, (2q-1)\}$. A graph which admits an odd graceful labeling is called an odd graceful graph.

Gnanajothi [2] proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs by showing that every graph with an α-labelings has an odd graceful labeling and every graph with an odd cycle is not odd graceful. In [2], the following graphs are proved to be odd graceful; the path P_n, the cycle C_n if and only if n is even, the comb $P_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of P_n), books, crowns $C_n \odot K_1$ (graphs obtained by joining a single pendant edge to each vertex of C_n), if and only if n is even, the disjoint union of copies of C_4, the one-point union of copies of C_4, caterpillars, rooted trees of height 2, the graphs obtained from $P_n (n \geq 3)$ by adding exactly two leaves at each vertex of degree 2 of P_n. Ibrahim Moussa [3] proved that the graph $C_m \cup P_n$ is odd graceful if m is even. Javid [4] proved that the disjoint union of cycle and H-isomorphic copies of paths is odd graceful. Joseph Gallian [1] has given a broad and a dynamic survey on various graph labeling methods including odd graceful labeling.

2. Main Result

In this section we will prove that the disjoint union of cycle and trees of diameter four is odd graceful.

Definition 1. A tree is a connected acyclic graph.

Definition 2. Diameter of tree is defined as a maximum length between any two vertices in a tree.

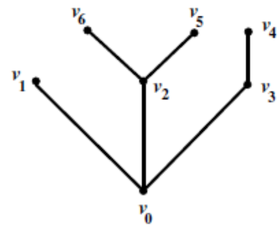

Figure 1. A Tree.

It is denoted by $diam(G)$. An example of a Trees of diameter four is given in Figure 1.

Theorem 1. *The disjoint union of cycle and trees of diameter four admits odd graceful labeling.*

Proof. Let G denote the disjoint union of a cycle C_m and trees of diameter four. The vertices in G are described as follows. Let T be a tree of diameter four with two levels. The rooted vertex of the tree T denoted as x. The vertices in the first level of tree T are denoted as $v_1, v_2, v_3, \ldots, v_n$. The vertices in the second level of tree T are denoted as $u_1, u_2, u_3, \ldots, u_R$ where $R = r_1 + r_2 + \ldots + r_n$.

Let $r_1, r_2, r_3, \ldots, r_n$ be the pendent vertices attached to the first level of vertices are $v_1, v_2, v_3, \ldots, v_n$. The vertices of cycle C_m are denoted as $u_{R+1}, u_{R+2}, u_{R+3}, \ldots, u_{R+\frac{m}{2}}$ and $w_1, w_2, w_3, \ldots, w_{\frac{m}{2}}$ as shown in Figure 2. Note that the graph G has $p = m + n + R + 1$ vertices and $q = m + n + R$ edges.

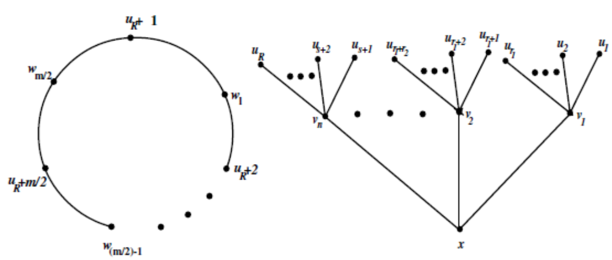

Figure 2. Disjoint Union of Cycle and Trees of diameter four.

When $n \leq m$.

The vertex label for the tree of diameter four is given by

$$f(x) = 0 \tag{2.1}$$

$$f(u_i) = 2i, \text{ for } 1 \leq i \leq r_1 \tag{2.2}$$

$$f\{u_{(\sum_{\alpha=1}^{\beta} r_\alpha)+l}\} = f\{u_{(\sum_{\alpha=1}^{\beta} r_\alpha)}\} + 4 + (2l - 2), \text{ for } 1 \leq \beta \leq n - 1 \tag{2.3}$$

$$1 \leq l \leq r_{\beta+1} \tag{2.4}$$

$$f(v_j) = 2q - (2n - (2j - 1)), \text{ for } 1 \leq j \leq n. \tag{2.5}$$

The vertex label for the cycle C_m is given by

$$f(u_{R+k}) = f(u_R) + 2k, \text{ for } 1 \leq k \leq \frac{m}{2} \tag{2.6}$$

$$f(w_k) = \begin{cases} f(u_{R+1}) - 1 - (2k - 2), & \text{for } 1 \leq k \leq \frac{m}{4} \\ f(u_{R+1}) - (2k - 1) - 3 & \text{for } \frac{m}{4} + 1 \leq k \leq \frac{m}{2} \end{cases} \tag{2.7}$$

From the vertex labels defined above, we see that the vertices receive distinct labels. The edge labels are given by

$$| f(v_j) - f(x) | = | 2q - (2j - 1) |, \text{ for } 1 \leq j \leq n \tag{2.8}$$

$$| f(v_1) - f(u_i) | = | 2q - (2n - 1) - 2i |, \text{ for } 1 \leq i \leq r_1 \tag{2.9}$$

$$| f(v_j) - f\{u_{(\sum_{\alpha=1}^{\beta} r_\alpha)+l}\} | = | f\{u_{(\sum_{\alpha=1}^{\beta} r_\alpha)}\} + 4 + (2l - 2) - 2q + (2n - (2j - 1)) |,$$

$$\text{for } 1 \leq \beta \leq n - 1, 1 \leq l \leq r_{\beta+1}, 2 \leq j \leq n \tag{2.10}$$

$$| f(w_k) - f(u_{R+k}) | = \begin{cases} f(u_{R+1}) - 1 - (2k - 2) - f(u_R) - 2k, & \text{for } 1 \leq k \leq \frac{m}{4} \\ f(u_{R+1}) - (2k - 1) - 3 - f(u_R) - 2k & \text{for } \frac{m}{4} + 1 \leq k \leq \frac{m}{2} \end{cases} \tag{2.11}$$

From the above computed edge labels it is observed that the edge labels for the disjoint union of cycle and trees of degree four are distinct.

The sets E_α, $1 \le \alpha \le 4$ consisting of the above computed edge labels, are given below

$$E_1 = \{| f(v_j) - f(x) |, \text{ for } 1 \le j \le n\}$$

$$= \{(2q-1), (2q-3), \ldots, (2q-2n-1)\} \quad (2.12)$$

$$E_2 = \{| f(v_1) - f(u_i) |, \text{ for } 1 \le i \le r_1\}$$

$$= \{(2q-2n-1), (2q-2n-3), \ldots, (2q-2n-2r_1+1)\} \quad (2.13)$$

$$E_3 = \{| f(v_j) - f(u_{(\sum_{\alpha=1}^{\beta} r_{\alpha+l})}) |, \text{ for } 1 \le \beta \le n-1, 1 \le l \le r_{\beta+1}, 2 \le j \le n\}$$

$$= | f\{u_{r_1}\} + 4 - 2q + (2n-3), f\{u_{r_1}\} + 6 - 2q + (2n-3), \ldots,$$

$$f\{u_{(\sum_{\alpha=1}^{\beta} r_\alpha)}\} + 4 + (2r_{\beta+1} - 2) - 2q + (2n - (2n-1)) | \quad (2.14)$$

$$E_4 = \{| f(w_k) - f(u_{R+k}) |, \text{ for } 1 \le k \le \frac{m}{2}\}$$

$$= \{f(u_{R+1}) - f(u_R) - 3, f(u_{R+1}) - f(u_R) - 7, \ldots, f(u_{R+1}) - f(u_R) - 2m - 1\}. \quad (2.15)$$

From the above equations it is observed that $\bigcup_{\alpha=1}^{4} E_\alpha = \{1, 3, 5, \ldots, (2q-1)\}$. Hence the vertices and edges are distinct in the graph G. Therefore the disjoint union of cycle and trees of diameter four admits odd graceful labeling.

3. Illustration

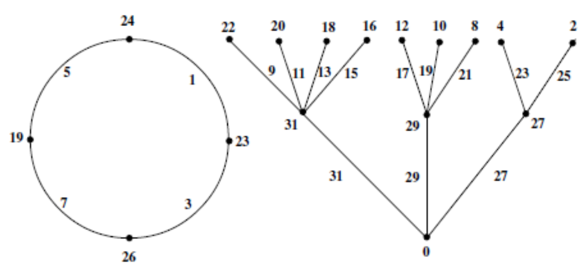

Figure 3. Disjoint Union of Cycle C_4 and trees of diameter four.

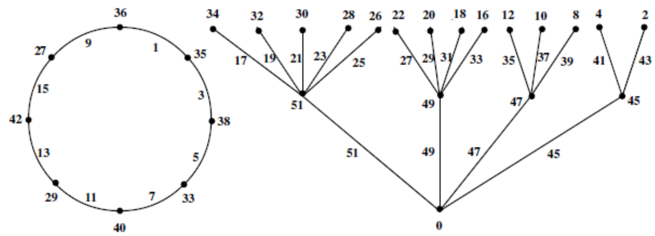

Figure 4. Disjoint Union of Cycle C_8 and trees of diameter four.

4. Conclusion

In this paper we have proved that the disjoint union of cycle C_m where $\mathbf{m} \equiv \mathbf{0} (\mathbf{mod}\, \mathbf{4})$ and trees of diameter four is odd graceful.

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