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ON THE MERSENNE AND MERSENNE-LUCAS HYBRID QUATERNIONS

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Abstract

In this communication, we define the Mersenne hybrid quaternions and give some of their properties. Further, we analyze the relations between the Mersenne hybrid quaternions and the Mersenne-Lucas hybrid quaternions which connected Mersenne quaternions and Mersenne-Lucas quaternions. Also, we give the Binet formulas and moreover, well known identities like Catalan identity, Cassini identity and d'Ocagne's identity for these quaternions.

1. Introduction

In [10], Ozdemir introduced hybrid numbers as a new type of numbers. Hybrid numbers are generalizations of complex, hyperbolic and dual numbers. A hyperbolic complex structure has many applications in both pure mathematics and various areas of Physics [2, 11]. Hybrid numbers can be connected with the family of Mersenne type numbers. Herewith, we recall hybrid number definition as

$$\mathcal{H} = a + bi + c\varepsilon + dh,\ a, b, c, d \in \mathbb{R},$$

$$i^2 = -1,\ \varepsilon^2 = 0,\ h^2 = 1,\ ih = -hi = \varepsilon + i.$$

The conjugate of the hybrid number $\mathcal{H}$ is denoted by

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$$\mathcal{H}^c = a - bi - c\varepsilon - dh.$$

A quaternion has an extension of the complex numbers was first defined by Hamilton [6]. The quaternion of sequences was first considered by Horadam [8]. A real quaternion is defined as

$$\mathcal{Q} = z_0 + z_1 i + z_2 j + z_3 k, \text{ where } z_0, z_1, z_2, z_3 \in \mathbb{R}.$$

Also i, j, k are the units of the real quaternions which satisfy the equalities

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The conjugate of the quaternion $\mathcal{Q}$ is denoted by

$$\overline{\mathcal{Q}} = z_0 - z_1 i - z_2 j - z_3 k.$$

Although, the advantages of the quaternions appeared in the fundamental equations of some field of science [3, 4, 5, 7]. Recently, many mathematicians are trying more and more to use algebraic properties of quaternions to make easy and efficient calculations [1, 9, 12, 13]. This system has a strong algebraic structure and it is a generalization of dual and hyperbolic quaternion. Moreover, hybrid quaternions are also generalized features of quaternions system such as inner product, vector product and norm.

The Mersenne hybrid numbers and Mersenne-Lucas hybrid numbers are defined as

$$\ddot{M}_n = M_n + M_{n+1} i + M_{n+2} \varepsilon + M_{n+3} h$$

$$\ddot{ML}_n = ML_n + ML_{n+1} i + ML_{n+2} \varepsilon + ML_{n+3} h$$

The Mersenne quaternions and Mersenne-Lucas quaternions are defined as

$$\widetilde{M_n} = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3}$$

$$\widetilde{ML_n} = ML_n + iML_{n+1} + jML_{n+2} + kML_{n+3}$$

Table 1. Notations.

Notations	Numbers
M_n	Mersenne numbers
ML_n	Mersenne-Lucas numbers
$\widetilde{M_n}$	Mersenne quaternions
$\widetilde{ML_n}$	Mersenne-Lucas quaternions
$\ddot{M}_n$	Mersenne hybrid numbers
$\ddot{M}L_n$	Mersenne-Lucas hybrid numbers
$\overbrace{M_n}$	Mersenne hybrid quaternions
$\overbrace{ML_n}$	Mersenne-Lucas hybrid quaternions

Definition. The n^{th} Mersenne hybrid quaternions $\overbrace{M_n}$ is defined by

$$\overbrace{M_n} = \ddot{M}_n + i\ddot{M}_{n+1} + j\ddot{M}_{n+2} + k\ddot{M}_{n+3}$$

where i, j, k are quaternion units.

We will restate $\overbrace{M_n}$ by

$$\overbrace{M_n} = \widetilde{M_n} + i\widetilde{M_{n+1}} + \varepsilon\widetilde{M_{n+2}} + h\widetilde{M_{n+3}}$$

Definition. The Mersenne-Lucas hybrid quaternions are defined as

$$\overbrace{ML_n} = \ddot{M}L_n + i\ddot{M}L_{n+1} + j\ddot{M}L_{n+2} + k\ddot{M}L_{n+3}$$

can be written as

$$\overbrace{ML_n} = \widetilde{ML_n} + i\widetilde{ML_{n+1}} + \varepsilon\widetilde{ML_{n+2}} + h\widetilde{ML_{n+3}}$$

Definition. Let $\overbrace{U_n}$ and $\overbrace{V_n}$ be the n^{th} terms of the Mersenne hybrid quaternion sequences such that

$$\widehat{U_n} = \ddot{U}_n + i\ddot{U}_{n+1} + j\ddot{U}_{n+2} + k\ddot{U}_{n+3} = \widetilde{U_n} + i\widetilde{U_{n+1}} + \varepsilon\widetilde{U_{n+2}} + h\widetilde{U_{n+3}}$$

and

$$\widehat{V_n} = \ddot{V}_n + i\ddot{V}_{n+1} + j\ddot{V}_{n+2} + k\ddot{V}_{n+3} = \widetilde{V_n} + i\widetilde{V_{n+1}} + \varepsilon\widetilde{V_{n+2}} + h\widetilde{V_{n+3}}$$

Then the addition and subtraction of the Mersenne hybrid quaternions are defined by

$$\widehat{U_n} \pm \widehat{V_n} = (\ddot{U}_n + i\ddot{U}_{n+1} + j\ddot{U}_{n+2} + k\ddot{U}_{n+3}) \pm (\ddot{V}_n + i\ddot{V}_{n+1} + j\ddot{V}_{n+2} + k\ddot{V}_{n+3})$$

$$= (\ddot{U}_n \pm \ddot{V}_n) + i(\ddot{U}_{n+1} \pm \ddot{V}_{n+1}) + j(\ddot{U}_{n+2} \pm \ddot{V}_{n+2}) + k(\ddot{U}_{n+3} \pm \ddot{V}_{n+3})$$

$$\widehat{U_n} \pm \widehat{V_n} = (\widetilde{U_n} + i\widetilde{U_{n+1}} + j\widetilde{U_{n+2}} + k\widetilde{U_{n+3}}) \pm (\widetilde{V_n} + i\widetilde{V_{n+1}} + j\widetilde{V_{n+2}} + k\widetilde{V_{n+3}})$$

$$= (\widetilde{U_n} \pm \widetilde{V_n}) + i(\widetilde{U_{n+1}} \pm \widetilde{V_{n+1}}) + j(\widetilde{U_{n+2}} \pm \widetilde{V_{n+2}}) + k(\widetilde{U_{n+3}} \pm \widetilde{V_{n+3}})$$

Definition. The multiplication of the Mersenne hybrid quaternions in terms of Mersenne hybrid numbers is defined as

$$\widehat{U_n}\widehat{V_n} = (\ddot{U}_n + i\ddot{U}_{n+1} + j\ddot{U}_{n+2} + k\ddot{U}_{n+3})(\ddot{V}_n + i\ddot{V}_{n+1} + j\ddot{V}_{n+2} + k\ddot{V}_{n+3})$$

$$= (\ddot{U}_n\ddot{V}_n - \ddot{U}_{n+1}\ddot{V}_{n+1} - \ddot{U}_{n+2}\ddot{V}_{n+2} - \ddot{U}_{n+3}\ddot{V}_{n+3})$$

$$+ i(\ddot{U}_n\ddot{V}_{n+1} + \ddot{U}_{n+1}\ddot{V}_n + \ddot{U}_{n+2}\ddot{V}_{n+3} - \ddot{U}_{n+3}\ddot{V}_{n+2})$$

$$+ j(\ddot{U}_n\ddot{V}_{n+2} + \ddot{U}_{n+1}\ddot{V}_{n+3} + \ddot{U}_{n+2}\ddot{V}_n - \ddot{U}_{n+3}\ddot{V}_{n+1})$$

$$+ k(\ddot{U}_n\ddot{V}_{n+3} + \ddot{U}_{n+1}\ddot{V}_{n+2} - \ddot{U}_{n+2}\ddot{V}_{n+1} + \ddot{U}_{n+3}\ddot{V}_n)$$

In terms of Mersenne quaternions we defined as

$$\widehat{U_n}\widehat{V_n} = (\widetilde{U_n} + i\widetilde{U_{n+1}} + \varepsilon\widetilde{U_{n+2}} + h\widetilde{U_{n+3}})(\widetilde{V_n} + i\widetilde{V_{n+1}} + \varepsilon\widetilde{V_{n+2}} + h\widetilde{V_{n+3}})$$

$$= (\widetilde{U_n}\widetilde{V_n} - \widetilde{U_{n+1}}\widetilde{V_{n+1}} + \widetilde{U_{n+3}}\widetilde{V_{n+3}} + \widetilde{U_{n+3}}\widetilde{V_{n+2}} + \widetilde{U_{n+2}}\widetilde{V_{n+1}})$$

$$+ i(\widetilde{U_n}\widetilde{V_{n+1}} - \widetilde{U_{n+1}}\widetilde{V_n} + \widetilde{U_{n+1}}\widetilde{V_{n+3}} - \widetilde{U_{n+3}}\widetilde{V_{n+1}})$$

$$+ \varepsilon(\widetilde{U_n}\widetilde{V_{n+2}} + \widetilde{U_{n+1}}\widetilde{V_{n+3}} + \widetilde{U_{n+2}}\widetilde{V_n} - \widetilde{U_{n+2}}\widetilde{V_{n+3}} - \widetilde{U_{n+3}}\widetilde{V_{n+1}} + \widetilde{U_{n+3}}\widetilde{V_{n+2}})$$

$$+ h(\widetilde{U_n}\widetilde{V_{n+3}} - \widetilde{U_{n+1}}\widetilde{V_{n+2}} + \widetilde{U_{n+2}}\widetilde{V_{n+1}} - \widetilde{U_{n+3}}\widetilde{V_n})$$

Definition. The conjugate of Mersenne hybrid quaternions is defined by

i. Quaternion conjugate:

$$\overline{\widehat{M_n}} = \overline{\widetilde{M_n}} + i\overline{\widetilde{M_{n+1}}} + \varepsilon\overline{\widetilde{M_{n+2}}} + h\overline{\widetilde{M_{n+3}}}$$

ii. Hybrid conjugate:

$$\widehat{M_n}^{c} = \widetilde{M_n} - i\widetilde{M_{n+1}} - \varepsilon\widetilde{M_{n+2}} - h\widetilde{M_{n+3}}$$

iii. Hybrid quaternion conjugate:

$$\overline{(\widehat{M_n})^c} = \overline{\widetilde{M_n}} + i\overline{\widetilde{M_{n+1}}} + \varepsilon\overline{\widetilde{M_{n+2}}} + h\overline{\widetilde{M_{n+3}}}$$

Theorem 1. *Let* $\widehat{M_n}$ *and* $\widehat{ML_n}$ *be Mersenne hybrid quaternion and Mersenne-Lucas hybrid quaternion. The Binet formulas for these hybrid quaternions are given as*

i. $\widehat{M_n} = \alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B}$

ii. $\widehat{ML_n} = \alpha^n \alpha^* \tilde{A} + \beta^n \beta^* \tilde{B}$

where $\alpha^* = 1 + i\alpha + \varepsilon\alpha^2 + h\alpha^3$, $\beta^* = 1 + i\beta + \varepsilon\beta^2 + h\beta^3$, $\tilde{A} = 1 + i\alpha + j\alpha^2 + k\alpha^3$

and $\tilde{B} = 1 + i\beta + j\beta^2 + k\beta^3$, $\alpha = 2, \beta = 1.$

Proof of Theorem 1. The Binet formulas for the Mersenne quaternions and Mersenne-Lucas quaternions are $\widetilde{M_n} = \alpha^n \tilde{A} - \beta^n \tilde{B}$ and $\widetilde{ML_n} = \alpha^n \tilde{A} - \beta^n \tilde{B}$

$$\widehat{M_n} = \widetilde{M_n} + i\widetilde{M_{n+1}} + \varepsilon\widetilde{M_{n+2}} + h\widetilde{M_{n+3}}$$

$$= (\alpha^n \tilde{A} - \beta^n \tilde{B}) + i(\alpha^{n+1}\tilde{A} - \beta^{n+1}\tilde{B}) + \varepsilon(\alpha^{n+2}\tilde{A} - \beta^{n+2}\tilde{B}) + h(\alpha^{n+3}\tilde{A} - \beta^{n+3}\tilde{B})$$

$$= \alpha^n \tilde{A}(1 + i\alpha + \varepsilon\alpha^2 + h\alpha^3) - \beta^n \tilde{B}(1 + i\beta + \varepsilon\beta^2 + h\beta^3)$$

$$= \alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B}$$

$$\overbrace{ML_n} = \widetilde{ML_n} + i\widetilde{ML_{n+1}} + \varepsilon\widetilde{ML_{n+2}} + h\widetilde{ML_{n+3}}$$

$$= (\alpha^n \tilde{A} - \beta^n \tilde{B}) + i(\alpha^{n+1}\tilde{A} - \beta^{n+1}\tilde{B}) + \varepsilon(\alpha^{n+2}\tilde{A} - \beta^{n+2}\tilde{B}) + h(\alpha^{n+3}\tilde{A} - \beta^{n+3}\tilde{B})$$

$$= \alpha^n \tilde{A}(1 + i\alpha + \varepsilon\alpha^2 + h\alpha^3) - \beta^n \tilde{B}(1 + i\beta + \varepsilon\beta^2 + h\beta^3)$$

$$= \alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B}$$

Theorem 2. *Let* $\overbrace{M_n}$ *and* $\overbrace{ML_n}$ *be Mersenne hybrid quaternion and Mersenne-Lucas hybrid quaternion. Then*

i. $\overbrace{M_n} + \overbrace{M_{n+1}} = 3\alpha^n \alpha^* \tilde{A} - 2\beta^n \beta^* \tilde{B}$

ii. $\overbrace{ML_n} + \overbrace{ML_{n+1}} = 3\alpha^n \alpha^* \tilde{A} - 2\beta^n \beta^* \tilde{B}$

Proof of Theorem 2. By theorem 1, we have $\overbrace{M_n} = \alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B}$

$$\overbrace{M_n} + \overbrace{M_{n+1}} = (\alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B}) + (\alpha^{n+1} \alpha^* \tilde{A} - \beta^{n+1} \beta^* \tilde{B})$$

$$= \alpha^n \alpha^* \tilde{A}(\alpha + 1) - \beta^n \beta^* \tilde{B}(\beta + 1)$$

$$= 3\alpha^n \alpha^* \tilde{A} - 2\beta^n \beta^* \tilde{B}$$

And by using $\overbrace{ML_n} = \alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B},$ we can prove (ii).

Theorem 3. (Catalan's Identity) *Let* $n, r \in \mathbb{Z},$ *then we have*

i. $\overbrace{M_{n-r}}\,\overbrace{M_{n+r}} - \overbrace{M_n^2} = 2^{n-r} M_r [\beta^r \alpha^* \beta^* \tilde{A}\tilde{B} - \alpha^r \beta^* \alpha^* \tilde{B}\tilde{A}]$

ii. $\overbrace{ML_{n-r}}\,\overbrace{ML_{n+r}} - \overbrace{ML_n^2} = 2^{n-r} M_r [\alpha^r \beta^* \alpha^* \tilde{B}\tilde{A} - \beta^r \alpha^* \beta^* \tilde{A}\tilde{B}]$

Proof of Theorem 3.

i. $\overbrace{M_{n-r}}\,\overbrace{M_{n+r}} - \overbrace{M_n^2}$

$$= (\alpha^{n-r} \alpha^* \tilde{A} - \beta^{n-r} \beta^* \tilde{B})(\alpha^{n+r} \alpha^* \tilde{A} - \beta^{n+r} \beta^* \tilde{B}) - (\alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B})^2$$

$$= \alpha^n \beta^{n-r} \beta^* \alpha^* \tilde{B}\tilde{A}(\beta^r - \alpha^r) - \alpha^{n-r} \beta^n \alpha^* \beta^* \tilde{A}\tilde{B}(\beta^r - \alpha^r)$$

$$= \alpha^{n-r} \beta^{n-r} [\beta^r \alpha^* \beta^* \tilde{A}\tilde{B}(2^r - 1) - \alpha^r \beta^* \alpha^* \tilde{B}\tilde{A}(2^r - 1)]$$

$$= 2^{n-r} M_r [\beta^r \alpha^* \beta^* \tilde{A}\tilde{B} - \alpha^r \beta^* \alpha^* \tilde{B}\tilde{A}]$$

Similarly, proceeding like this we obtain identity (ii).

By substituting $r = 1$ in Theorem 3, we get Cassini's Identity.

Theorem 4. (Cassini's Identity)**.** *Let n be any integer then*

i. $\overbrace{M_{n-1}}\overbrace{M_{n+1}} - \overbrace{M_n^2} = 2^{n-1}[\beta\alpha^* \beta^* \tilde{A}\tilde{B} - \alpha\beta^* \alpha^* \tilde{B}\tilde{A}]$

ii. $\overbrace{ML_{n-1}}\overbrace{ML_{n+1}} - \overbrace{ML_n^2} = 2^{n-1}[\alpha\beta^* \alpha^* \tilde{B}\tilde{A} - \beta\alpha^* \beta^* \tilde{A}\tilde{B}]$.

Theorem 5. (d'Ocagne's Identity)**.** *Let* m, n *be any integers then*

i. $\overbrace{M_m}\overbrace{M_{n+1}} - \overbrace{M_{n+1}}\overbrace{M_n} = \alpha^m \beta^n \alpha^* \beta^* \tilde{A}\tilde{B} - \alpha^n \beta^m \beta^* \alpha^* \tilde{B}\tilde{A}$

ii. $\overbrace{ML_m}\overbrace{ML_{n+1}} - \overbrace{ML_{n+1}}\overbrace{ML_n} = \alpha^n \beta^m \beta^* \alpha^* \tilde{B}\tilde{A} - \alpha^m \beta^n \alpha^* \beta^* \tilde{A}\tilde{B}$.

Proof of Theorem 5.

i. $\overbrace{M_m}\overbrace{M_{n+1}} - \overbrace{M_{n+1}}\overbrace{M_n} = (\alpha^m \alpha^* \tilde{A} - \beta^m \beta^* \tilde{B})(\alpha^{n+1} \alpha^* \tilde{A} - \beta^{n+1} \beta^* \tilde{B})$

$$-(\alpha^{m+1} \alpha^* \tilde{A} - \beta^{m+1} \beta^* \tilde{B})(\alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B})$$

$$= \alpha^n \beta^m \beta^* \alpha^* \tilde{B}\tilde{A}(\beta - \alpha) - \alpha^m \beta^n \alpha^* \beta^* \tilde{A}\tilde{B}(\beta - \alpha)$$

$$= \alpha^m \beta^n \alpha^* \beta^* \tilde{A}\tilde{B} - \alpha^n \beta^m \beta^* \alpha^* \tilde{B}\tilde{A}$$

In a similar way, the second identity can be proved.

Theorem 6. (Vajda Identity)**.** *Let* k, n, r *be any integers then*

i. $\overbrace{M_{n+r}}\overbrace{M_{n+k}} - \overbrace{M_n}\overbrace{M_{n+r+k}} = 2^n M_r [2^k \beta^* \alpha^* \tilde{B}\tilde{A} - \alpha^* \beta^* \tilde{A}\tilde{B}]$

ii. $\overbrace{ML_{n+r}}\overbrace{ML_{n+k}} - \overbrace{ML_n}\overbrace{ML_{n+r+k}} = 2^n M_r [\alpha^* \beta^* \tilde{A}\tilde{B} - 2^k \beta^* \alpha^* \tilde{B}\tilde{A}]$.

Proof of Theorem 6.

i. $\overbrace{ML_{n+r}}\overbrace{ML_{n+k}} - \overbrace{ML_n}\overbrace{ML_{n+r+k}}$

$= (2^{n+r}\alpha^*\tilde{A} - \beta^*\tilde{B})(2^{n+k}\alpha^*\tilde{A} - \beta^*\tilde{B}) - (2^n\alpha^*\tilde{A} - \beta^*\tilde{B})(2^{n+r+k}\alpha^*\tilde{A} - \beta^*\tilde{B})$

$= 2^{n+k}\beta^*\alpha^*\tilde{B}\tilde{A}(2^r - 1) - 2^n\alpha^*\beta^*\tilde{A}\tilde{B}(2^r - 1)$

$= 2^n M_r[2^k\beta^*\alpha^*\tilde{B}\tilde{A} - \alpha^*\beta^*\tilde{A}\tilde{B}]$

The identity (ii) can be proved similarly by using Binet formula.

Theorem 7. (Honsberger Identity)**.** *Let* m, n *be any integers then*

i. $\overbrace{M_n}\overbrace{M_m} + \overbrace{M_{n+1}}\overbrace{M_{m+1}} = 2^{n+m}(5)(\alpha^*)^2(\tilde{A})^2 -$

$2^m(3)\beta^*\alpha^*\tilde{B}\tilde{A} - 2^n(3)\alpha^*\beta^*\tilde{A}\tilde{B} + 2(\beta^*)^2(\tilde{B})^2$

ii. $\overbrace{ML_n}\overbrace{ML_m} + \overbrace{ML_{n+1}}\overbrace{ML_{m+1}} = 2^{n+m}(5)(\alpha^*)^2(\tilde{A})^2 +$

$2^m(3)\beta^*\alpha^*\tilde{B}\tilde{A} + 2^n(3)\alpha^*\beta^*\tilde{A}\tilde{B} + 2(\beta^*)^2(\tilde{B})^2.$

Proof of Theorem 7.

i. $\overbrace{M_n}\overbrace{M_m} - \overbrace{M_{n+1}}\overbrace{M_{m+1}}$

$= (2^n\alpha^*\tilde{A} - \beta^*\tilde{B})(2^m\alpha^*\tilde{A} - \beta^*\tilde{B}) - (2^{n+1}\alpha^*\tilde{A} - \beta^*\tilde{B})(2^{m+1}\alpha^*\tilde{A} - \beta^*\tilde{B})$

$= 2^{n+m}(\alpha^*)^2(\tilde{A})^2(2^r - 1) - 2^m\beta^*\alpha^*\tilde{B}\tilde{A}(2+1) - 2^n\alpha^*\beta^*\tilde{A}\tilde{B}(2+1) + 2(\beta^*)^2(\tilde{B})^2$

$= 2^{n+m}(5)(\alpha^*)^2(\tilde{A})^2 - 2^m(3)\beta^*\alpha^*\tilde{B}\tilde{A} - 2^n(3)\alpha^*\beta^*\tilde{A}\tilde{B} + 2(\beta^*)^2(\tilde{B})^2$

In the same way, using Binet formula one can prove (ii).

Theorem 8. *Let* $\overbrace{M_n}$ *be the* n^{th} *term of the Mersenne hybrid quaternion sequence, then*

$$3\overbrace{M_{n+1}} - 2\overbrace{M_n} = \overbrace{M_{n+2}}.$$

Proof of Theorem 8. First, we prove this relation by using Mersenne hybrid numbers

$$3\overbrace{M_{n+1}} - 2\overbrace{M_n} = 3(\ddot{M}_{n+1} + i\ddot{M}_{n+2} + \varepsilon\ddot{M}_{n+3} + h\ddot{M}_{n+4})$$

$$-2(\ddot{M}_n + i\ddot{M}_{n+1} + \varepsilon\ddot{M}_{n+2} + h\ddot{M}_{n+3})$$

$$= (3\ddot{M}_{n+1} - 2\ddot{M}_n) + i(3\ddot{M}_{n+2} - 2\ddot{M}_{n+1}) + \varepsilon(3\ddot{M}_{n+3} - 2\ddot{M}_{n+2}) + h(3\ddot{M}_{n+4} - 2\ddot{M}_{n+3})$$

$$= \ddot{M}_{n+2} + i\ddot{M}_{n+3} + \varepsilon\ddot{M}_{n+4} + h\ddot{M}_{n+5}$$

$$= \overbrace{M_{n+2}}.$$

Next, by using Mersenne quaternions

$$3\overbrace{M_{n+1}} - 2\overbrace{M_n} = 3(\widetilde{M_{n+1}} + i\widetilde{M_{n+2}} + \varepsilon\widetilde{M_{n+3}} + h\widetilde{M_{n+4}})$$

$$-2(\widetilde{M_n} + i\widetilde{M_{n+1}} + \varepsilon\widetilde{M_{n+2}} + h\widetilde{M_{n+3}})$$

$$= (3\widetilde{M_{n+1}} - 2\widetilde{M_n}) + i(3\widetilde{M_{n+2}} - 2\widetilde{M_{n+1}}) + \varepsilon(3\widetilde{M_{n+3}} - 2\widetilde{M_{n+2}}) + h(3\widetilde{M_{n+4}} - 2\widetilde{M_{n+3}})$$

$$= \widetilde{M_{n+2}} + i\widetilde{M_{n+3}} + \varepsilon\widetilde{M_{n+4}} + h\widetilde{M_{n+5}}$$

$$= \overbrace{M_{n+2}}.$$

Theorem 9. *Let* $\overbrace{M_n}$ *and* $\overbrace{ML_n}$ *be Mersenne hybrid quaternion and Mersenne-Lucas hybrid quaternion. Then*

$$2\overbrace{ML_{n+1}} - 3\overbrace{ML_n} = \overbrace{M_n}.$$

Proof of Theorem 9.

$$2\overbrace{ML_{n+1}} - 3\overbrace{ML_n} = 2(\alpha^{n+1}\alpha^*\tilde{A} + \beta^{n+1}\beta^*\tilde{B}) - 3(\alpha^n\alpha^*\tilde{A} + \beta^n\beta^*\tilde{B})$$

$$= 2\alpha^{n+1}\alpha^*\tilde{A} + 2\beta^{n+1}\beta^*\tilde{B} - 3\alpha^n\alpha^*\tilde{A} - 3\beta^n\beta^*\tilde{B}$$

$$= \alpha^n\alpha^*\tilde{A}(2\alpha - 3) + \beta^n\beta^*\tilde{B}(2\beta - 3)$$

$$= \alpha^n\alpha^*\tilde{A} - \beta^{\eta}\beta^*\tilde{B} = \overbrace{M_n}.$$

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