ON THE MERSENNE AND MERSENNE-LUCAS HYBRID QUATERNIONS

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Abstract

In this communication, we define the Mersenne hybrid quaternions and give some of their properties. Further, we analyze the relations between the Mersenne hybrid quaternions and the Mersenne-Lucas hybrid quaternions which connected Mersenne quaternions and Mersenne-Lucas quaternions. Also, we give the Binet formulas and moreover, well known identities like Catalan identity, Cassini identity and d'Ocagne's identity for these quaternions.

1. Introduction

In [10], Ozdemir introduced hybrid numbers as a new type of numbers. Hybrid numbers are generalizations of complex, hyperbolic and dual numbers. A hyperbolic complex structure has many applications in both pure mathematics and various areas of Physics [2, 11]. Hybrid numbers can be connected with the family of Mersenne type numbers. Herewith, we recall hybrid number definition as

$$\mathcal{H} = a + bi + c\varepsilon + dh, \ a, b, c, d \in \mathbb{R},$$

$$\varepsilon^2 = -1, \ v^2 = 0, \ h^2 = 1, \ ih = -hi = \varepsilon + i.$$  

The conjugate of the hybrid number $\mathcal{H}$ is denoted by
A quaternion has an extension of the complex numbers was first defined by Hamilton [6]. The quaternion of sequences was first considered by Horadam [8]. A real quaternion is defined as

\[ Q = z_0 + z_1i + z_2j + z_3k, \text{ where } z_0, z_1, z_2, z_3 \in \mathbb{R}. \]

Also \( i, j, k \) are the units of the real quaternions which satisfy the equalities

\[ i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \]

The conjugate of the quaternion \( Q \) is denoted by

\[ \overline{Q} = z_0 - z_1i - z_2j - z_3k. \]

Although, the advantages of the quaternions appeared in the fundamental equations of some field of science [3, 4, 5, 7]. Recently, many mathematicians are trying more and more to use algebraic properties of quaternions to make easy and efficient calculations [1, 9, 12, 13]. This system has a strong algebraic structure and it is a generalization of dual and hyperbolic quaternion. Moreover, hybrid quaternions are also generalized features of quaternions system such as inner product, vector product and norm.

The Mersenne hybrid numbers and Mersenne-Lucas hybrid numbers are defined as

\[ \tilde{M}_n = M_n + M_{n+1}i + M_{n+2}e + M_{n+3}h \]

\[ \tilde{ML}_n = ML_n + ML_{n+1}i + ML_{n+2}e + ML_{n+3}h \]

The Mersenne quaternions and Mersenne-Lucas quaternions are defined as

\[ \overline{M}_n = M_n + iM_{n+1} + jM_{n+2} + kM_{n+3} \]

\[ \overline{ML}_n = ML_n + iML_{n+1} + jML_{n+2} + kML_{n+3} \]
Table 1. Notations.

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**Definition.** The $n^{th}$ Mersenne hybrid quaternions $\widehat{M}_n$ is defined by

$$\widehat{M}_n = \hat{M}_n + i\hat{M}_{n+1} + j\hat{M}_{n+2} + k\hat{M}_{n+3}$$

where $i$, $j$, $k$ are quaternion units.

We will restate $\widehat{M}_n$ by

$$\widehat{M}_n = \overline{M}_n + i\overline{M}_{n+1} + j\overline{M}_{n+2} + k\overline{M}_{n+3}$$

**Definition.** The Mersenne-Lucas hybrid quaternions are defined as

$$\widehat{ML}_n = \hat{ML}_n + i\hat{ML}_{n+1} + j\hat{ML}_{n+2} + k\hat{ML}_{n+3}$$

can be written as

$$\overline{\hat{ML}}_n = \overline{ML}_n + i\overline{ML}_{n+1} + j\overline{ML}_{n+2} + k\overline{ML}_{n+3}$$

**Definition.** Let $\overline{U}_n$ and $\overline{V}_n$ be the $n^{th}$ terms of the Mersenne hybrid quaternion sequences such that
\[ \widehat{U}_n = \overline{U}_n + i\overline{U}_{n+1} + j\overline{U}_{n+2} + k\overline{U}_{n+3} = \overline{U}_n + i\overline{U}_{n+1} + \varepsilon\overline{U}_{n+2} + h\overline{U}_{n+3} \]

and

\[ \widehat{V}_n = \overline{V}_n + i\overline{V}_{n+1} + j\overline{V}_{n+2} + k\overline{V}_{n+3} = \overline{V}_n + i\overline{V}_{n+1} + \varepsilon\overline{V}_{n+2} + h\overline{V}_{n+3} \]

Then the addition and subtraction of the Mersenne hybrid quaternions are defined by

\[ \widehat{U}_n \pm \widehat{V}_n = (\overline{U}_n + i\overline{U}_{n+1} + j\overline{U}_{n+2} + k\overline{U}_{n+3}) \pm (\overline{V}_n + i\overline{V}_{n+1} + j\overline{V}_{n+2} + k\overline{V}_{n+3}) \]

\[ = (\overline{U}_n \pm \overline{V}_n) + i(\overline{U}_{n+1} \pm \overline{V}_{n+1}) + j(\overline{U}_{n+2} \pm \overline{V}_{n+2}) + k(\overline{U}_{n+3} \pm \overline{V}_{n+3}) \]

\[ \widehat{U}_n \pm \widehat{V}_n = (\overline{U}_n + i\overline{U}_{n+1} + j\overline{U}_{n+2} + k\overline{U}_{n+3}) \pm (\overline{V}_n + i\overline{V}_{n+1} + j\overline{V}_{n+2} + k\overline{V}_{n+3}) \]

\[ = (\overline{U}_n \pm \overline{V}_n) + i(\overline{U}_{n+1} \pm \overline{V}_{n+1}) + j(\overline{U}_{n+2} \pm \overline{V}_{n+2}) + k(\overline{U}_{n+3} \pm \overline{V}_{n+3}) \]

**Definition.** The multiplication of the Mersenne hybrid quaternions in terms of Mersenne hybrid numbers is defined as

\[ \widehat{U}_n \widehat{V}_n = (\overline{U}_n + i\overline{U}_{n+1} + j\overline{U}_{n+2} + k\overline{U}_{n+3})(\overline{V}_n + i\overline{V}_{n+1} + j\overline{V}_{n+2} + k\overline{V}_{n+3}) \]

\[ = (\overline{U}_n\overline{V}_n - \overline{U}_{n+1}\overline{V}_{n+1} - \overline{U}_{n+2}\overline{V}_{n+2} - \overline{U}_{n+3}\overline{V}_{n+3}) \]

\[ + i(\overline{U}_n\overline{V}_{n+1} + \overline{U}_{n+1}\overline{V}_n + \overline{U}_{n+2}\overline{V}_{n+3} - \overline{U}_{n+3}\overline{V}_{n+2}) \]

\[ + j(\overline{U}_n\overline{V}_{n+2} + \overline{U}_{n+1}\overline{V}_{n+3} + \overline{U}_{n+2}\overline{V}_n - \overline{U}_{n+3}\overline{V}_{n+1}) \]

\[ + k(\overline{U}_n\overline{V}_{n+3} + \overline{U}_{n+1}\overline{V}_{n+2} + \overline{U}_{n+2}\overline{V}_{n+1} + \overline{U}_{n+3}\overline{V}_n) \]

In terms of Mersenne quaternions we defined as

\[ \widehat{U}_n \widehat{V}_n = (\overline{U}_n + i\overline{U}_{n+1} + \varepsilon\overline{U}_{n+2} + h\overline{U}_{n+3})(\overline{V}_n + i\overline{V}_{n+1} + \varepsilon\overline{V}_{n+2} + h\overline{V}_{n+3}) \]

\[ = (\overline{U}_n\overline{V}_n - \overline{U}_{n+1}\overline{V}_{n+1} + \overline{U}_{n+3}\overline{V}_{n+3} + \overline{U}_{n+2}\overline{V}_{n+2} + \overline{U}_{n+1}\overline{V}_{n+1}) \]

\[ + i(\overline{U}_n\overline{V}_{n+1} - \overline{U}_{n+1}\overline{V}_n + \overline{U}_{n+2}\overline{V}_{n+3} - \overline{U}_{n+3}\overline{V}_{n+2}) \]

\[ + j(\overline{U}_n\overline{V}_{n+2} + \overline{U}_{n+1}\overline{V}_{n+3} + \overline{U}_{n+2}\overline{V}_n - \overline{U}_{n+3}\overline{V}_{n+1}) \]

\[ + k(\overline{U}_n\overline{V}_{n+3} + \overline{U}_{n+1}\overline{V}_{n+2} + \overline{U}_{n+2}\overline{V}_{n+1} - \overline{U}_{n+3}\overline{V}_n) \]
**Definition.** The conjugate of Mersenne hybrid quaternions is defined by

i. Quaternion conjugate:

\[ \overline{M_n} = M_n + iM_{n+1} + eM_{n+2} + hM_{n+3} \]

ii. Hybrid conjugate:

\[ \overline{M_n^c} = M_n - iM_{n+1} - eM_{n+2} - hM_{n+3} \]

iii. Hybrid quaternion conjugate:

\[ (\overline{M_n^c})^c = M_n + iM_{n+1} + eM_{n+2} + hM_{n+3} \]

**Theorem 1.** Let \( \overline{M_n} \) and \( \overline{ML_n} \) be Mersenne hybrid quaternion and Mersenne-Lucas hybrid quaternion. The Binet formulas for these hybrid quaternions are given as

i. \( \overline{M_n} = \alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B} \)

ii. \( \overline{ML_n} = \alpha^n \alpha^* \tilde{A} + \beta^n \beta^* \tilde{B} \)

where \( \alpha^* = 1 + ia + e\alpha^2 + ha^3, \beta^* = 1 + ib + e\beta^2 + h\beta^3, \tilde{A} = 1 + ia + j\alpha^2 + k\alpha^3 \)

and \( \tilde{B} = 1 + ib + j\beta^2 + k\beta^3, \alpha = 2, \beta = 1. \)

**Proof of Theorem 1.** The Binet formulas for the Mersenne quaternions and Mersenne-Lucas quaternions are \( \overline{M_n} = \alpha^n \tilde{A} - \beta^n \tilde{B} \) and \( \overline{ML_n} = \alpha^n \tilde{A} - \beta^n \tilde{B} \)

\[ \overline{M_n} = M_n + i\overline{M_{n+1}} + e\overline{M_{n+2}} + h\overline{M_{n+3}} \]

\[ = (\alpha^n \tilde{A} - \beta^n \tilde{B}) + i(\alpha^{n+1} \tilde{A} - \beta^{n+1} \tilde{B}) + e(\alpha^{n+2} \tilde{A} - \beta^{n+2} \tilde{B}) + h(\alpha^{n+3} \tilde{A} - \beta^{n+3} \tilde{B}) \]

\[ = \alpha^n \tilde{A}(1 + ia + e\alpha^2 + ha^3) - \beta^n \tilde{B}(1 + ib + e\beta^2 + h\beta^3) \]

\[ = \alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B} \]

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\[ ML_n = ML_n + iML_{n+1} + eML_{n+2} + hML_{n+3} \]
\[ = (\alpha^n \hat{A} - \beta^n \hat{B}) + i(\alpha^{n+1} \hat{A} - \beta^{n+1} \hat{B}) + e(\alpha^{n+2} \hat{A} - \beta^{n+2} \hat{B}) + h(\alpha^{n+3} \hat{A} - \beta^{n+3} \hat{B}) \]
\[ = \alpha^n \hat{A}(1 + i\alpha + e\alpha^2 + h\alpha^3) - \beta^n \hat{B}(1 + i\beta + e\beta^2 + h\beta^3) \]
\[ = \alpha^n \alpha^* \hat{A} - \beta^n \beta^* \hat{B} \]

**Theorem 2.** Let \( \hat{M}_n \) and \( \hat{ML}_n \) be Mersenne hybrid quaternion and Mersenne-Lucas hybrid quaternion. Then

i. \( \hat{M}_n + \hat{M}_{n+1} = 3\alpha^n \alpha^* \hat{A} - 2\beta^n \beta^* \hat{B} \)

ii. \( \hat{ML}_n + \hat{ML}_{n+1} = 3\alpha^n \alpha^* \hat{A} - 2\beta^n \beta^* \hat{B} \)

**Proof of Theorem 2.** By theorem 1, we have \( \hat{M}_n = \alpha^n \alpha^* \hat{A} - \beta^n \beta^* \hat{B} \)

\[ \hat{M}_n + \hat{M}_{n+1} = (\alpha^n \alpha^* \hat{A} - \beta^n \beta^* \hat{B}) + (\alpha^{n+1} \alpha^* \hat{A} - \beta^{n+1} \beta^* \hat{B}) \]
\[ = \alpha^n \alpha^* \hat{A}(\alpha + 1) - \beta^n \beta^* \hat{B}(\beta + 1) \]
\[ = 3\alpha^n \alpha^* \hat{A} - 2\beta^n \beta^* \hat{B} \]

And by using \( \hat{ML}_n = \alpha^n \alpha^* \hat{A} - \beta^n \beta^* \hat{B} \), we can prove (ii).

**Theorem 3.** (Catalan’s Identity) Let \( n, r \in \mathbb{Z} \), then we have

i. \( \hat{M}_{n-r} \hat{M}_{n+r} - \hat{M}_n^2 = 2^{n-r} M_r [\beta^r \alpha^* \hat{A} \hat{B} - \alpha^r \beta^* \hat{A} \hat{A}] \)

ii. \( \hat{ML}_{n-r} \hat{ML}_{n+r} - \hat{ML}_n^2 = 2^{n-r} M_r [\alpha^r \beta^* \hat{A} \hat{B} - \beta^r \alpha^* \hat{A} \hat{A}] \)

**Proof of Theorem 3.**

i. \( \hat{M}_{n-r} \hat{M}_{n+r} - \hat{M}_n^2 \)
\[ = (\alpha^{n-r} \alpha^* \hat{A} - \beta^{n-r} \beta^* \hat{B})(\alpha^{n+r} \alpha^* \hat{A} - \beta^{n+r} \beta^* \hat{B}) - (\alpha^n \alpha^* \hat{A} - \beta^n \beta^* \hat{B})^2 \]
\[ \alpha^n \beta^n - \beta^* \alpha^* \tilde{B} \tilde{A}(\beta^r - \alpha^r) - \alpha^n \beta^n \alpha^* \beta^* \tilde{A} \tilde{B}(\beta^r - \alpha^r) \]
\[ = \alpha^{n-r} \beta^n \alpha^* \beta^* \tilde{A} \tilde{B}(2^r - 1) - \alpha^r \beta^* \alpha^* \tilde{B} \tilde{A}(2^r - 1) \]
\[ = 2^{n-r} M_r [\beta^r \alpha^* \beta^* \tilde{A} \tilde{B} - \alpha^r \beta^* \alpha^* \tilde{B} \tilde{A}] \]

Similarly, proceeding like this we obtain identity (ii).

By substituting \( r = 1 \) in Theorem 3, we get Cassini’s Identity.

**Theorem 4.** (Cassini’s Identity). Let \( n \) be any integer then

\[
\begin{align*}
\text{i. } & \quad \tilde{M}_{n-1} \tilde{M}_{n+1} - \tilde{M}_n^2 = 2^{n-1} [\beta \alpha^* \beta^* \tilde{A} \tilde{B} - \alpha \beta^* \alpha^* \tilde{B} \tilde{A}] \\
\text{ii. } & \quad \tilde{M}_{L_{n-1}} \tilde{M}_{L_{n+1}} - \tilde{M}_{L_n}^2 = 2^{n-1} [\alpha \beta^* \alpha^* \tilde{B} \tilde{A} - \beta \alpha^* \beta^* \tilde{A} \tilde{B}].
\end{align*}
\]

**Theorem 5.** (d’Ocagne’s Identity). Let \( m, n \) be any integers then

\[
\begin{align*}
\text{i. } & \quad \tilde{M}_m \tilde{M}_{n+1} - \tilde{M}_{n+1} \tilde{M}_n = \alpha^m \beta^n \alpha^* \beta^* \tilde{A} \tilde{B} - \alpha^n \beta^m \beta^* \alpha^* \tilde{B} \tilde{A} \\
\text{ii. } & \quad \tilde{M}_L_m \tilde{M}_{L_{n+1}} - \tilde{M}_{L_{n+1}} \tilde{M}_{L_n} = \alpha^m \beta^n \beta^* \alpha^* \tilde{B} \tilde{A} - \alpha^n \beta^m \alpha^* \beta^* \tilde{A} \tilde{B}.
\end{align*}
\]

**Proof of Theorem 5.**

\[
\begin{align*}
\text{i. } & \quad \tilde{M}_m \tilde{M}_{n+1} - \tilde{M}_{n+1} \tilde{M}_n = \left( \alpha^m \alpha^* \tilde{A} - \beta^m \beta^* \tilde{B} \right) \left( \alpha^{n+1} \alpha^* \tilde{A} - \beta^{n+1} \beta^* \tilde{B} \right) \\
& \quad - \left( \alpha^{m+1} \alpha^* \tilde{A} - \beta^{m+1} \beta^* \tilde{B} \right) \left( \alpha^n \alpha^* \tilde{A} - \beta^n \beta^* \tilde{B} \right) \\
& \quad = \alpha^n \beta^m \beta^* \alpha^* \tilde{B} \tilde{A}(\beta - \alpha) - \alpha^m \beta^n \alpha^* \beta^* \tilde{A} \tilde{B}(\beta - \alpha) \\
& \quad = \alpha^n \beta^m \beta^* \alpha^* \tilde{B} \tilde{A} - \alpha^m \beta^n \alpha^* \beta^* \tilde{A} \tilde{B}.
\end{align*}
\]

In a similar way, the second identity can be proved.

**Theorem 6.** (Vajda Identity). Let \( k, n, r \) be any integers then

\[
\begin{align*}
\text{i. } & \quad \tilde{M}_{n^r} \tilde{M}_{n+k} - \tilde{M}_{n+k} \tilde{M}_{n^r+k} = 2^n M_r \left[ 2^k \beta^* \alpha^* \tilde{B} \tilde{A} - \alpha^* \beta^* \tilde{A} \tilde{B} \right] \\
\text{ii. } & \quad \tilde{M}_{L_{n^r}} \tilde{M}_{L_{n+k}} - \tilde{M}_{L_{n+k}} \tilde{M}_{L_{n^r+k}} = 2^n M_r \left[ \alpha^* \beta^* \tilde{A} \tilde{B} - 2^k \beta^* \alpha^* \tilde{B} \tilde{A} \right].
\end{align*}
\]
Proof of Theorem 6.

i. \( \overline{\text{ML}_{n+r} \overline{\text{ML}_{n+k}}} - \overline{\text{ML}_{n} \overline{\text{ML}_{n+r+k}}} \)

\[ = (2^{n+r} a^* \tilde{A} \beta^* \tilde{B})(2^{n+k} a^* \tilde{A} \beta^* \tilde{B}) - (2^n a^* \tilde{A} \beta^* \tilde{B})(2^{n+r+k} a^* \tilde{A} \beta^* \tilde{B}) \]

\[ = 2^{n+k} \beta^* a^* \tilde{B} \tilde{A}(2r-1) - 2^n a^* \beta^* \tilde{A} \tilde{B}(2^r-1) \]

\[ = 2^n M_r [2^k \beta^* a^* \tilde{B} \tilde{A} - a^* \beta^* \tilde{A} \tilde{B}] \]

The identity (ii) can be proved similarly by using Binet formula.

Theorem 7. (Honsberger Identity). Let \( m, n \) be any integers then

i. \( \overline{\text{M}_n \overline{\text{M}_m}} + \overline{\text{M}_{n+1} \overline{\text{M}_{m+1}}} = 2^{n+m} (5)(\alpha^*)^2 (\tilde{A})^2 - 2^m (3) \beta^* \alpha^* \tilde{B} \tilde{A} - 2^n (3) \alpha^* \beta^* \tilde{A} \tilde{B} + 2(\beta^*)^2 (\tilde{B})^2 \)

ii. \( \overline{\text{ML}_n \overline{\text{ML}_m}} + \overline{\text{ML}_{n+1} \overline{\text{ML}_{m+1}}} = 2^{n+m} (5)(\alpha^*)^2 (\tilde{A})^2 + 2^m (3) \beta^* \alpha^* \tilde{B} \tilde{A} + 2^n (3) \alpha^* \beta^* \tilde{A} \tilde{B} + 2(\beta^*)^2 (\tilde{B})^2 \).

Proof of Theorem 7.

i. \( \overline{\text{M}_n \overline{\text{M}_m}} - \overline{\text{M}_{n+1} \overline{\text{M}_{m+1}}} \)

\[ = (2^n a^* \tilde{A} \beta^* \tilde{B})(2^m a^* \tilde{A} \beta^* \tilde{B}) - (2^{n+1} a^* \tilde{A} \beta^* \tilde{B})(2^{m+1} a^* \tilde{A} \beta^* \tilde{B}) \]

\[ = 2^{n+m} (\alpha^*)^2 (\tilde{A})^2 (2^r - 1) - 2^n \alpha^* \beta^* \tilde{B} \tilde{A}(2+1) - 2^n a^* \beta^* \tilde{A} \tilde{B}(2+1) + 2(\beta^*)^2 (\tilde{B})^2 \]

\[ = 2^{n+m} (5)(\alpha^*)^2 (\tilde{A})^2 - 2^m (3) \beta^* \alpha^* \tilde{B} \tilde{A} - 2^n (3) \alpha^* \beta^* \tilde{A} \tilde{B} + 2(\beta^*)^2 (\tilde{B})^2 \]

In the same way, using Binet formula one can prove (ii).

Theorem 8. Let \( \overline{\text{M}_n} \) be the \( n \)th term of the Mersenne hybrid quaternion sequence, then

\[ 3 \overline{\text{M}_{n+1}} - 2 \overline{\text{M}_n} = \overline{\text{M}_{n+2}}. \]

Proof of Theorem 8. First, we prove this relation by using Mersenne
hybrid numbers

\[
3\overline{M_{n+1}} - 2\overline{M_n} = 3(\overline{M_{n+1}} + i\overline{M_{n+2}} + \epsilon\overline{M_{n+3}} + h\overline{M_{n+4}})
- 2(\overline{M_{n+1}} + i\overline{M_{n+2}} + \epsilon\overline{M_{n+3}} + h\overline{M_{n+4}})
\]

\[
= (3\overline{M_{n+1}} - 2\overline{M_n}) + i(3\overline{M_{n+2}} - 2\overline{M_{n+1}}) + \epsilon(3\overline{M_{n+3}} - 2\overline{M_{n+2}}) + h(3\overline{M_{n+4}} - 2\overline{M_{n+3}})
\]

\[
= \overline{M_{n+2}} + i\overline{M_{n+3}} + \epsilon\overline{M_{n+4}} + h\overline{M_{n+5}}
\]

\[
= \overline{M_{n+2}}.
\]

Next, by using Mersenne quaternions

\[
3\overline{M_{n+1}} - 2\overline{M_n} = 3(\overline{M_{n+1}} + i\overline{M_{n+2}} + \epsilon\overline{M_{n+3}} + h\overline{M_{n+4}})
- 2(\overline{M_{n+1}} + i\overline{M_{n+2}} + \epsilon\overline{M_{n+3}} + h\overline{M_{n+4}})
\]

\[
= (3\overline{M_{n+1}} - 2\overline{M_n}) + i(3\overline{M_{n+2}} - 2\overline{M_{n+1}}) + \epsilon(3\overline{M_{n+3}} - 2\overline{M_{n+2}}) + h(3\overline{M_{n+4}} - 2\overline{M_{n+3}})
\]

\[
= \overline{M_{n+2}} + i\overline{M_{n+3}} + \epsilon\overline{M_{n+4}} + h\overline{M_{n+5}}
\]

\[
= \overline{M_{n+2}}.
\]

**Theorem 9.** Let \( \overline{M_n} \) and \( \overline{ML_n} \) be Mersenne hybrid quaternion and Mersenne-Lucas hybrid quaternion. Then

\[
2\overline{ML_{n+1}} - 3\overline{ML_n} = \overline{M_n}.
\]

**Proof of Theorem 9.**

\[
2\overline{ML_{n+1}} - 3\overline{ML_n} = 2(\alpha^{n+1}a^*\tilde{A} + \beta^{n+1}b^*\tilde{B}) - 3(\alpha^n a^*\tilde{A} + \beta^n b^*\tilde{B})
\]

\[
= 2\alpha^{n+1}a^*\tilde{A} + 2\beta^{n+1}b^*\tilde{B} - 3\alpha^n a^*\tilde{A} - 3\beta^n b^*\tilde{B}
\]

\[
= \alpha^n a^*\tilde{A}(2\alpha - 3) + \beta^n b^*\tilde{B}(2\beta - 3)
\]

\[
= \alpha^n a^*\tilde{A} - \beta^n b^*\tilde{B} = \overline{M_n}.
\]
References


