NEW RESULTS FOR THE DESCARTES-FRENICLE-SORLI CONJECTURE ON ODD PERFECT NUMBERS

JOSE ARNALDO B. DRIS

Far Eastern University
Manila, Philippines
E-mail: josearnaldodris@gmail.com

Abstract

If \( N = q^k n^2 \) is an odd perfect number given in Eulerian form, then the Descartes-Frenicle-Sorli conjecture predicts that \( k = 1 \). Brown [5] has recently announced a proof for the inequality \( q < n \), and a partial proof that \( q^k < n \) holds under many cases. In this article, we give a strategy for strengthening Brown’s result to \( q^2 < n \).

1. Introduction

If \( N \) is a positive integer, then we write \( \sigma(N) \) for the sum of the divisors of \( N \). A number \( N \) is perfect if \( \sigma(N) = 2N \). It is currently unknown whether there are infinitely many even perfect numbers, or whether any odd perfect numbers (OPNs) exist. Ochem and Rao recently proved [12] that, if \( N \) is an odd perfect number, then \( N > 10^{1500} \) and that the largest component (i.e., divisor \( p^a \) with \( p \) prime) of \( N \) is bigger than \( 10^{62} \). This improves on previous results by Brent, Cohen and te Riele [3] in 1991 \( (N > 10^{300}) \) and Cohen [7] in 1987 \( (\text{largest component } p^a > 10^{20}) \).

An odd perfect number \( N = q^k n^2 \) is said to be given in Eulerian form if \( q \) is prime with \( q \equiv k \equiv 1 \pmod{4} \) and \( \gcd(q, n) = 1 \). (The number \( q \) is called the Euler prime, while the component \( q^k \) is referred to as the Euler factor. Note that, since \( q \) is prime and \( q \equiv 1 \pmod{4} \), then \( q \geq 5 \).

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We denote the abundancy index $I$ of the positive integer $x$ as

$$I(x) = \frac{\sigma(x)}{x}.$$ 

In his Ph.D. thesis, Sorli [13] conjectured that $k = 1$, after testing large numbers with 8 distinct prime factors for perfection. (More recently, Beasley [2] points out that Descartes was the first to conjecture $k = 1$ “in a letter to Mersenne in 1638, with Frenicle’s subsequent observation occurring in 1657.”)

In the M.Sc. thesis [11], it was conjectured that the components $q^k$ and $n$ are related by the inequality $q^k < n$. This conjecture was made on the basis of the result $I(q^k) < I(n)$. Recently, Brown [5] announced a proof for the inequality $q < n$, and a partial proof that $q^k < n$ holds under many cases.

2. Conditions Sufficient for Sorli’s Conjecture

Some sufficient conditions for Sorli’s conjecture were given in [9]. We reproduce these conditions here.

**Lemma 1.** Let $N = q^k n^2$ be an odd perfect number given in Eulerian form. If $n < q$, then $k = 1$.

**Remark 2.** The proof of Lemma 1 follows from the inequality $q^k < n^2$ and the congruence $k \equiv 1 \pmod{4}$ (see [9]). (Note the related inequality

$$I(q^k) < I(n^2)$$

for the abundancy indices of the components $q^k$ and $n^2$).

**Lemma 3.** Let $N = q^k n^2$ be an odd perfect number given in Eulerian form. If

$$\sigma(n) \leq \sigma(q),$$

then $k = 1$.

**Lemma 4.** Let $N = q^k n^2$ be an odd perfect number given in Eulerian form. If
\[
\frac{\sigma(n)}{q} < \frac{\sigma(q)}{n},
\]

then \( k = 1. \)

**Remark 5.** Notice that, if

\[
\frac{\sigma(n)}{q} < \frac{\sigma(q)}{n},
\]

then it follows that

\[
\frac{\sigma(n)}{q^k} = \frac{\sigma(n)}{q} < \frac{\sigma(q)}{n} = \frac{\sigma(q^k)}{n}.
\]

Consequently, by the contrapositive, if

\[
\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k},
\]

then

\[
\frac{\sigma(q)}{n} < \frac{\sigma(q^k)}{n^k} < \frac{\sigma(n)}{q^k} < \frac{\sigma(n)}{q}.
\]

**Remark 6.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Suppose that

\[
\frac{\sigma(q)}{n} = \frac{\sigma(n)}{q}.
\]

Then we know that:

\[q\sigma(q) = n\sigma(n).\]

Since \( \gcd(q, n) = 1 \), then \( q | \sigma(n) \) and \( n | \sigma(q) \). Therefore, it follows that \( \frac{\sigma(q)}{n} \) and \( \frac{\sigma(n)}{q} \) are equal positive integers.

This is a contradiction, as:

\[
1 < I(q) = \frac{\sigma(q)}{q} = 1 + \frac{1}{q} \leq \frac{6}{5} < \sqrt[5]{3} < I(n) < I(q)I(n) = I(qn) < 2
\]
which implies that:

\[ 1 < \sqrt{\frac{5}{3}} < I(n) < I(q)I(n) = I(qn) = \left[ \frac{\sigma(q)}{q} \right] \left[ \frac{\sigma(n)}{n} \right] = \left[ \frac{\sigma(q)}{n} \right] \left[ \frac{\sigma(n)}{q} \right] < 2. \]

Consequently,

\[ \frac{\sigma(q)}{n} \neq \frac{\sigma(n)}{q}. \]

Similarly, we can prove that

\[ \frac{\sigma(q^k)}{n} \neq \frac{\sigma(n)}{q^k}. \]

**Lemma 7.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Then \( n < q \) if and only if \( N < q^3 \).

**Proof.** Suppose that \( N = q^k n^2 \) is an odd perfect number given in Eulerian form. If \( n < q \), then assuming to the contrary that \( q^3 < N \), we get that

\[ q^3 < N = qn^2 < q \cdot q^2 = q^3 \]

since \( n < q \) implies \( k = 1 \), by Lemma 1. For the other direction, if \( N < q^3 \), then \( q^k n^2 < q^3 \), so that we have

\[ n^2 < q^{3-k} \leq q^2 \]

since \( k \equiv 1 \pmod{4} \) implies that \( k \geq 1 \). Consequently, \( n < q \), and we are done. \( \square \)

**Corollary 8.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Then \( n < q^{5/2} \) if and only if \( N < q^6 \).

**Proof.** First we show that \( n < q^{5/2} \) implies \( k = 1 \). To this end, assuming \( n < q^{5/2} \), since \( q^k < n^2 \) (see [9]), we then have that:

\[ q \leq q^k < n^2 < q^5. \]
The last chain of inequalities implies that

\[ 1 \leq k < 5. \]

This inequality, together with the condition \( k \equiv 1 \text{ (mod 4)} \), implies that \( k = 1 \).

We now prove the claim in Corollary 8. If \( n < q^{5/2} \), then assuming to the contrary that \( q^6 < N \), we get that

\[ q^6 < N = qn^2 < q \cdot q^5 = q^6. \]

This is a contradiction. For the other direction, if \( N < q^6 \), then \( q^k n^2 < q^6 \), so that we have

\[ n^2 < q^{6-k} \leq q^5 \]

since \( k \equiv 1 \text{ (mod 4)} \) implies that \( k \geq 1 \). Consequently, \( n < q^{5/2} \), and we are done. \( \square \)

**Remark 9.** A recent result by Acquaah and Konyagin [1] almost disproves \( n < q \). They obtained the estimate \( y < (3N)^{1/3} \) for all the prime factors \( y \) of an odd perfect number \( N \). In particular, if \( N = q^k n^2 \) is an odd perfect number given in Eulerian form, then letting \( y = q \) and assuming \( k = 1 \) gives:

\[ q < (3N)^{1/3} = (3qn^2)^{1/3} \Rightarrow q^3 < 3qn^2 \Rightarrow q < n \sqrt[3]{3}. \]

Since the contrapositive of the implication \( n < q \Rightarrow k = 1 \) is \( k > 1 \Rightarrow q < n \), it follows that the inequality

\[ q < n \sqrt[3]{3} \]

holds unconditionally, regardless of the status of Sorli’s conjecture.

More recently, Brown [5] claims a proof for the inequality \( q < n \), and a partial proof that \( q^k < n \) holds under many cases.

We now give a condition that is weaker than \( n < q \), which also implies \( k = 1 \).
Lemma 10. Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Then
\[
n < \left( \frac{3}{2} q^5 \right)^{1/2}
\]
implies \( k = 1 \).

Proof. Suppose that \( N = q^k n^2 \) is an odd perfect number given in Eulerian form. Let
\[
n < \left( \frac{3}{2} q^5 \right)^{1/2}
\]
and assume to the contrary that \( k \neq 1 \). Since \( k \equiv 1 \pmod{4} \), this means that \( k \geq 5 \). Additionally, from [9], we have that
\[
q^k < \sigma(q^k) \leq \frac{2}{3} n^2.
\]
Consequently, we have the following chain of inequalities:
\[
q^5 \leq q^k < \frac{2}{3} \left( \frac{3}{2} q^5 \right)^{1/2} < q^5.
\]
This is a contradiction. \( \square \)

We also have the following corollary to Lemma 10, and this uses a result from [4].

Corollary 11. Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Then
\[
n < \left( \frac{315}{2} q^5 \right)^{1/2}
\]
implies \( k = 1 \).

Proof. The proof is very similar to that of Lemma 10, except that it uses the improved bound
\[
\sigma(q^k) \leq \frac{2}{315} n^2
\]
(see [4]) instead of
\[ \sigma(q^k) \leq \frac{2}{3} n^2 \]

(see [9]).

**Remark 12.** Similar to the proofs of Lemma 7 and Corollary 8, we can show that the following biconditionals are true:

\[ n < \left( \frac{3}{2} q^5 \right)^{1/2} \iff N < \frac{3}{2} q^6 \]

\[ n < \left( \frac{315}{2} q^5 \right)^{1/2} \iff N < \frac{315}{2} q^6. \]

**Remark 13.** Chen and Chen [6] has a relatively recent paper which further improves on Broughan et al.’s results (see [4]). They also pose a related open problem.

### 3. New Results Related to Sorli’s Conjecture

First, we reproduce the following lemma from [9], as we will be using these results later.

**Lemma 14.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. The following series of inequalities hold:

- If \( k = 1 \), then \( 1 < I(q^k) = I(q) \leq \frac{6}{5} < \sqrt[5]{\frac{5}{3}} < I(n) < 2. \)

- If \( k \geq 1 \), then \( 1 < I(q^k) < \frac{5}{4} < \sqrt[8]{\frac{8}{5}} < I(n) < 2. \)

We have the following (slightly) stronger inequality from [9].

**Lemma 15.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Then \( (I(q^k))^2 < I(n^2). \)

**Proof.** The proof follows from the inequality \( I(q^k) < \sqrt[5]{2} \) and the equation \( 2 = I(q^k) I(n^2). \)

**Remark 16.** Another proof of Lemma 15 is as follows:
In fact, if 

\[(I(q^k))^y < \left(\frac{5}{4}\right)^y \leq \frac{8}{5} < I(n^2)\]

then 

\[y \leq 3 \log 2 - \log 5 \] 
\[\log 5 - 2 \log 2.

Thus, if we let 

\[z = \frac{3 \log 2 - \log 5 \log 5 - 2 \log 2}{\log 5 - 2 \log 2} \approx 2.1062837195\]

then 

\[(I(q^k))^z \leq \frac{8}{5} < I(n^2).

Next, we derive a lower bound for \(I(q^k) + I(n)\).

**Lemma 17.** Let \(N = q^k n^2\) be an odd perfect number given in Eulerian form. The following inequality holds:

\[I(q^k) + I(n) \geq I(q) + I(n) > 1 + \sqrt{2}.

**Proof.** Let \(N = q^k n^2\) be an odd perfect number given in Eulerian form. Then we have the following:

\[I(q^k) + I(n) \geq I(q) + I(n) \geq 1 + \frac{1}{q} + \sqrt{\frac{2(q-1)}{q}}.

But

\[f(q) = 1 + \frac{1}{q} + \sqrt{\frac{2(q-1)}{q}}\]

is a decreasing function of \(q\). Consequently,
\[
\lim_{q \to \infty} \left( 1 + \frac{1}{q} + \frac{2(q-1)}{q} \right) = 1 + \sqrt{2}.
\]

\[\text{Remark 18.} \] The following result was communicated to the author (via e-mail, by Pascal Ochem) in April of 2013. If \( N = q^k n^2 \) is an odd perfect number given in Eulerian form, then

\[I(n) > \left( \frac{8}{5} \right)^{\ln(4/3) \over \ln(13/9)} \approx 1.44440557\]

(Note that \( \left( \frac{8}{5} \right)^{\ln(4/3) \over \ln(13/9)} > \sqrt{2} \))

Further to Remark 18 and Lemma 15, we have the following related result.

**Lemma 19.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Then \( (I(q))^2 < I(n) \).

**Proof.** By Lemma 14,

\[I(q) \leq {6 \over 5} \Rightarrow (I(q))^2 \leq {36 \over 25} = 1.44.\]

The conclusion follows from the result \( I(n) > 1.44440557 \) in Remark 18.

In fact, if

\[I(q)^u < \left( {6 \over 5} \right)^u \leq \left( {8 \over 5} \right)^{\ln(4/3) \over \ln(13/9)}\]

Then

\[u \leq \frac{(2 \log (2) - \log (3))(3 \log (2) - \log (5))}{(\log (2) + \log (3) - \log (5))(2 \log (3) - \log (13))}.
\]

Thus, if we let

\[v = \frac{(2 \log (2) - \log (3))(3 \log (2) - \log (5))}{(\log (2) + \log (3) - \log (5))(2 \log (3) - \log (13))} \approx 2.0168\]

then
(I(q))^\nu \leq \left( \frac{8}{5} \right)^{\frac{\ln(4/3)}{\ln(13/9)}} < I(n). \quad \square

**Remark 20.** As pointed out by Ochem to the author (via the same e-mail mentioned in Remark 18), a case-by-case analysis yields a sharper lower bound for $I(q^k) + I(n)$:

- If $q = 5$, then $I(q^k) + I(n) \geq I(q) + I(n) \geq (6/5) + (8/5)^{\ln(4/3)/\ln(13/9)} \approx 2.6444055$

- If $q \geq 13$, then $I(q^k) + I(n) \geq I(q) + I(n) \geq (14/13) + (24/13)^{\ln(4/3)/\ln(13/9)} \approx 2.6924318$

Therefore, we have the lower bound

$$I(q^k) + I(n) \geq I(q) + I(n) \geq \frac{6}{5} + \left( \frac{8}{5} \right)^{\frac{\ln(4/3)}{\ln(13/9)}} \approx 2.6444055$$

We now state and prove the following theorem, which provides conditions equivalent to the conjecture mentioned in the introduction.

**Theorem 21.** If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then the following biconditional is true:

$$q^k < n \iff \sigma(q^k) < \sigma(n).$$

In preparation for the proof of Theorem 21, we derive the following results.

**Lemma 22.** Let $N = q^k n^2$ be an odd perfect number given in Eulerian form. If

$$I(q^k) + I(n) < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k},$$

then

$$q^k < n \iff \sigma(q^k) < \sigma(n).$$
Proof. Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Assume that

\[
I(q^k) + I(n) < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}.
\]

It follows that

\[
I(q^k) + I(n) < \left( \frac{q^k}{n} \right) I(q^k) + \left( \frac{n}{q^k} \right) I(n).
\]

Consequently,

\[
q^k n(I(q^k) + I(n)) < q^{2k} I(q^k) + n^2 I(n).
\]

Thus,

\[
n[q^k - n]I(n) < q^k[q^k - n]I(q^k).
\]

If \( q^k < n \), then \( q^k - n < 0 \). Hence,

\[
q^k < n \Rightarrow q^k I(q^k) < nI(n) \Rightarrow \sigma(q^k) < \sigma(n).
\]

If \( n < q^k \), then \( 0 < q^k - n \). Hence,

\[
n < q^k \Rightarrow nI(n) < q^k I(q^k) \Rightarrow \sigma(n) < \sigma(q^k).
\]

Consequently, we have

\[
q^k < n \iff \sigma(q^k) < \sigma(n),
\]

as desired. \( \square \)

Lemma 23. Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. If

\[
\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n),
\]

then

\[
q^k < n \iff \sigma(n) < \sigma(q^k).
\]
Proof. The proof of Lemma 23 is very similar to the proof of Lemma 22. □

Now, assume that
\[
\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n).
\]

Consider the conclusion of the implication in Lemma 23 in light of the result \( I(q^k) < I(n) \):
\[
q^k < n \iff \sigma(n) < \sigma(q^k).
\]

If \( q^k < n \), then since \( I(q^k) < I(n) \) implies that
\[
\frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n},
\]
we have
\[
\frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n} < 1,
\]
which further implies that \( \sigma(q^k) < \sigma(n) \). This contradicts Lemma 23. Similarly, if \( \sigma(n) < \sigma(q^k) \), then
\[
1 < \frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n},
\]
from which it follows that \( n < q^k \). Again, this contradicts Lemma 23. Hence, we know that
\[
n < q^k < \sigma(q^k) < \sigma(n)
\]
must hold, under the given assumption. Assuming Brown’s proof for \( q^k < n \) is completed, this case is ruled out. Consequently, the inequality
\[
\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n)
\]
cannot be true. Therefore, the reverse inequality
\[ I(q^k) + I(n) \leq \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} \]
must be true.

It remains to consider the case when
\[ I(q^k) + I(n) = \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}. \]
Notice that this is true if and only if
\[ \sigma(q^k) = \sigma(n), \]
(because \( q^k \neq n \)). Thus, since \( I(q^k) < I(n) \), this implies that \( n < q^k \).

Again, assuming Brown's proof for \( q^k < n \) is completed, this case is ruled out.

In other words (by Lemma 22), we have Theorem 21 (and the corollary that follows).

**Corollary 24.** If \( N = q^k n^2 \) is an odd perfect number given in Eulerian form, then the following biconditional is true:
\[ q^k < n \iff \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}. \]

We now give another condition that is equivalent to the author's conjecture (mentioned in the introduction).

**Theorem 25.** If \( N = q^k n^2 \) is an odd perfect number given in Eulerian form, then the following biconditional is true:
\[ \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k} \iff \frac{q^k}{n} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}. \]

**Proof.** Let \( N \) be an odd perfect number given in Eulerian form. Then \( N = q^k n^2 \) where \( q \equiv k \equiv 1(\text{mod } 4) \) and \( \gcd (q, n) = 1 \).
First, we show that
\[
\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}
\]
implies
\[
\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}.
\]

Since \(I(q^k) < I(n)\), we have that
\[
\frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n}.
\]

On the other hand, the inequality
\[
\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}
\]
gives us that
\[
\frac{\sigma(q^k)}{\sigma(n)} < \frac{n}{q^k}.
\]

This in turn implies that
\[
\frac{q^k}{n} < \frac{\sigma(n)}{\sigma(q^k)}.
\]

Putting these inequalities together, we have the series
\[
\frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n} < \frac{\sigma(n)}{\sigma(q^k)}.
\]

Now consider the product
\[
\left(\frac{\sigma(q^k)}{\sigma(n)} - \frac{q^k}{n}\right)\left(\frac{\sigma(n)}{\sigma(q^k)} - \frac{q^k}{n}\right).
\]

This product is negative. Consequently we have
\[
\left( \frac{\sigma(q^k)}{\sigma(n)} \right) \left( \frac{\sigma(n)}{\sigma(q^k)} \right) - \left( \frac{q^k}{n} \right) \left( \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)} \right) + \left( \frac{q^k}{n} \right)^2 < 0,
\]
from which it follows that
\[
1 + \left( \frac{q^k}{n} \right)^2 < \left( \frac{q^k}{n} \right) \left( \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)} \right).
\]
Therefore, we obtain
\[
\frac{n}{q^k} + \frac{q^k}{n} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}
\]
as desired.

Next, assume that
\[
\frac{\sigma(n)}{q^k} < \frac{\sigma(q^k)}{n}.
\]
Since \( I(q^k) < I(n) \), we obtain
\[
\frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} < \frac{q^k}{n}.
\]
Now consider the product
\[
\left( \frac{n}{q^k} - \frac{\sigma(q^k)}{\sigma(n)} \right) \left( \frac{q^k}{n} - \frac{\sigma(q^k)}{\sigma(n)} \right).
\]
This product is negative. Therefore, we obtain
\[
\left( \frac{n}{q^k} \right) \left( \frac{q^k}{n} \right) - \left( \frac{\sigma(q^k)}{\sigma(n)} \right) \left( \frac{n}{q^k} + \frac{q^k}{n} \right) + \left( \frac{\sigma(q^k)}{\sigma(n)} \right)^2 < 0,
\]
from which we get
\[
1 + \left( \frac{\sigma(q^k)}{\sigma(n)} \right)^2 < \left( \frac{\sigma(q^k)}{\sigma(n)} \right) \left( \frac{n}{q^k} + \frac{q^k}{n} \right).
\]
Consequently, we have
Together with the result in the previous paragraph, this shows that
\[
\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}
\]
is equivalent to
\[
\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)}.
\]

**Remark 26.** Let \( N = q^kn^2 \) be an odd perfect number given in Eulerian form.

Note that, in general, it is true that
\[
\frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k},
\]
and
\[
\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}.
\]
Therefore,
\[
\frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}
\]
is equivalent to
\[
\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q^k)} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k},
\]
while
\[
\frac{\sigma(n)}{q^k} < \frac{\sigma(q^k)}{n}
\]
is equivalent to
\[
\frac{\sigma(q_k)}{\sigma(n)} + \frac{\sigma(n)}{\sigma(q_k)} < \frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q_k)}{n} + \frac{\sigma(n)}{q^k}.
\]

At this point, we dispose of the following lemma:

**Lemma 27.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Then at least one of the following sets of inequalities is true:

\begin{itemize}
  \item A : \( q^k < \sigma(q^k) < n < \sigma(n) \)
  \item B : \( q^k < n < \sigma(q^k) < \sigma(n) \)
  \item C : \( n < q^k < \sigma(n) < \sigma(q^k) \)
  \item D : \( n < \sigma(n) \leq q^k < \sigma(q^k) \).
\end{itemize}

Lemma 27 is proved by listing all possible permutations of the set \( \{q^k, n, \sigma(q^k), \sigma(n)\} \) and then using Theorem 21.

Note that Brown’s result that \( q^k < n \), when completed, would rule out cases C and D in Lemma 27. Also, notice that by assuming \( k = 1 \), case B is also ruled out.

Consequently, we have the following theorem.

**Theorem 28.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. If \( k = 1 \), then \( \sigma(q^k) < n \).

As a corollary, by the contrapositive to Theorem 28, we have:

**Corollary 29.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. If \( n < \sigma(q^k) \), then \( k > 1 \).

**Remark 30.** If one could show the biconditional

\[
n < q^{k+1} \iff n < \sigma(q^k),
\]

then one would be able to show that

\[
n < q^{k+1} \Rightarrow k > 1.
\]
By the contrapositive, one would then have

\[ k = 1 \Rightarrow q^{k+1} < n \Rightarrow q^2 < n. \]

However, we know that

\[ n < q^2 \Rightarrow k = 1. \]

Consequently,

\[ n < q^2 \Rightarrow k = 1 \Rightarrow q^2 < n \]

which proves that \( q^2 < n \), strengthening Brown’s result.

4. Final Analysis of the New Results

The new results presented in this article seem to imply the following conjecture (see [10]).

**Conjecture 31.** Let \( N = q^k n^2 \) be an odd perfect number given in Eulerian form. Then the Descartes-Frenicle-Sorli conjecture is false. (That is, \( k > 1 \) must hold).

**Remark 32.** Notice how all of the implications in the Lemmas 1, 3 and 4 in Section 2 become vacuously true, given Brown’s result that \( q < n \). Also, notice that, in Section 3, we could specialize Theorem 21 (and its consequences) to the case \( k = 1 \) and still get the same results, as follows:

\[ q < n \iff \sigma(q) < \sigma(n) \iff \frac{\sigma(q)}{n} < \frac{\sigma(n)}{q}. \]

5. Conclusion

An improvement to the currently known upper bound of \( I(n) < 2 \) will be considered a major breakthrough. In the sequel (http://arxiv.org/abs/1303.2329), a viable approach towards improving the inequality \( I(n) < 2 \) will be presented, which may necessitate the use of ideas from the paper [14].
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References


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