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SCHWARZ BOUNDARY VALUE PROBLEM ON THE QUARTER PLANE

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Abstract

We will give solution of Schwarz problem of higher order after applying different combinations of boundary value conditions on the quarter plane.

1. Introduction

Here we will write the solution of inhomogeneous polyanalytic equations [1] with Schwarz Boundary value conditions on the Quarter plane. In the case of unbounded domains (like quarter plane and upper half plane), while finding the solution of differential equations under different boundary conditions, the technique of using iteration method gets failed due to arising of unbounded integrals. While in case of bounded domains, it works well [2, 3, 4, 6, 7, 11, 12, 13]. Therefore, higher order representation of Gauss theorem and Cauchy-Pompeiu formula are first developed on the Quarter plane to solve these problems [14, 15, 16, 17] under different boundary conditions. Similar technique of using Gauss theorem and Cauchy-Pompeiu formula is also developed on the upper half plane [8, 9, 10].

2. Schwarz Boundary Value Problem

For a function $w \in C^1(\mathbb{Q}_1; \mathbb{C}) \cap C(\overline{\mathbb{Q}_1}; \mathbb{C})$ satisfying $w \in L_1(\mathbb{Q}_1; \mathbb{C})$ and

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for which $z^{\delta} w(z)$ for some $0 < \delta$ is bounded in $\mathbb{Q}_1$ besides the Cauchy-Pompeiu representation formula (2.1) according to Gauss theorem [3, 4, 14] and because $\bar{z}, -\bar{z}, -z \notin \mathbb{Q}_1$ if $z \in \mathbb{Q}_1$ the relations

$$w(z) = \frac{1}{2\pi i}\int_0^{+\infty} w(s)\frac{ds}{s-z} - \frac{1}{2\pi i}\int_0^{+\infty} w(is)\frac{ds}{s+iz} - \frac{1}{\pi}\int_{\mathbb{Q}_1} w_{\bar{\varsigma}}\frac{d\sigma d\rho}{\varsigma - z} = 0 \tag{2.1}$$

$$-\frac{1}{2\pi i}\int_0^{+\infty} \overline{w(s)}\frac{ds}{s-z} + \frac{1}{2\pi i}\int_0^{+\infty} \overline{w(is)}\frac{ds}{s-iz} - \frac{1}{\pi}\int_{\mathbb{Q}_1} \overline{w_{\bar{\varsigma}}(\varsigma)}\frac{d\sigma d\rho}{\bar{\varsigma} + z} = 0 \tag{2.2}$$

$$-\frac{1}{2\pi i}\int_0^{+\infty} \overline{w(s)}\frac{ds}{s+z} + \frac{1}{2\pi i}\int_0^{+\infty} \overline{w(is)}\frac{ds}{s+iz} - \frac{1}{\pi}\int_{\mathbb{Q}_1} \overline{w_{\bar{\varsigma}}(\varsigma)}\frac{d\sigma d\rho}{\bar{\varsigma} + z} = 0 \tag{2.3}$$

$$\frac{1}{2\pi i}\int_0^{+\infty} w(s)\frac{ds}{s+z} - \frac{1}{2\pi i}\int_0^{+\infty} w(is)\frac{ds}{s-iz} - \frac{1}{\pi}\int_{\mathbb{Q}_1} \overline{w_{\bar{\varsigma}}(\varsigma)}\frac{d\sigma d\rho}{\varsigma + z} = 0 \tag{2.4}$$

hold for $z \in \mathbb{Q}_1$. Adding equation (2.3) to equation (2.1) and subtracting (2.2) and (2.4) from this sum gives

$$\begin{aligned}
w(z) &= \frac{1}{2\pi i}\int_0^{\infty}\left[w(s)\left(\frac{1}{s-z} - \frac{1}{s+z}\right) + \overline{w(s)}\left(\frac{1}{s-z} - \frac{1}{s+z}\right)\right]ds \\
&- \frac{1}{2\pi i}\int_0^{\infty}\left[w(is)\left(\frac{1}{s+iz} - \frac{1}{s-iz}\right) + \overline{w(is)}\left(\frac{1}{s-iz} - \frac{1}{s+iz}\right)\right]ds \\
&- \frac{1}{\pi}\int_{\mathbb{Q}_1}\left[w_{\bar{\varsigma}}(\varsigma)\left(\frac{1}{\varsigma - z} - \frac{1}{\varsigma + z}\right) + \overline{w_{\bar{\varsigma}}(\varsigma)}\left(\frac{1}{\varsigma - z} - \frac{1}{\varsigma + z}\right)\right]d\sigma d\rho \\
&= \frac{1}{\pi i}\int_0^{\infty} \operatorname{Re} w(s)\frac{z}{s^2 - z^2}\,ds - \frac{2}{\pi i}\int_0^{\infty} \operatorname{Im} w(is)\frac{z}{s^2 + z^2}\,ds \\
&- \frac{2}{\pi}\int_{\mathbb{Q}_1}\left[w_{\bar{\varsigma}}(\varsigma)\frac{1}{\varsigma^2 - z^2} - \overline{w_{\bar{\varsigma}}(\varsigma)}\frac{1}{\varsigma^2 - z^2}\right]d\sigma d\rho
\end{aligned} \tag{2.5}$$

If instead equations (2.3) and (2.4) are subtracted from the sum of (2.1) and (2.2) similarly

$$w(z) = \frac{1}{2\pi i}\int_0^{\infty} \operatorname{Im} w(s)\frac{z}{s^2 - z^2}\,ds + \frac{2}{\pi i}\int_0^{\infty} \operatorname{Re} w(is)\frac{z}{s^2 + z^2}\,ds$$

$$-\frac{2}{\pi}\int_{\mathbb{Q}_1}\left[w_{\bar{\varsigma}}(\varsigma)\frac{z}{\varsigma^2-z^2}-\overline{w_{\bar{\varsigma}}(\varsigma)}\frac{z}{\varsigma^2-z^2}\right]d\sigma d\rho \tag{2.6}$$

The representations (2.5) and (2.6) suggest two formulations of a Schwarz boundary condition. They are dual to one another as can be seen by replacing w by $-iw$.

Theorem 2.1. *The Schwarz problem is uniquely solvable. The solution is*

$$w(z)=\frac{1}{2\pi}\int_0^\infty \gamma_1(s)\frac{z}{s^2-z^2}ds-\frac{2}{\pi i}\int_0^\infty \gamma_2(s)\frac{z}{s^2+z^2}ds$$

$$-\frac{2}{\pi}\int_{\mathbb{Q}_1}\left[\frac{zf(\varsigma)}{\varsigma^2-z^2}-\frac{\overline{zf(\varsigma)}}{\varsigma^2-z^2}\right]d\sigma d\rho \tag{2.7}$$

Proof. By the assumptions all the improper integrals involved exist. From

$$\operatorname{Re} w(z)=\frac{2}{\pi}\int_0^\infty \gamma_1(s)\operatorname{Im}\frac{z}{s^2-z^2}ds-\frac{2}{\pi i}\int_0^\infty \gamma_2(s)\operatorname{Im}\frac{z}{s^2+z^2}ds$$

$$-\frac{2}{\pi}\int_{\mathbb{Q}_1}\operatorname{Re}\left[\frac{zf(\varsigma)}{\varsigma^2-z^2}-\frac{\overline{zf(\varsigma)}}{\varsigma^2-z^2}\right]d\sigma d\rho$$

and

$$\operatorname{Im} w(z)=-\frac{2}{\pi}\int_0^\infty \gamma_1(s)\operatorname{Re}\frac{z}{s^2-z^2}ds-\frac{2}{\pi i}\int_0^\infty \gamma_2(s)\operatorname{Re}\frac{z}{s^2+z^2}ds$$

$$-\frac{2}{\pi}\int_{\mathbb{Q}_1}\operatorname{Im}\left[\frac{zf(\varsigma)}{\varsigma^2-z^2}-\frac{\overline{zf(\varsigma)}}{\varsigma^2-z^2}\right]d\sigma d\rho$$

follows for $s_0\in\mathbb{R}^+$

$$\lim_{z\to s_0}\operatorname{Re} w(z)=\lim_{z\to s_0}\frac{2}{\pi}\int_0^\infty \gamma_1(s)\frac{y}{|s-z|^2}\frac{s^2+|z|^2}{|s+z|^2}ds=\gamma_1(s_0)$$

$$\lim_{z\to s_0}\operatorname{Im} w(z)=\lim_{z\to s_0}\frac{2}{\pi}\int_0^\infty \gamma_2(s)\frac{x}{|s_iz|^2}\frac{s^2+|z|^2}{|s-iz|^2}ds=\gamma_2(s_0)$$

Moreover, as all terms in (2.7) are analytic in $\mathbb{Q}_1$ up to

$$T\ f(z) = -\frac{1}{\pi}\int_{\mathbb{Q}_1} f(\varsigma)\frac{d\sigma d\rho}{\varsigma - z}$$

and $\partial_{\bar{z}} T\ f = f$ in the weak sense (2.7) is seen to be a weak solution to the differential equation. □

In an analogue way a Schwarz problem can be formulated and solved on the basis of representation (2.6). Before writing expression for higher orders we recall the following theorem, for proof see [14].

Theorem 2.2. *Let F_k be the space of functions in $W^{k,1}(\mathbb{Q}_1, \mathbb{C})$ for which*

$$\lim_{R\leftarrow\infty} R^{\lambda} M(\partial_{\bar{z}}^{\lambda} w,\ R) = 0,\ 0 \le \lambda \le k-1 \quad where \quad M(\partial_{\bar{z}}^{\lambda} w,\ R) = \max_{\substack{|z|=R \\ 0\le I_m Z}} |\ \partial_{\bar{z}}^{\lambda} w(z)\ |$$

and $\bar{z}^{k-2}\partial_{\bar{z}}^{k} w \in L^1(\mathbb{Q}_1, \mathbb{C})$. Then every $w \in F_k$ is representable as

$$w(z) = \sum_{\lambda=0}^{k-1}\frac{1}{2\pi i}\int_0^{+\infty}\frac{1}{\lambda!}\frac{(\overline{z-s})^{\lambda}}{(s-z)}\partial_{\bar{\varsigma}}^{\lambda} w(s)\,ds - \sum_{\lambda=0}^{k-1}\frac{1}{2\pi i}\int_0^{+\infty}\frac{1}{\lambda!}\frac{(\bar{z}+is)^{\lambda}}{(s+iz)}\partial_{\bar{\varsigma}}^{\lambda} w(s)\,ds$$

$$-\frac{1}{\pi}\int_{\mathbb{Q}_1}\frac{1}{(k-1)!}\frac{(\overline{z-\varsigma})^{k-1}}{(\varsigma - z)}\partial_{\bar{\varsigma}}^{k} w(\varsigma)\,d\xi d\eta, \qquad (2.8)$$

for $z \in \mathbb{Q}_1$. □

Lemma 2.3. *For a function $w \in C^1(\mathbb{Q}_1;\ \mathbb{C}) \cap C(\overline{\mathbb{Q}_1};\ \mathbb{C})$ satisfying $\bar{z}^{n-2}\partial_{\bar{z}}^{n} w(z) \in L_1(\mathbb{Q}_1;\ \mathbb{C})$ and for which $z^{\delta} w(z)$ for some $0 < \delta$ is bounded in $\mathbb{Q}_1$ besides the representation formula* (2.9), *according to Gauss theorem and because $-\bar{z}, \bar{z}, -z \notin \mathbb{Q}_1$ and if $z \in \mathbb{Q}_1$ then the relations* (2.10), (2.11) *and* (2.12) *hold for $z \in \mathbb{Q}_1$.*

$$w(z) = \sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda}}{\lambda!}\int_0^{+\infty}\partial_{\bar{\varsigma}}^{\lambda} w(s)(s-\bar{z})^{\lambda}\frac{ds}{(s-z)}$$

$$-\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{1}{\lambda!}\int_0^{+\infty}\partial_{\bar{\varsigma}}^{\lambda} w(is)(is+\bar{z})^{\lambda}\frac{ds}{(s+iz)}$$

$$+\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1} f(\varsigma)(\overline{\varsigma - z})^{n-1}\frac{d\sigma d\rho}{\varsigma - z}, \tag{2.9}$$

$$\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{\lambda!}\int_0^{+\infty}\partial_{\bar\varsigma}^{\lambda} w(s)(s-z)^\lambda \frac{ds}{(s-\bar z)}$$

$$-\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{1}{\lambda!}\int_0^{+\infty}\partial_{\bar\varsigma}^{\lambda} w(is)(is+z)^\lambda \frac{ds}{(s+i\bar z)}$$

$$+\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1} f(\varsigma)(\bar\varsigma - z)^{n-1}\frac{d\sigma d\rho}{\varsigma - \bar z} = 0, \tag{2.10}$$

$$\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{\lambda!}\int_0^{+\infty}\partial_{\bar\varsigma}^{\lambda} w(s)(s+z)^\lambda \frac{ds}{(s+\bar z)}$$

$$-\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{1}{\lambda!}\int_0^{+\infty}\partial_{\bar\varsigma}^{\lambda} w(is)(z-is)^\lambda \frac{ds}{(s-i\bar z)}$$

$$+\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1} f(\varsigma)(\bar\varsigma + z)^{n-1}\frac{d\sigma d\rho}{\varsigma + \bar z} = 0, \tag{2.11}$$

$$\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{(-1)^\lambda}{\lambda!}\int_0^{+\infty}\partial_{\bar\varsigma}^{\lambda} w(s)(s+\bar z)^\lambda \frac{ds}{(s+z)}$$

$$-\sum_{\lambda=0}^{n-1}\frac{1}{2\pi i}\frac{1}{\lambda!}\int_0^{+\infty}\partial_{\bar\varsigma}^{\lambda} w(is)(-is+z)^\lambda \frac{ds}{(s-iz)}$$

$$+\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1} f(\varsigma)(\overline{\varsigma + z})^{n-1}\frac{d\sigma d\rho}{\varsigma + z} = 0, \tag{2.12}$$

Proof. Equations (2.10), (2.11) and (2.12) are clear for $n = 1$ by Gauss theorem. Assume $z \in \mathbb{Q}_{1R}$

$$\frac{(-1)^{n-v}}{(n-1-v)!}\frac{1}{\pi}\int_{\mathbb{Q}_{1,R}}\partial_{\bar\varsigma}^{n} w(\varsigma)(\bar\varsigma - z)^{n-1-v}\frac{d\sigma d\rho}{\varsigma - \bar z}$$

$$= \sum_{\lambda=v}^{n-1} \frac{(-1)^{\lambda-v}}{2\pi i(\lambda-v)!} \frac{1}{\pi} \int_{\partial \mathbb{Q}_{1,\ R}} \partial_{\bar{\varsigma}}^{v} w(\varsigma)(\bar{\varsigma}-z)^{\lambda-v} \frac{d\sigma d\rho}{\varsigma-\bar{z}}$$

$$\frac{(-1)^{n+1-v}}{(n-v)!} \frac{1}{\pi} \int_{\mathbb{Q}_{1,\ R}} \partial_{\bar{\varsigma}}^{n+1} w(\varsigma)(\bar{\varsigma}-z)^{n-v} \frac{d\sigma d\rho}{\varsigma-\bar{z}}$$

$$= (-1)^{n+1-v}\left[\frac{1}{(n-v)!} \frac{1}{\pi} \int_{\mathbb{Q}_{1,\ R}} \frac{\partial}{\partial \bar{\varsigma}}\left(\frac{\partial_{\bar{\varsigma}}^{n} w(\varsigma)(\bar{\varsigma}-z)^{n-v}}{\varsigma-\bar{z}}\right) d\sigma d\rho\right.$$

$$\left.-\frac{1}{(n-v-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_{1,\ R}} \partial_{\bar{\varsigma}}^{n} w(\varsigma) \frac{(\bar{\varsigma}-z)^{n-v-1}}{\varsigma-\bar{z}} d\sigma d\rho\right]$$

$$= \frac{(-1)^{n+1-v}}{(n-v)!} \frac{1}{2\pi i} \int_{\mathbb{Q}_{1,\ R}} \partial_{\bar{\varsigma}}^{n} w(\varsigma) \frac{(\bar{\varsigma}-z)^{n-v}}{\varsigma-\bar{z}} d\varsigma$$

$$+ \sum_{\lambda=v}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda-v}}{(\lambda-v)!} \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1,\ R}} \partial_{\bar{\varsigma}}^{n} w(\varsigma)(\bar{\varsigma}-z)^{n-v} \frac{d\varsigma}{\varsigma-\bar{z}} \tag{2.13}$$

which is (2.10) for $n+1$

$$\left|\frac{1}{2\pi i} \int_{\substack{|z|=R \\ 0<x \\ 0<y}} \partial_{\bar{\varsigma}}^{\lambda} w(\varsigma)(\bar{\varsigma}-z)^{\lambda-v} \frac{d\varsigma}{\varsigma-\bar{z}}\right| \leq \frac{(R+|z|)^{\lambda-v}}{R-|z|} M(R, \partial_{\bar{z}}^{\lambda} w) R$$

which tends to zero as R tends to 1. Now, applying $R \leftarrow \infty$ in (2.13), we have

$$\sum_{v=0}^{k} \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \frac{1}{v!} \frac{(\overline{z-\varsigma})^{v}}{(\varsigma-z)} \partial_{\bar{\varsigma}}^{v} w(\varsigma) d\varsigma$$

$$= \sum_{v=0}^{k-1}\left[\frac{1}{2\pi i} \int_{0}^{+\infty} \frac{1}{v!} \frac{(\overline{z-s})^{v}}{(s-z)} \partial_{\bar{\varsigma}}^{v} w(s) ds - \frac{1}{2\pi i} \int_{0}^{+\infty} \frac{1}{v!} \frac{(\bar{z}+is)^{v}}{(s+iz)} \partial_{\bar{\varsigma}}^{v} w(is) ds\right]$$

which exists by the respective assumptions. Hence the required result. □

Now, adding (2.9) and the complex conjugate of (2.11) and subtracting (2.12) and complex conjugate of (2.10) leads to

$$w(z) = \sum_{\lambda=0}^{n-1} \frac{1}{\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \operatorname{Re} \partial_{\bar{\varsigma}}^{\lambda} w(s) \left\{ \frac{(s-\bar{z})^{\lambda}}{(s-z)} - \frac{(s+\bar{z})^{\lambda}}{(s+z)} \right\} ds$$

$$- \sum_{\lambda=0,\, 2,\, 4,\, 6,\, \ldots}^{n-1} \frac{1}{\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \operatorname{Im} \partial_{\bar{\varsigma}}^{\lambda} w(s) \left\{ \frac{(is+\bar{z})^{\lambda}}{(s+iz)} - \frac{(-is+\bar{z})^{\lambda}}{(s-iz)} \right\} ds$$

$$- \sum_{\lambda=1,\, 3,\, 5,\, 7,\, \ldots}^{n-1} \frac{1}{\pi} \frac{1}{\lambda!} \int_0^{+\infty} \operatorname{Re} \partial_{\bar{\varsigma}}^{\lambda} w(is) \left\{ \frac{(is+\bar{z})^{\lambda}}{(s+iz)} - \frac{(-is+\bar{z})^{\lambda}}{(s-iz)} \right\} ds$$

$$+ \frac{(-1)^n}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_1} \left[f(\varsigma) \left\{ \frac{(\overline{\varsigma - z})^{n-1}}{(\varsigma - z)} - \frac{(\overline{\varsigma + z})^{n-1}}{(\varsigma + z)} \right\} \right.$$

$$\left. - \overline{f(\varsigma)} \left\{ \frac{(\varsigma - \bar{z})^{n-1}}{(\bar{\varsigma} - z)} - \frac{(\varsigma + \bar{z})^{n-1}}{(\bar{\varsigma} + z)} \right\} \right] d\sigma d\rho \tag{2.14}$$

Similarly adding (2.9) and the complex conjugate of (2.10) and subtracting (2.12) and complex conjugate of (2.11) shows

$$w(z) = \sum_{\lambda=0}^{n-1} \frac{1}{\pi} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \operatorname{Im} \partial_{\bar{\varsigma}}^{\lambda} w(s) \left\{ \frac{(s-\bar{z})^{\lambda}}{(s-z)} - \frac{(s+\bar{z})^{\lambda}}{(s+z)} \right\} ds$$

$$- \sum_{\lambda=0,\, 2,\, 4,\, 6,\, \ldots}^{n-1} \frac{1}{\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \operatorname{Re} \partial_{\bar{\varsigma}}^{\lambda} w(is) \left\{ \frac{(is+\bar{z})^{\lambda}}{(s+iz)} - \frac{(-is+\bar{z})^{\lambda}}{(s-iz)} \right\} ds$$

$$- \sum_{\lambda=1,\, 3,\, 5,\, 7,\, \ldots}^{n-1} \frac{1}{\pi} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \operatorname{Im} \partial_{\bar{\varsigma}}^{\lambda} w(is) \left\{ \frac{(is+\bar{z})^{\lambda}}{(s+iz)} - \frac{(-is+\bar{z})^{\lambda}}{(s-iz)} \right\} ds$$

$$+ \frac{(-1)^n}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_1} \left[f(\varsigma) \left\{ \frac{(\overline{\varsigma - z})^{n-1}}{(\varsigma - z)} - \frac{(\overline{\varsigma + z})^{n-1}}{(\varsigma + z)} \right\} \right.$$

$$\left. + \overline{f(\varsigma)} \left\{ \frac{(\varsigma - \bar{z})^{n-1}}{(\bar{\varsigma} - z)} - \frac{(\varsigma + \bar{z})^{n-1}}{(\bar{\varsigma} + z)} \right\} \right] d\sigma d\rho \tag{2.15}$$

These representation formulas (2.13) and (2.14) suggest Schwarz Boundary value problem for inhomogeneous polyanalytic equation.

Theorem 2.4. *Let w be as in Lemma 2.1 then the Schwarz problem $\partial_{\bar{z}}^{n} w(z) = f(z)$ in $\mathbb{Q}_1$, $Re\partial_{\bar{z}}^{\lambda} w = \varphi\lambda$ for $0 < x < +\infty$, $y = 0 \,\forall\, 0 \le \lambda \le n-1$, $Im\partial_{\bar{z}}^{n} w(z) = \psi\lambda$ for $0 < y < +\infty$, $x = 0 \,\forall\, \lambda = 0, 2, 4, 6, \ldots$, $Re\partial_{\bar{z}}^{\lambda} w = \phi_\lambda$ for $0 < y < +\infty$, $x = 0 \,\forall\, \lambda = 1, 3, 5, 7, \ldots$ is uniquely weakly solvable for $f \in L_1(\mathbb{Q}_1; \mathbb{C})$, $\varphi_\lambda, \psi_\lambda, \phi_\lambda \in C(\mathbb{R}^+; \mathbb{R})$, such that $s^{\delta+\lambda}\varphi_\lambda(s)$, $s^{\delta+\lambda}\psi_\lambda(s)$, $s^{\delta+\lambda}\phi_\lambda(s)$ are bounded on $\mathbb{R}^+ = [0, +\infty)$ for some $0 < \delta$ and $0 \le \lambda \le n-1$. The solution is*

$$w(z) = \sum_{\lambda=0}^{n-1} \frac{1}{\pi i} \frac{(-1)^\lambda}{\lambda!} \int_0^{+\infty} \phi_\lambda(s) \left\{ \frac{(s-\bar{z})^\lambda}{(s-z)} - \frac{(s+\bar{z})^\lambda}{(s+z)} \right\} ds$$

$$- \sum_{\lambda=0,\,2,\,4,\,6,\,\ldots}^{n-1} \frac{1}{\pi i} \frac{(-1)^\lambda}{\lambda!} \int_0^{+\infty} \psi_\lambda \left\{ \frac{(is+\bar{z})^\lambda}{(s+iz)} - \frac{(-is+\bar{z})^\lambda}{(s-iz)} \right\} ds$$

$$- \sum_{\lambda=1,\,3,\,5,\,7,\,\ldots}^{n-1} \frac{1}{\pi} \frac{(-1)^\lambda}{\lambda!} \int_0^{+\infty} \phi_\lambda \left\{ \frac{(is+\bar{z})^\lambda}{(s+iz)} - \frac{(-is+\bar{z})^\lambda}{(s-iz)} \right\} ds$$

$$+ \frac{(-1)^n}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_1} \left[f(\varsigma) \left\{ \frac{(\overline{\varsigma - z})^{n-1}}{(\varsigma - z)} - \frac{(\overline{\varsigma + z})^{n-1}}{(\varsigma + z)} \right\} \right.$$

$$\left. - \overline{f(\varsigma)} \left\{ \frac{(\varsigma - \bar{z})^{n-1}}{(\bar{\varsigma} - z)} - \frac{(\varsigma + \bar{z})^{n-1}}{(\bar{\varsigma} + z)} \right\} \right] d\sigma d\rho \tag{2.16}$$

Theorem 2.5. *Let w be as in Lemma 2.1 then the Schwarz problem $\partial_{\bar{z}}^{n} w(z) = f(z)$ in $\mathbb{Q}_1$, $Im\partial_{\bar{z}}^{\lambda} w = \varphi\lambda$ for $0 < x < +\infty$, $y = 0 \,\forall\, 0 \le \lambda \le n-1$, $Re\partial_{\bar{z}}^{n} w(z) = \psi\lambda$ for $0 < y < +\infty$, $x = 0 \,\forall\, \lambda = 0, 2, 4, 6, \ldots$, $Im\partial_{\bar{z}}^{\lambda} w = \phi_\lambda$ for $0 < y < +\infty$, $x = 0 \,\forall\, \lambda = 1, 3, 5, 7, \ldots$ is uniquely weakly solvable for $f \in L_1(\mathbb{Q}_1; \mathbb{C})$, $\varphi_\lambda, \psi_\lambda, \phi_\lambda \in C(\mathbb{R}^+; \mathbb{R})$, such that $s^{\delta+\lambda}\varphi_\lambda(s)$, $s^{\delta+\lambda}\psi_\lambda(s)$, $s^{\delta+\lambda}\phi_\lambda(s)$ are bounded on $\mathbb{R}^+ = [0, +\infty)$ for some $0 < \delta$ and $0 \le \lambda \le n-1$. The solution is*

$$w(z) = \sum_{\lambda=0}^{n-1} \frac{1}{\pi} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \varphi_{\lambda}(s) \left\{ \frac{(s-\bar{z})^{\lambda}}{(s-z)} - \frac{(s+\bar{z})^{\lambda}}{(s+z)} \right\} ds$$

$$- \sum_{\lambda=0,\,2,\,4,\,6,\,\dots}^{n-1} \frac{1}{\pi i} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \psi_{\lambda}(s) \left\{ \frac{(is+\bar{z})^{\lambda}}{(s+iz)} - \frac{(-is+\bar{z})^{\lambda}}{(s-iz)} \right\} ds$$

$$- \sum_{\lambda=1,\,3,\,5,\,7,\,\dots}^{n-1} \frac{1}{\pi} \frac{(-1)^{\lambda}}{\lambda!} \int_0^{+\infty} \phi_{\lambda}(s) \left\{ \frac{(is+\bar{z})^{\lambda}}{(s+iz)} - \frac{(-is+\bar{z})^{\lambda}}{(s-iz)} \right\} ds$$

$$+ \frac{(-1)^n}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{Q}_1} \Big[f(\varsigma) \left\{ \frac{(\overline{\varsigma - z})^{n-1}}{(\varsigma - z)} - \frac{(\overline{\varsigma + z})^{n-1}}{(\varsigma + z)} \right\}$$

$$+ \overline{f(\varsigma)} \left\{ \frac{(\varsigma - \bar{z})^{n-1}}{(\bar{\varsigma} - z)} - \frac{(\varsigma + \bar{z})^{n-1}}{(\bar{\varsigma} + z)} \right\} \Big] d\sigma d\rho \tag{2.17}$$

Solution of all other combinations of Schwarz boundary conditions can also be written in the similar way.

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