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ON THE SUMS OF k-LUCAS NUMBERS

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Abstract

In this paper, we present some identities for the sums of k-Lucas numbers with $m+1$ consecutive members of k-Lucas numbers and the same thing for even, for odd, for their product and alternating sums of adjacent k-Lucas numbers. Mainly, Binet's formula will be used to establish properties of k-Lucas numbers.

1. Introduction

Fibonacci and Lucas sequences are the two most well-known linear homogeneous recurrence relations of order two with constant coefficients.

The sequence of Fibonacci numbers F_n is defined by

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \text{ with } F_0 = 0,\ F_2 = 1. \tag{1.1}$$

The sequence of Lucas numbers L_n is defined by

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$$L_n = L_{n-1} + L_{n-2},\ n \geq 2 \text{ with } L_0 = 2,\ L_2 = 1. \tag{1.2}$$

The second order recurrence sequence has been generalized in two ways mainly, first by preserving the initial conditions and second by preserving the recurrence relation.

Let us remember the k-Lucas numbers defined by Falcon [5], themselves as well as looking at its close relationship with the k-Fibonacci numbers, the k-Lucas sequence is defined recurrently by

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1},\ n \geq 1 \text{ with } L_{k,0} = 2,\ L_{k,1} = k. \tag{1.3}$$

First few generalized Lucas numbers are

$$\{L_{k,n}\} = \{2,\ k,\ k^2 + 2,\ k^3 + 3k,\ k^4 + 4^2 + 2,\ k^5 + 5k^3 + 5k,\ \ldots\}.$$

Particular cases: On the k-Lucas numbers

- For $k = 1,$ the classical Lucas sequence appears: $\{2,1,3,4,7,11,18,\ldots\}$
- For $k = 2,$ we obtain the Pell-Lucas sequence: $\{2,2,6,14,34,82,198,\ldots\}$.

Among other properties, the Binnet Identity establishes:

$$L_{k,n} = \mathfrak{R}_1^n + \mathfrak{R}_2^n \tag{1.4}$$

Being $\mathfrak{R}_1 = \dfrac{k + \sqrt{k^2 + 4}}{2}$ and $\mathfrak{R}_2 = \dfrac{k - \sqrt{k^2 + 4}}{2}$ the characteristic roots of the recurrence equation $x^2 = kx + 1$.

Evidently,

$$\mathfrak{R}_1 + \mathfrak{R}_2 = k,\ \mathfrak{R}_1\mathfrak{R}_2 = -1,\ \mathfrak{R}_1 - \mathfrak{R}_2 = \sqrt{k^2+4},\ \mathfrak{R}_1^2 - 1 = k\mathfrak{R}_1,\ \mathfrak{R}_2^2 - 1 = k\mathfrak{R}_2.$$

Falcon [7], present Lucas triangle and its relationship with the k-Lucas numbers, combinatorial formula for k-Lucas numbers, generating function and defined Properties of the diagonals of the Lucas triangle and the rows of the Lucas triangle. Falcon [5], study the properties of the k-Lucas numbers and will prove these properties will be related with the k-Fibonaci numbers. From a special sequence of squares of k-Fibonacci numbers, the k-Lucas sequences are obtained in a natural form. Also examine some of the interesting properties of the k-Lucas numbers themselves as well as looking at its close relationship with the k-Fibonacci numbers. The k-Lucas numbers have lots of properties, similar to those of k-Fibonacci numbers and often occur in various formulae simultaneously with latter. Falcon [6], study the k-Lucas numbers of arithmetic indexes of the form $an+r$ and present a formula for the sum of the square of the k-Fibonacci even numbers by mean of the k-Lucas numbers.

2. Preliminary

Čerin [16], defines some sums of squares of odd and even terms of Lucas sequence. Rajesh and Leversha [11], define some properties of Fibonacci numbers in odd terms. Čerin [15], consider alternating sums of squares of odd and even terms of the Lucas sequence and alternating sums of their products. Čerin [17], improve some results on sums of squares of odd terms of the Fibonacci sequence by Rajesh and Leversha. Belbachir and Bencherif [3], recover and extend all result of Čerin [15], Čerin and Gianella [19]. Yazlik et al. [14], investigate some properties additive of k-Fibonacci and k-Lucas sequences and obtain new identities on sums of powers these sequences and obtain the recurrence relations for powers of k-Fibonacci and k-Lucas sequences. Also they will be given new formulas for the powers of k-Fibonacci and k-Lucas sequences. Gnanam and Anitha [1], present some identities for the sums of squares of Fibonacci and Lucas numbers with consecutive primes, using maximal prime gap $G(x) \sim \log^2 x$, as indices. Panwar et al. [12], present the sum of consecutive members of k-Fibonacci numbers. Panwar and Gupta [13], define the sum of consecutive members of Fibonacci sequence and the same thing for even and for odd and their product of adjacent Fibonacci numbers. In this paper we present the sum of consecutive members of k-Lucas numbers.

3. Main Results

In this section, we prove some identities for sums of a finite number of consecutive terms of the k-Lucas numbers.

Theorem 3.1. *For* $m \geq 0$ *and* $v \geq 0$ *the following equality holds*:

$$\sum_{i=0}^{m} L_{k,v+i} = \frac{1}{k}[L_{k,v+m} + L_{k,v+m+1} - L_{k,v} - L_{k,v-1}]. \tag{3.1}$$

Proof. By Binet's formula, we have

$$\sum_{i=0}^{m} L_{k,v+i} = \sum_{i=0}^{m} (\mathfrak{R}_1^{v+i} + \mathfrak{R}_2^{v+i})$$

$$= \left[\frac{\mathfrak{R}_1^{v+m+1} - \mathfrak{R}_1^{v}}{\mathfrak{R}_1 - 1} + \frac{\mathfrak{R}_2^{v+m+1} - \mathfrak{R}_2^{v}}{\mathfrak{R}_2 - 1}\right]$$

$$= \left[\frac{(\mathfrak{R}_2 - 1) - (\mathfrak{R}_1^{v+m+1} - \mathfrak{R}_1^{v}) + (\mathfrak{R}_1 - 1)(\mathfrak{R}_2^{v+m+1} - \mathfrak{R}_2^{v})}{(\mathfrak{R}_1 - 1)(\mathfrak{R}_2 - 1)}\right],$$

$$= \left[\frac{\begin{array}{c}\mathfrak{R}_1^{v+m}(\mathfrak{R}_1\mathfrak{R}_2) - \mathfrak{R}_1^{v}\mathfrak{R}_2 - \mathfrak{R}_1^{v+m+1} + \mathfrak{R}_1^{v} \\ + \mathfrak{R}_2^{v+m}(\mathfrak{R}_1\mathfrak{R}_2) - \mathfrak{R}_1\mathfrak{R}_2^{v} - \mathfrak{R}_2^{v+m+1} + \mathfrak{R}_2^{v}\end{array}}{\mathfrak{R}_1\mathfrak{R}_2 - \mathfrak{R}_1 - \mathfrak{R}_2 + 1}\right]$$

$$= \left[\frac{\begin{array}{c}-(\mathfrak{R}_1^{v+m} + \mathfrak{R}_2^{v+m}) - (\mathfrak{R}_1^{v}\mathfrak{R}_2 + \mathfrak{R}_1\mathfrak{R}_2^{v}) \\ -(\mathfrak{R}_1^{v+m+1} + \mathfrak{R}_2^{v+m+1}) + (\mathfrak{R}_1^{v} + \mathfrak{R}_2^{v})\end{array}}{-(\mathfrak{R}_1 + \mathfrak{R}_2)}\right]$$

$$= \frac{1}{k}[(\mathfrak{R}_1^{v+m} + \mathfrak{R}_2^{v+m}) - (\mathfrak{R}_1^{v-1} + \mathfrak{R}_2^{v-1})$$

$$+ (\mathfrak{R}_1^{v+m+1} + \mathfrak{R}_2^{v+m+1}) + (\mathfrak{R}_1^{v} + \mathfrak{R}_2^{v})]$$

$$\sum_{i=0}^{m} L_{k,v+i} = \frac{1}{k}[L_{k,v+m} + L_{k,v+m+1} - L_{k,v} - L_{k,v-1}].$$

This completes the proof.

Theorem 3.2. *For* $m \geq 0$ *and* $v \geq 0$ *the following equality holds*:

$$\sum_{i=0}^{m} L_{k,\,2v+2i} = \frac{1}{k}\left(L_{k,\,2v+2m+1} - L_{k,\,2v-1}\right). \tag{3.2}$$

Proof. By Binet's formula, we have

$$\sum_{i=0}^{m} L_{k,\,2v+2i} = \sum_{i=0}^{m} \left(\Re_1^{2v+2i} + \Re_2^{2v+2i}\right)$$

$$= \left[\frac{\Re_1^{2v+2m+2} - \Re_1^{2v}}{k\Re_1} + \frac{\Re_2^{2v+2m+2} - \Re_2^{2v}}{k\Re_2}\right]$$

$$= \frac{1}{k}\left[\frac{\Re_1\Re_2\left(\Re_1^{2v+2m+1} - \Re_1^{2v+2m+1}\right) - \left(\Re_1^{v}\Re_2 - \Re_1\Re_2^{v}\right)}{\Re_1\Re_2}\right]$$

$$\sum_{i=0}^{m} L_{k,\,2v+2i} = \frac{1}{k}\left(L_{k,\,2v+2m+1} - L_{k,\,2v-1}\right).$$

This completes the proof.

The alternating sums of consecutive k-Lucas number are treated in the following theorem.

Theorem 3.3. *For* $m \geq 0$ *and* $v \geq 0$ *the following equality holds*:

$$\sum_{i=0}^{m} (-1)^i L_{k,\,v+i} = \frac{1}{k}\left[(-1)^m\left(L_{k,\,v+m+1} - L_{k,\,v+m}\right) - \left(L_{k,\,v-1} - L_{k,\,v}\right)\right]. \tag{3.3}$$

Theorem 3.4. *For* $m \geq 0$ *and* $v \geq 0$ *the following equality holds*:

$$\sum_{i=0}^{m} (-1)^i L_{k,\,2v+2i} = \frac{1}{k^2+4}\left(L_{k,\,2v-2} - L_{k,\,2v+2m} + L_{k,\,2v} - L_{k,\,2v+2m+2}\right). \tag{3.4}$$

The sum of squares of consecutive k-Lucas number is treated in the following theorem.

Theorem 3.5. *For* $m \geq 0$ *and* $v \geq 0$ *the following equality holds*:

$$\sum_{i=0}^{m} L_{k,\,2v+2i}^{2} = \frac{1}{k}\left[L_{k,\,2v+2m+1} - L_{k,\,2v-1} + k\{(-1)^{v+m+1} - (-1)^{v}\}\right]. \tag{3.5}$$

Proof. By Binet's formula, we have

$$\sum_{i=0}^{m} L_{k,\,v+i}^{2} = \sum_{i=0}^{m} (\Re_1^{v+i} + \Re_2^{v+i})^2$$

$$= \sum_{i=0}^{m} \{\Re_1^{2v+2i} + \Re_2^{2v+2i} + 2(\Re_1\Re_2)^{v+i}\}$$

$$= \left[\Re_1^{2v}\,\frac{\Re_1^{2m+2} - 1}{\Re_1^2 - 1} + \Re_2^{2v}\,\frac{\Re_2^{2m+2} - 1}{\Re_2^2 - 1} + 2(\Re_1\Re_2)^{v}\,\frac{(\Re_1\Re_2)^{m+1} - 1}{\Re_1\Re_2 - 1}\right]$$

$$= \left[\frac{\Re_1^{2v+2m+2} - \Re_1^{2v}}{k\Re_1} + \frac{\Re_2^{2v+2m+2} - \Re_2^{2v}}{k\Re_2} + 2\left\{\frac{(-1)^{v+m+1} - (-1)^{v}}{\Re_1\Re_2 - 1}\right\}\right]$$

$$= \frac{1}{k}\Big[\frac{\Re_1\Re_2(\Re_1^{2v+2m+1} + \Re_2^{2v+2m+1}) + (\Re_1^{2v-1} + \Re_2^{2v-1})}{\Re_1\Re_2}$$

$$- k\{(-1)^{v+m+1} - (-1)^{v}\}\Big]$$

$$\sum_{i=0}^{m} L_{k,\,2v+2i}^{2} = \frac{1}{k}\left[L_{k,\,2v+2m+1} - L_{k,\,2v-1} + k\{(-1)^{v+m+1} - (-1)^{v}\}\right].$$

This completes the proof.

Identities on Sums of Squares of k-Lucas Numbers in terms of k-Fibonacci numbers.

The k-Fibonacci numbers defined by Falcon and Plaza [8, 9 and 10], depending only on one integer parameter k as follows, for any positive real number k, the k-Fibonacci sequence is defined recurrently by

$$F_{k,\,n+1} = kF_{k,\,n} + F_{k,\,n-1},\ n \geq 1 \text{ with } F_{k,\,0} = 0,\ F_{k,1} = 1. \tag{3.6}$$

The Binet's formula for k-Fibonacci Sequence is given by

$$F_{k,\,n} = \frac{\Re_1^n - \Re_2^n}{\Re_1 - \Re_2}. \tag{3.7}$$

Here the following formulae are repeatedly used.

- $L_{k,i}^2 + L_{k,i+1}^2 = (k^2 + 4)F_{k,2i+1}$
- $L_{k,i+1} + L_{k,i-1} = (k^2 + 4)F_{k,i}$
- $L_{k,n+1}L_{k,i} + L_{k,n}L_{k,i-1} = (k^2 + 4)F_{k,n+i}$.

Theorem 3.6. *For* $m \geq 0$ *and* $v \geq 0$ *the following equality holds*:

$$\sum_{i=0}^{m}(L_{k,i}^2 + L_{k,i+1}^2) = \left(\frac{k^2+4}{k}\right)F_{k,2m+2}. \tag{3.8}$$

Theorem 3.7. *For* $m \geq 0$ *and* $v \geq 0$ *the following equality holds*:

$$\sum_{i=0}^{m}(L_{k,i+1} + L_{k,i-1}) = \left(\frac{k^2+4}{k}\right)[F_{k,m+1} + F_{k,m} - 1]. \tag{3.9}$$

Theorem 3.8. *For* $m \geq 0$ *and* $v \geq 0$ *the following equality holds*:

$$\sum_{i=0}^{m}(L_{k,n+1}L_{k,i} + L_{k,n}L_{k,i-1}) = \left(\frac{k^2+4}{k}\right)$$

$$[(F_{k,m+n+1} + F_{k,m+n}) - (F_{k,n-1} + F_{k,n})]. \tag{3.10}$$

Conclusion

In this paper, we have stated and derived many identities. We define the sum of $m+1$ consecutive members of k-Lucas numbers and the same thing for even, for odd, for their product and alternating sums of adjacent k-Lucas numbers.

References

[1] A. Gnanam and B. Anitha, Sums of Squares of Fibonacci Numbers with Prime Indices, Journal of Applied Mathematics and Physics 3 (2015), 1619-1623. http://dx.doi.org/10.4236/jamp.2015.312186

[2] D. Jennings, On sums the reciprocals of Fibonacci and Lucas Numbers, The Fibonacci Quarterly 32(1) (1994), 18-21.

[3] H. Belbachir and F. Bencherif, Sums of products of generalized Fibonacci and Lucas numbers, arXiv: 0708.2347v1 [math.NT], (2007).

[4] S. Clarly and D. Hemenway, On sums of cubes of Fibonacci Numbers, In Applications of Fibonacci Numbers 5 (1993), 123-136. https://doi.org/10.1007/978-94-011-2058-6_12

[5] S. Falcon, On the k-Lucas numbers, International Journal of Contemporary Mathematical Sciences 6(21) (2011), 1039-1050.

[6] S. Falcon, On the k-Lucas numbers of arithmetic indexes, Applied Mathematics 3 (2019) 1202-1206. http://dx.doi.org/10.4236/am.2012.310175

[7] S. Falcon, On the Lucas Triangle and its Relationship with the k-Lucas numbers, Journal of Mathematical and Computational Science 2(3) (2012), 425-434.

[8] S. Falcon and A. Plaza, On the Fibonacci k-numbers, Chaos, Solitons and Fractals 32(5) (2007), 1615-1624. doi:10.1016/j.chaos.2006.09.022

[9] S. Falcon and A. Plaza, The k-Fibonacci hyperbolic functions, Chaos, Solitons and Fractals 38(2) (2008) 409-420. https://doi.org/10.1016/j.chaos.2006.11.019

[10] S. Falcon and A. Plaza, The k-Fibonacci sequence and the Pascal 2-triangle, Chaos, Solitons and Fractals 33(1) (2007), 38-49. doi:10.1016/j.chaos.2006.10.022

[11] V. Rajesh and G. Leversha, Some properties of odd terms of the Fibonacci sequence, Mathematical Gazette 88(511) (2004), 85-86. doi:10.1017/S0025557200174285

[12] Y. K. Panwar, G. P. S. Rathore and R. Chawla, On sums of odd and even terms of the K-Fibonacci numbers, Global Journal of Mathematical Analysis 2(3) (2014), 115-119. doi: 10.14419/gjma.v2i3.2587

[13] Y. K. Panwar and V. K. Gupta, On Sums of Odd and Even Terms of the Fibonacci Sequence, J. Ana. Num. Theor. 4(1) (2016), 81-84. http://dx.doi.org/10.18576/jant/040112

[14] Y. Yazlik, N. Yilmaz and N. Taskara, On the Sums of Powers of k-Fibonacci and k-Lucas Sequences, Selçuk J. Appl. Math., Special Issue (2012), 47-50.

[15] Z. Cerin, Some Alternating sums of Lucas numbers, Central European Journal of Mathematics 3(1) (2005), 1-13. https://doi.org/10.2478/BF02475651

[16] Z. Čerin, On sums of squares of odd and even terms of the Lucas sequence, Proceedings of the 11th Fibonacci Conference, (to appear).

[17] Z. Čerin, Properties of odd and even terms of the Fibonacci sequence, Demonstratio Mathematica 39(1) (2006), 55-60. DOI: 10.1515/dema-2006-0107.

[18] Z. Čerin and G. M. Gianella, On sums of Pell numbers, Acc. Sc. Torino-Atti Sc. Fis. 140, xx-xx. web.math.pmf.unizg.hr/~cerin/c165.pdf, (2006).

[19] Z. Čerin and G. M. Gianella, On sums of squares of Pell-Lucas numbers, INTEGERS: Electronic Journal of Combinatorial Number Theory, 6, #A15, (2006).