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# SOME SPECIAL ANALYSATION ON A PYTHAGOREAN TRIANGLE WHICH SATISFIES $\lambda((\text{Hypotonuse} \times \text{Perimeter}) - 4(\text{Area})) = \mu^2(\text{perimeter})$

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## Abstract

We obtain non-trivial integer solutions for the sides of the Pythagorean triangle, for some particular values of $\alpha$ which satisfies $\lambda((\text{Hypotonuse} \times \text{Perimeter}) - 4(\text{Area})) = \mu^2(\text{perimeter})$. A few interesting relations between the sides of the Pythagorean triangle are presented.

## 1. Motivation and Main Results

One well known set of solutions of the Pythagorean equation $x^2 + y^2 = z^2$ are $x = 2uv, y = u^2 - v^2$ and $z = u^2 + v^2$, many mathematicians has been used this set of solutions to obtain the non-zero integral values for $x$, $y$, and $z$ [1-3]. As a new approach, in this paper we introduce another set of solutions $x = 2T - 1$, $y = 2T^2 - 2T$ and $z = 2T^2 - 2T + 1$ for the equation $x^2 + y^2 = z^2$. By

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using this solution we obtain three non-zero integers $x, y$ and $z$ under certain relations satisfying the equation $x^2 + y^2 = z^2$ [4-6]. In this communication we, present yet another interest Pythagorean Triangles, where in each of which, $\lambda((\text{Hypotonuse} \times \text{Perimeter}) - 4(\text{Area})) = \mu^2(\text{perimeter})$. A few interesting relations are also given. In addition, the recurrence relations for the sides of the triangle are presented.

Taking $T > 0$, to be the generators of the Pythagorean triangle $(x, y, z)$, the assumption that $\lambda((\text{Hypotonuse} \times \text{Perimeter}) - 4(\text{Area})) = \mu^2(\text{perimeter})$ becomes $\lambda((2T^2 - 2T + 1) \times (2T - 1 + 2T^2 - 2T + 2T^2 - 2T + 1)) - 4(\frac{1}{2}(2U + 1)$ $(2T^2 - 2T) = \mu^2(2T - 1 + 2T^2 - 2T + 2T^2 + 2T + 1)$ and which leads to the Pellian equation

$$X^2 = DY^2 - \lambda \tag{1}$$

where $D = 2\lambda$, not a perfect square $\mu = X$ and $T = Y$.

Setting $\lambda = 1$, so that $D = 2$.

The equation $X^2 = DY^2 - \lambda$ becomes

$$X^2 = 2Y^2 - 1 \tag{2}$$

$(X_0, Y_0) = (1, 1)$ be the initial solution of (2)

Consider the Pellian

$$X^2 = 2Y^2 + 1. \tag{3}$$

Let $(\overline{x_0}, \overline{y_0}) = (3, 2)$ be the initial solution of (3)

Using Brahmagupta lemma, $\bar{x}_n + \sqrt{D}\,\bar{y}_n = (\overline{x_0} + \sqrt{D}\,\overline{y_0})^{n+1}$, $n = 0, 1, 2, 3, \ldots$ leads to $\bar{x}_n + \sqrt{2}\,\bar{y}_n = (3 + 2\sqrt{2})^{n+1}$, $n = 0, 1, 2, 3, \ldots$.

Since irrational roots occurs in pairs, $\bar{x}_n - \sqrt{2}\,\bar{y}_n = (3 - 2\sqrt{2})^{n+1}$, $n = 0, 1, 2, 3, \ldots$.

Thus the solution of equation (2) can be represented by,

$$X_{n+1} = X_0 \bar{x}_n + DY_0 \bar{y}_n, \; n = 0, 1, 2, 3, \ldots \tag{4}$$

$$Y_{n+1} = X_0 \bar{x}_n + Y_0 \bar{x}_n, \; n = 0, 1, 2, 3, \ldots \tag{5}$$

Where,

$$\bar{x}_n = \frac{1}{2}[(\overline{x_0} + \overline{y_0}\sqrt{D})^{n+1} + (\overline{x_0} + \overline{y_0}\sqrt{D})^{n+1}], \; n = 0, 1, 2, 3, \ldots$$

$$\bar{y}_n = \frac{1}{2\sqrt{D}}[(\overline{x_0} + \overline{y_0}\sqrt{D})^{n+1} - (\overline{x_0} - \overline{y_0}\sqrt{D})^{n+1}], \; n = 0, 1, 2, 3, \ldots.$$

Thus,

$$X_{n+1} = \bar{x}_n + 2\bar{y}_n, \; n = 0, 1, 2, 3, \ldots \tag{6}$$

$$Y_{n+1} = \bar{y}_n + \bar{x}_n, \; n = 0, 1, 2, 3, \ldots. \tag{7}$$

Where, $\bar{x}_n = \frac{1}{2}[(3 + 2\sqrt{2})^{n+1} + (3 - 2\sqrt{2})^{n+1}], \; n = 0, 1, 2, 3, \ldots$

$$\bar{y}_n = \frac{1}{2\sqrt{2}}[(3 + 2\sqrt{2})^{n+1} - (3 - 2\sqrt{2})^{n+1}], \; n = 0, 1, 2, 3, \ldots.$$

Take $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$, so that

$$\alpha^2 = 2\alpha + 1 = 3 + 2\sqrt{2} \text{ and } \beta^2 = 2\beta + 1 = 3 - 2\sqrt{2}.$$

Substituting $\alpha$, $\beta$, $\alpha^2$ and $\beta^2$ in $\bar{x}_n$ and $\bar{y}_n$ we get,

$$\bar{x}_n = \frac{1}{2}[(\alpha^2)^{n+1} + (\beta^2)^{n+1}], \; n = 0, 1, 2, 3, \ldots.$$

$$\bar{y}_n = \frac{1}{2\sqrt{2}}[(\alpha^2)^{n+1} - (\beta^2)^{n+1}], \; n = 0, 1, 2, 3, \ldots.$$

Therefore, equation (6) and (7) can be written as,

$$X_{n+1} = \frac{\alpha^{2n+3} + \beta^{2n+3}}{\alpha + \beta}, \; n = 0, 1, 2, 3, \ldots \tag{8}$$

$$Y_{n+1} = \frac{\alpha^{2n+3} - \beta^{2n+3}}{\alpha - \beta}, \; n = 0, 1, 2, 3, \ldots \tag{9}$$

Numerical examples

| n | $X_{n+1}$ | $T_{n+1} = Y_{n+1}$ | $x_{n+1} = 2T_{n+1} - 1$ | $y_{n+1} = 2T_{n+1}^2 - 2T_{n+1}$ | $z_{n+1} = 2T_{n+1}^2 - 2T_{n+1} + 1$ | Area $A_{n+1} = \frac{1}{2}x_{n+1}y_{n+1}$ |
|---|---|---|---|---|---|---|
| 0 | 7 | 5 | 9 | 40 | 41 | 180 |
| 1 | 41 | 29 | 57 | 1624 | 1625 | 46284 |
| 2 | 239 | 169 | 337 | 56784 | 56785 | 9568104 |
| 3 | 1393 | 985 | 1969 | 1938480 | 1938481 | 1908433560 |
| 4 | 8119 | 5741 | 11481 | 65906680 | 65906681 | 378337296540 |
| 5 | 47321 | 33461 | 66921 | 2239210120 | 2239210121 | 74925090220260 |
| 6 | 275807 | 195025 | 390049 | 76069111200 | 76069111201 | 14835340377224400 |
| 7 | 1607521 | 1136689 | 2273377 | 2584121492064 | 2584121492065 | 2937341182631990000 |
| 8 | 9369319 | 6625109 | 13250217 | 87784125273544 | 87784125273545 | 5815793545148210000 0 |
| 9 | 54608393 | 38613965 | 77227929 | 2982076508814 520 | 29820765088145 20 | 1151497964476480000 0000 |
| 10 | 318281039 | 225058681 | 450117361 | 1013028193368 02000 | 10130281933680 2000 | 2279907885087060000 000000 |

**Properties:**

1. Recurrence relation for $X$ and $Y$ are

$$X_{n+3} - 6X_{n+2} + X_{n+1} = 0,$$

$$Y_{n+3} - 6Y_{n+2} + Y_{n+1} = 0.$$

2. For all values of $n$, $X$ and $Y$ are odd.

3. For all values of $n$, $X_{n+3} + X_{n+1} \equiv 0 (\text{mod } 6)$

4. For all values of $n$, $Y_{n+3} + Y_n \equiv 0 (\text{mod } 6)$.

5. From the table, the Pythagorean triple $(x, y, z)$ is a Twin Pythagorean triple.

6. From the table, the values of $X$ and $Y$ Pell-Lucas and Pell Numbers respectively.

## 2. Remarks

**Remark 1.** For large values of $n$, the ratio $\frac{X_{n+1}}{X_n}$ approaches to $\alpha^2$ (Approximately equal to 6).

**Proof.** Since $X_{n+1} = \dfrac{\alpha^{2n+3} + \beta^{2n+2}}{\alpha + \beta}$, and $X_n = \dfrac{\alpha^{2n+1} + \beta^{2n+1}}{\alpha + \beta}$. Now, $\dfrac{\beta}{\alpha} = \dfrac{1 - \sqrt{2}}{1 + \sqrt{2}} = 2\sqrt{2} - 3$ and $\dfrac{\beta}{\alpha} \to 0$ as $n \to \infty$.

$$\frac{X_{n+1}}{X_n} = \frac{\dfrac{\alpha^{2n+3} + \beta^{2n+3}}{\alpha + \beta}}{\dfrac{\alpha^{2n+1} + \beta^{2n+1}}{\alpha + \beta}} = \frac{\alpha^{2n+3} + \beta^{2n+3}}{\alpha^{2n+1} + \beta^{2n+1}}$$

$$= \frac{\alpha^{2n+3}}{\alpha^{2n+1}} \frac{1 + \left(\dfrac{\beta}{\alpha}\right)^{2n+3}}{1 + \left(\dfrac{\beta}{\alpha}\right)^{2n+1}}$$

$$= \alpha^2 \left( \frac{1 + \left(\dfrac{\beta}{\alpha}\right)^{2n+3}}{1 + \left(\dfrac{\beta}{\alpha}\right)^{2n+1}} \right).$$

Thus,

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \alpha^2 \left( \frac{1 + \left(\dfrac{\beta}{\alpha}\right)^{2n+3}}{1 + \left(\dfrac{\beta}{\alpha}\right)^{2n+1}} \right)$$

$$= \alpha^2 \left( \frac{1 + 0}{1 + 0} \right) = \alpha^2$$

$$= 3 + 2\sqrt{2}$$

$$= 5.828427125 \cong 6.$$

**Remark 2.** For large values of $n$, the ratio $\dfrac{Y_{n+1}}{Y_n}$ approaches to $\alpha^2$ (Approximately equal to 6).

**Proof.** Since $Y_{n+1} = \dfrac{\alpha^{2n+3} - \beta^{2n+3}}{\alpha - \beta}$, and $Y_n = \dfrac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}$. Now,

$\frac{\beta}{\alpha} = \frac{1-\sqrt{2}}{1+\sqrt{2}} = 2\sqrt{2} - 3$ and $\frac{\beta}{\alpha} \to 0$ as $n \to \infty$.

$$\frac{Y_{n+1}}{Y_n} = \frac{\frac{\alpha^{2n+3} - \beta^{2n+3}}{\alpha - \beta}}{\frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}} = \frac{\alpha^{2n+3} - \beta^{2n+3}}{\alpha^{2n+1} - \beta^{2n+1}}$$

$$= \frac{\alpha^{2n+3}}{\alpha^{2n+1}}\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+3}}{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}}\right)$$

$$= \alpha^2\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+3}}{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}}\right).$$

Thus,

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \alpha^2\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+3}}{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}}\right)$$

$$= \alpha^2\left(\frac{1-0}{1-0}\right) = \alpha^2$$

$$= 3 + 2\sqrt{2}$$

$$= 5.828427125 \cong 6.$$

**Remark 3.** For large values of $n$, the ratio $\frac{Y_{n+1}}{X_{n+1}}$ approaches to $\frac{1}{\sqrt{2}}$.

**Proof.**

$$\lim_{n\to\infty} \frac{Y_{n+1}}{X_{n+1}} = \frac{\frac{\alpha^{2n+3} + \beta^{2n+3}}{\alpha - \beta}}{\frac{\alpha^{2n+3} + \beta^{2n+3}}{\alpha + \beta}} = \left(\frac{2}{2\sqrt{2}}\right)\left(\frac{\alpha^{2n+3}}{\alpha^{2n+3}}\right)\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+3}}{1+\left(\frac{\beta}{\alpha}\right)^{2n+3}}\right).$$

Since $\left(\frac{\beta}{\alpha}\right)^{2n+3} \to 0$ as $n \to \infty$.

Thus, $\lim_{n\to\infty} \frac{Y_{n+1}}{X_{n+1}} = \frac{1}{\sqrt{2}} \cong 0.707$. Similarly, $\lim_{n\to\infty} \frac{X_{n+1}}{Y_{n+1}} = \sqrt{2}$.

**Remark 4.** For large values of $n$, the ratio $\frac{A_{n+1}}{A_n}$ approaches to $\alpha^6$. (Approximately equal to 198).

**Proof.** Area can be represented by the formula $A_n = \frac{1}{2} x_n y_n$.

Where $x_n = 2T_n - 1 = 2Y_2 - 1$ and $y_n = 2T_n^2 - 2T_n = 2Y_n^2 - 2Y_n$.

So that

$$x_n = 2\left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}\right) - 1$$

$$= \frac{2(\alpha^{2n+1} - \beta^{2n+1}) - (\alpha - \beta)}{\alpha - \beta}$$

and

$$y_n = 2Y_n^2 - 2Y_n$$

$$= 2Y_n^2(Y_n - 1)$$

$$= 2\left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}\right)\left(\frac{2(\alpha^{2n+1} - \beta^{2n+1}) - (\alpha - \beta)}{\alpha - \beta}\right).$$

Area

$$A_n = \frac{1}{2} x_n y_n = \left(\frac{1}{2}\right)\left(\frac{2(\alpha^{2n+1} - \beta^{2n+1}) - (\alpha - \beta)}{\alpha - \beta}\right)$$

$$\left(2\left(\frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}\right)\left(\frac{(\alpha^{2n+1} - \beta^{2n+1}) - (\alpha - \beta)}{\alpha - \beta}\right)\right)$$

$$A_n = \frac{2}{(\alpha - \beta)^3}(\alpha^{2n+1} - \beta^{2n+1} - \frac{1}{2}(\alpha - \beta))(\alpha^{2n+1} - \beta^{2n+1})(\alpha^{2n+1} - \beta^{2n+1} - (\alpha - \beta)).$$

Similarly,

$$A_{n+1}=\frac{2}{(\alpha-\beta)^3}(\alpha^{2n+3}-\beta^{2n+3}-\frac{1}{2}(\alpha-\beta))(\alpha^{2n+3}-\beta^{2n+3})(\alpha^{2n+3}-\beta^{2n+3}-(\alpha-\beta)).$$

Now,

$$\frac{A_{n+1}}{A_n}=\frac{\frac{2}{(\alpha-\beta)^3}(\alpha^{2n+3}-\beta^{2n+3}-\frac{1}{2}(\alpha-\beta))(\alpha^{2n+3}-\beta^{2n+3})(\alpha^{2n+3}-\beta^{2n+3}-(\alpha-\beta))}{\frac{2}{(\alpha-\beta)^3}(\alpha^{2n+1}-\beta^{2n+1}-\frac{1}{2}(\alpha-\beta))(\alpha^{2n+1}-\beta^{2n+1})(\alpha^{2n+1}-\beta^{2n+1}-(\alpha-\beta))}$$

$$=\left(\frac{\alpha^{2n+3}}{\alpha^{2n+1}}\right)\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+3}\frac{(\alpha-\beta)}{2\alpha^{2n+3}}}{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}\frac{(\alpha-\beta)}{2\alpha^{2n+1}}}\right)\left(\frac{\alpha^{2n+3}}{\alpha^{2n+1}}\right)\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+3}}{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}}\right)$$

$$\left(\frac{\alpha^{2n+3}}{\alpha^{2n+1}}\right)\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{2n+3}\frac{(\alpha-\beta)}{\alpha^{2n+3}}}{1-\left(\frac{\beta}{\alpha}\right)^{2n+1}\frac{(\alpha-\beta)}{\alpha^{2n+1}}}\right).$$

As $n\to\infty,\left(\frac{\beta}{\alpha}\right)^n\to 0$ and $\alpha^n\to\infty\Rightarrow\frac{1}{\alpha^n}\to\lim_{n\to\infty}\frac{A_{n+1}}{A_n}=\left(\frac{\alpha^3}{\alpha}\right)$ $\left(\frac{\alpha^3}{\alpha}\right)\left(\frac{\alpha^3}{\alpha}\right)=\alpha^6=197.9949494\cong 198$. Similarly, $\lim_{n\to\infty}\frac{A_n}{A_{n+1}}=\frac{1}{\alpha^6}\cong 0$.

**Remark 5.** For all values of $n$, $X_n^2-2Y_n^2=-1$.

**Proof.** Since, $X_n=\frac{\alpha^{2n+1}+\beta^{2n+1}}{\alpha+\beta}$ and $Y_n=\frac{\alpha^{2n+1}-\beta^{2n+1}}{\alpha-\beta}$

$$X_n^2=\frac{\alpha^{4n+2}+\beta^{4n+2}+2(\alpha\beta)^{2n+1}}{(\alpha+\beta)^2}\Rightarrow 4X_n^2=\alpha^{4n+2}+\beta^{4n+2}+2(-1)^{2n+1}.$$

And, $Y_n^2=\frac{\alpha^{4n+2}+\beta^{4n+2}-2(\alpha\beta)^{2n+1}}{(\alpha+\beta)^2}\Rightarrow 8Y_n^2=\alpha^{4n+2}+\beta^{4n+2}-2(-1)^{2n+1}$.

$$4X_n^2-8Y_n^2=4(-1)^{2n+1}\Rightarrow X_n^2-2Y_n^2=(-1)^{2n+1}.$$

Thus, $X_n^2 - 2Y_n^2 = -1.$

## 3. Conclusion

In this paper, we observed the non-zero integer solutions of the problem which are exactly the Pell and Pell-Lucas numbers and it is interesting to see that the researcher can also proceed for further results in this problem.

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