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RADIUS PROBLEM FOR A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY FIRST ORDER DIFFERENTIAL INEQUALITY

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Abstract

Let $\mathcal{A}$ denote the class of all normalized analytic functions in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and Ω denote the subclass of $\mathcal{A}$ consisting of functions f satisfying the first order differential inequality

$$|zf'(z) - f(z)| < 1/2,\ z \in \Delta.$$

In this paper we obtain radius results for a combination of functions in the class $\mathcal{A}$ to be in the subclass Ω under certain conditions on their coefficients.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f defined on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with Taylor's series expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ containing univalent functions. De Branges Theorem states that the Taylor's coefficients of functions in $\mathcal{S}$ satisfy the inequality $|a_n| \le n,\ n \ge 2$ (see [2]). But the converse of the theorem is not true.

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For example, $f(z) = z + 2z^2$ is not a member of the class $\mathcal{S}$ though its coefficient satisfies de Branges Theorem. Therefore it is interesting to determine the radius of the largest disk inside the unit disk for which the converse holds. This problem is known as the radius problem for the class $\mathcal{S}$.

In general, if $\mathcal{P}$ is a property which the functions in a class $\mathcal{M}$ may or may not possess in the disk $|z| < r$, the radius of the property $\mathcal{P}$ in $\mathcal{M}$ is the largest $R > 0$ such that every function in $\mathcal{M}$ has the property $\mathcal{P}$ in each disk $\{z \in C : |z| < r\}$ for every $r < R$ [3]. If $\mathcal{F}$ and $\mathcal{G}$ are two subsets of $\mathcal{A}$ then the $\mathcal{G}$-radius of f in $\mathcal{F}$ is the largest R such that for each $f \in \mathcal{F}$, $r^{-1}f(rz) \in \mathcal{G}$ for each $r \leq R$. In [12, 4] the authors had considered this problem for various standard subclasses of $\mathcal{A}$. Recent interest among the researchers has been to solve the radii problems for certain combinations of functions in certain subclass to belong to another subclass [5, 6, 7, 8, 11].

A subclass Ω of $\mathcal{A}$ satisfying certain first order differential inequality was introduced by Peng et al. and its various properties were examined [9, 10]. In this paper we determine the radii for certain combinations of functions in the class $\mathcal{A}$ to be in the subclass Ω. We now recall the definition of the class Ω.

Definition 1.1 [9]. A function f in $\mathcal{A}$ belongs to the class Ω if

$$|zf'(z) - f(z)| < 1/2, \; z \in \Delta.$$

2. Main Results

Let f and g be two functions in the class $\mathcal{A}$ having the Taylor's series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (1)$$

respectively. Define

$$G(z) = \frac{f(z)g(z)}{z}. \quad (2)$$

Then $G \in \mathcal{A}$ and

$$| zG'(z) - G(z) | = \left| f'(z)g(z) + g'(z)f(z) - 2\left(\frac{f(z)g(z)}{z}\right)\right|.$$

$$| zG'(z) - G(z) | \leq | f'(z) || g(z) | + | g'(z) || f(z) | + 2| f(z) | \left| \frac{g(z)}{z} \right|. \tag{3}$$

Substituting the Taylor's series for the functions *f* and *g* we get

$$| zG'(z) - G(z) | \leq \left(1 + \sum_{n=2}^{\infty} n| a_n |r^{n-1}\right)\left(r + \sum_{n=2}^{\infty} | b_n |r^n\right) + \left(r + \sum_{n=2}^{\infty} | a_n |r^n\right)$$

$$\left(1 + \sum_{n=2}^{\infty} n| b_n |r^{n-1}\right) + 2\left(r + \sum_{n=2}^{\infty} | a_n |r^n\right)\left(1 + \sum_{n=2}^{\infty} | b_n |r^{n-1}\right). \tag{4}$$

Theorem 2.1. *Let* f, g *and* G *be given by* (1) *and* (2) *respectively*. *Let* $| a_n | \leq 1$ *for all* $n \geq 2$.

(i) *If* $| b_n | \leq 1$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $| z | < R_1$ *where* R_1 *is the root of the equation*

$$(1 - r)(1 - 2r + r^2) + 4r^2 - 8r = 0$$

in $(0, 1)$.

(ii) *If* $| b_n | \leq n$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $| z | < R_2$ *where* R_2 *is the root of the equation*

$$(1 - r)^2(1 - 2r + r^2) + r^2 - 4r = 0$$

in $(0, 1)$.

(iii) *If* $| b_n | \leq \frac{2}{n}$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $| z | < R_3$ *where* R_3 *is the root of the equation*

$$(1 - r)^2 + (12 - 8r)\log(1 - r) - 2r(3r - 2) = 0$$

in $(0, 1)$.

(iv) *If* $|b_n| \leq \frac{n+1}{2}$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $|z| < R_4$ *where* R_4 *is the root of the equation*

$$(1-r)^4 - 2r^3 + 7r^2 - 8r = 0$$

in $(0, 1)$.

Proof. (i) Applying the conditions $|a_n| \leq 1$ and $|b_n| \leq 1$ in (4) and using standard sums we get

$$|zG'(z) - G(z)| \leq \frac{2r}{(1-r)^3} + \frac{2r}{(1-r)^2}$$

$$= \frac{2r(2-r)}{(1-r)^3}$$

$$= \frac{1}{2} - \frac{1}{2(1-r)^3}((1-r)(1-2r+r^2) - 8r + 4r^2)$$

which is less than $\frac{1}{2}$ provided $\phi_1(r) > 0$ where

$$\phi_1(r) = (1-r)(1-2r+r^2) - 8r + 4r^2.$$

Thus $|zG'(z) - G(z)| < 1/2$ in the disk $|z| < r$, $0 \leq r < R_1$, where R_1 is a root of the equation

$$(1-r)(1-2r+r^2) + 4r^2 - 8r = 0.$$

in $(0, 1)$.

(ii) Applying the conditions $|a_n| \leq 1$ and $|b_n| \leq n$ in (4) we get

$$|zG'(z) - G(z)| \leq \frac{r}{(1-r)^4} + \frac{r(1+r)}{(1-r)^4} + \frac{2r}{(1-r)^3}$$

$$= \frac{2r(4-r)}{(1-r)^4}$$

$$= \frac{1}{2} - \frac{1}{2(1-r)^4}((1-r)^2(1-2r+r^2) - 4r + r^2)$$

which is less than $\frac{1}{2}$ if $\phi_2(r) > 0$, where

$$\phi_2(r) = (1-r)^2(1-2r+r^2) - 4r + r^2.$$

Thus $|zG'(z) - G(z)| < 1/2$ in the disk $|z| < r$, $0 \le r < R_2$, where R_2 is a root of the equation

$$(1-r)^2(1-2r+r^2) + r^2 - 4r = 0$$

in $(0, 1)$.

(iii) Applying the conditions $|a_n| \le 1$ and $|b_n| \le \frac{2}{n}$ in (4) we get

$$|zG'(z) - G(z)| \le \frac{-2\log(1-r) - r}{(1-r)^2} + \frac{r(1+r)}{(1-r)^2} + \frac{2(-2\log(1-r) - r)}{(1-r)}$$

$$= \frac{(-6+4r)\log(1-r) + r(3r-2)}{(1-r)^2}$$

$$= \frac{1}{2} - \frac{1}{2(1-r)^2}((1-r)^2 + (12-8r)\log(1-r) - 2r(3r-2).$$

which is less than $\frac{1}{2}$ if $\phi_3(r) > 0$ where

$$\phi_3(r) = (1-r)^2 + (12-8r)\log(1-r) - 2r(3r-2).$$

Thus $|zG'(z) - G(z)| < 1/2$ in the disk $|z| < r$, $0 \le r < R_3$, where R_3 is a root of the equation

$$(1-r)^2 + (12-8r)\log(1-r) - 2r(3r-2) = 0$$

in $(0, 1)$.

(iv) Applying the conditions $|a_n| \le 1$ and $|b_n| \le \frac{n+1}{2}$ in (4) we get

$$|zG'(z) - G(z)| \leq \frac{r(2+r)}{2(1-r)^4} + \frac{2r}{(1-r)^3} + \frac{r}{(1-r)^2}$$

$$= \frac{r(8-7r+2r^2)}{2(1-r)^4}$$

$$= \frac{1}{2} - \frac{1}{2(1-r)^4}((1-r)^4 - 8r + 7r^2 - 2r^3))$$

which is less than $\frac{1}{2}$ provided $\phi_4(r) > 0$ where

$$\phi_4(r) = (1-r)^4 - 2r^3 + 7r^2 - 8r.$$

Thus $|zG'(z) - G(z)| < 1/2$ in the disk $|z| < r,\ 0 \leq r < R_4$, where R_4 is a root of the equation

$$(1-r)^4 - 2r^3 + 7r^2 - 8r = 0$$

in $(0, 1)$. □

Theorem 2.2. *Let* f, g *and* G *be given by* (1) *and* (2) *respectively*. *Let* $|a_n| \leq n$ *for all* $n \geq 2$.

(i) *If* $|b_n| \leq n$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $|z| < R_5$ *where* R_5 *is the root of the equation*

$$(1-r)^5 - 8r = 0$$

in $(0, 1)$.

(ii) *If* $|b_n| \leq \frac{2}{n}$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $|z| < R_6$ *where* R_6 *is the root of the equation*

$$(1-r)(1-2r+r^2) + (12-4r)\log(1-r) + 4r(1-r) = 0$$

in $(0, 1)$.

(iii) *If* $|b_n| \leq \frac{n+1}{2}$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $|z| < R_7$ *where* R_7 *is the root of the equation*

$$(1-r)^5 + 2r^4 - 7r^3 + 9r^2 - 8r = 0$$

in $(0, 1)$.

Proof. (i) Applying the conditions $|a_n| \le n$ and $|b_n| \le n$ in (4) we get

$$|zG'(z) - G(z)| \le \frac{2r(1-r)}{(1-r)^5} + \frac{2r}{(1-r)^4}$$

$$= \frac{4r}{(1-r)^5}$$

$$= \frac{1}{2} - \frac{1}{2(1-r)^5}((1-r)^5 - 8r)$$

which is less than $\frac{1}{2}$ provided $\phi_5(r) > 0$ where $\phi_5(r) = (1-r)^5 - 8r$. Thus $|zG'(z) - G(z)| < 1/2$ in the disk $|z| < r$, $0 \le r < R_5$, where R_5 is a root of the equation

$$(1-r)^5 - 8r = 0$$

in $(0, 1)$.

(ii) Applying the conditions $|a_n| \le n$ and $|b_n| \le \frac{2}{n}$ in (4) we get

$$|zG'(z) - G(z)| \le \frac{-2\log(1-r)(1+-r)}{(1-r)^3} + \frac{-4\log(1+r)}{(1-r)^2} + \frac{2r}{(1-r)^2}$$

$$= \frac{(-6+2r)\log(1-r) - 2r(1-r)}{(1-r)^3}$$

$$= \frac{1}{2} - \frac{1}{2(1-r)^3}((1-r)(1-2r+r^2) + (12-4r)\log(1-r) + 4r(1-r)).$$

which is less than $\frac{1}{2}$ provided $\phi_6(r) > 0$ where

$$\phi_6(r) = (1-r)(1-2r+r^2) + (12-4r)\log(1-r) + 4r(1-r).$$

Thus $|zG'(z) - G(z)| < 1/2$ in the disk $|z| < r$, $0 \le r < R_6$, where R_6 is a root of the equation

$$(1-r)(1-2r+r^2)+(12-4r)\log(1-r)+4r(1-r).$$

in $(0, 1)$.

(iii) Applying the conditions $|a_n| \leq n$ and $|b_n| \leq \frac{n+1}{2}$ in (4) we get

$$|zG'(z)-G(z)| \leq \frac{r+r^2+2r(1-r)^2}{2(1-r)^5}+\frac{r+r^2+2r(1-r)^2}{2(1-r)^4}+\frac{r}{(1-r)^5}$$

$$=\frac{8r-9r^2+7r^3-2r^4}{2(1-r)^5}$$

$$=\frac{1}{2}-\frac{1}{2(1-r)^5}((1-r)^5+2r^4-7r^3+9r^2-8r)$$

which is less than $\frac{1}{2}$ if $\phi_7(r) > 0$ where

$$\phi_7(r)=(1-r)^5+2r^4-7r^3+9r^2-8r.$$

Thus $|zG'(z)-G(z)| < 1/2$ in the disk $|z| < r$, $0 \leq r < R_7$, where R_7 is a root of the equation $(1-r)^5+2r^4-7r^3+9r^2-8r=0.$ in $(0, 1)$. □

Theorem 2.3. *Let* f, g *and* G *be given by* (1) *and* (2) *respectively. Let* $|a_n| \leq \frac{2}{n}$ *for all* $n \geq 2$.

(i) *If* $|b_n| \leq \frac{2}{n}$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $|z| < R_8$ *where* R_8 *is the root of the equation*

$$8r^3-r^2+r+24r^2\log(1-r)+24r\log(1-r)-32\log(1-r)=0$$

in $(0, 1)$.

(ii) *If* $|b_n| \leq \frac{n+1}{2}$ *for all* $n \geq 2$, *then* $G \in \Omega$ *in the disk* $|z| < R_9$ *where* R_9 *is the root of the equation*

$$(1-r)^3+3r^3-9r^2+4r-4r^2\log(1-r)-16r\log(1-r)+12\log(1-r)=0$$

in $(0, 1)$.

Proof. (i) Applying the conditions $|a_n| \leq \frac{2}{n}$ and $|b_n| \leq \frac{2}{n}$ in (4) we get

$$|zG'(z) - G(z)| \leq \frac{2(-\log(1-r) - r)(2r^2 + 2r\log(1-r) - 2\log(1-r))}{r(1-r)}$$

$$= \frac{16\log(1-r) - 12r\log(1-r) - 12r^2\log(1-r) - 4r^3}{r(1-r)}$$

$$= \frac{1}{2} - \frac{1}{2r(1-r)}(8r^3 - r^2 + r + 24r^2\log(1-r) + 24r\log(1-r) - 32\log(1-r)).$$

which is less than $\frac{1}{2}$ provided $\phi_8(r) > 0$, where

$$\phi_8(r) = 8r^3 - r^2 + r + 24r^2\log(1-r) + 24r\log(1-r) - 32\log(1-r).$$

Thus $|zG'(z) - G(z)| < 1/2$ in the disk $|z| < r$, $0 \leq r < R_8$, where R_8 is a root of the equation $8r^3 - r^2 + r + 24r^2\log(1-r) + 24r\log(1-r) - 32\log(1-r) = 0.$ in $(0, 1)$.

(ii) Applying the conditions $|a_n| \leq \frac{2}{n}$ and $|b_n| \leq \frac{n+1}{2}$ in (4) we get

$$|zG'(z) - G(z)| \leq \frac{r(1+r)(2-r)}{2(1-r)^3} + \frac{(-2\log(1-r) - r)(3-r)}{(1-r)^2}$$

$$= \frac{-4r + 9r^2 - 3r^3 - 12\log(1-r) + 16r\log(1-r) - 4r^2\log(1-r)}{2(1-r)^3}$$

$$= \frac{1}{2} - \frac{1}{2(1-r)^3}((1-r)^3 + 3r^3 - 9r^2 + 4r - 4r^2\log(1-r) - 16r\log(1-r)$$

$$+ 12\log(1-r)).$$

which is less than $\frac{1}{2}$ provided $\phi_9(r) > 0$, where

$$\phi_9(r) = (1-r)^3 + 3r^3 - 9r^2 + 4r - 4r^2\log(1-r) - 16r\log(1-r) + 12\log(1-r).$$

Thus $|zG'(z) - G(z)| < 1/2$ in the disk $|z| < r$, $0 \leq r < R_9$, where R_9 is a root of the equation $(1-r)^3 + 3r^3 - 9r^2 + 4r - 4r^2 \log(1-r) - 16r \log(1-r)$ $+12 \log(1-r) = 0$ in $(0, 1)$. □

Theorem 2.4. *Let* f, g *and* G *be given by* (1) *and* (2) *respectively.*

If $|a_n| \leq \frac{n+1}{2}$ *and* $|b_n| \leq \frac{n+1}{2}$ *for all* $n \geq 2$, *then* $F \in \Omega$ *in the disk* $|z| < R_{10}$ *where* R_{10} *is the root of the equation*

$$(1-r)^5 - r(2-r)(4-3r+r^2) = 0$$

in $(0, 1)$.

Proof. By proceeding as in previous theorems and by applying the conditions $|a_n| \leq \frac{n+1}{2}$ and $|b_n| \leq \frac{n+1}{2}$ in (4) we get

$$|zG'(z) - G(z)| \leq \frac{r(2-r)}{2(1-r)^5} + \frac{r(2-r)^2}{2(1-r)^4}$$

$$= \frac{r(2-r)(4-3r+r^2)}{2(1-r)^5}$$

$$= \frac{1}{2} - \frac{1}{2(1-r)^5}((1-r)^5 - r(2-r)(4-3r+r^2))$$

which is less than $\frac{1}{2}$ if $\phi_{10}(r) > 0$ where

$$\phi_{10}(r) = (1-r)^5 - r(2-r)(4-3r+r^2).$$

Thus $|zG'(z) - G(z)| < 1/2$ in $|z| < r$, $0 \leq r < R_{10}$, where R_{10} is a root of the equation

$$(1-r)^5 - r(2-r)(4-3r+r^2) = 0$$

in $(0, 1)$.

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