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FULLY FUZZY ECONOMIC INVENTORY MODEL WITH BACKORDERS USING GENERALIZED QUADRILATERAL FUZZY NUMBERS

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Abstract

In every business scenario, inventory plays a vital role to maximize the profit and to minimize the expenditure. In every classical case, some of the inventory models are existed for different situations. When the situations become vague, it is very difficult to optimize the problems through classical inventory models. In this paper, we reviewed the concept of generalized quadrilateral fuzzy numbers (GQFN) and its arithmetic operations. We construct the fuzzy economic inventory model with backorders through GQFN's in the fully fuzzified manner. Also few numerical examples are provided to analyze the inventory model.

1. Introduction

Inventory models are the essential tools for the businessman to run the business successfully. It is simply a mathematical model to maintain the level of inventories. The model can be established for the two main sectors such as what are the materials to order and how many units to order. In every classical inventory models, the major objective is to minimize the total cost by determining the optimal economic order quantity or economic production

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quantity. But when the situation becomes imprecise, the classical inventory models are ineffectual. For this situation fuzzy inventory models are the beneficial one to handle the problems and to find the optimal solutions. These fuzzy concepts are initially introduced by Zadeh [13]. When we enter into the fuzzy case, fuzzy numbers are the important one to proceed. Dubois and Prade [1] discussed the concept of fuzzy numbers and its arithmetic operations. Stephen Dinagar and Abirami [6] introduced the more generalized interval valued fuzzy numbers with trapezoidal fuzzy numbers. Pathinathan and Santhoshkumar [5] introduced the quadrilateral fuzzy numbers as a generalization of pentagonal fuzzy numbers. Stephen Dinagar and Christopar Raj [7] proposed the concept of generalized quadrilateral fuzzy numbers and this is a more generalized version of trapezoidal fuzzy numbers. Stephen Dinagar and Manvizhi [8] studied the concept of generalized quadrilateral fuzzy numbers and its arithmetic operations through classical equivalent fuzzy mean. By using the triangular fuzzy numbers Yao and Wu [4] discussed the fuzzy inventory models by fuzzifying the order quantity and shortage quantity. Yao and Lee [10, 11] proposed the fuzzy inventory models with backorder by using the triangular fuzzy numbers and by the Nelder-Mead method. They also presented the fuzzy inventory model by using trapezoidal fuzzy numbers. Yao and Su [12] developed the fuzzy inventory model with backorder by using interval valued fuzzy set. Kazemi, Ehsani and Jaber [3] proposed the fuzzy inventory model with backorder by fuzzifying all the input parameters by triangular and trapezoidal fuzzy numbers. In this paper, we derived the fuzzy economic inventory model with backorders by generalized quadrilateral fuzzy numbers (GQFN's) in the fully fuzzified view. The paper organized as follows: section-2 deals with the preliminaries. In section-3, the arithmetic operations based on classical equivalent fuzzy mean are reviewed. In section-4, we discussed the concept of classical inventory models with backorders. In section-5, we presents the brief introduction to Karush Kuhn-Tucker (KKT) conditions. In section-6, we proposed the inventory model in the fully fuzzified case by GQFN's. In section-7, numerical examples are provided to verify the fuzzy inventory model. Finally, conclusion is also included.

2. Preliminaries

Definition 2.1. Fuzzy set A fuzzy set A in X characterized by a membership function $\mu_A(x)$ which mapping from the elements of X to the interval [0, 1].

The value of $\mu_A(x)$ is called the membership grade of x in A.

Definition 2.2. Fuzzy number

A fuzzy set $\widetilde{A}$ defined on R is said to be a fuzzy number if

(i) $\widetilde{A}s$ normal

(ii) $\widetilde{A}$ is convex

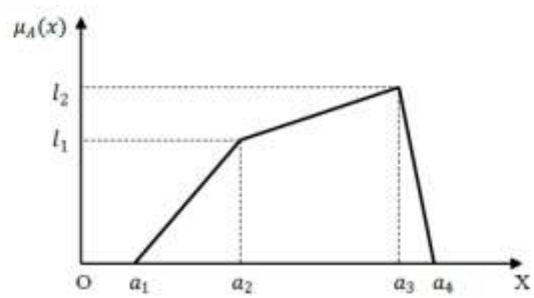
(iii) Supp $\widetilde{A}$ is closed and bounded.

Definition 2.3. Generalized quadrilateral fuzzy number.

A fuzzy number $\widetilde{A}$ is said to be a generalized quadrilateral fuzzy number (GQFN) represented by $\widetilde{A} = [a_1, a_2, a_3, a_4; l_1, l_2]$, where $a_1 \leq a_2 \leq a_3 \leq a_4$, when its membership function is given as

$$\mu_A(x) = \begin{cases} l_1 \dfrac{x - a_1}{a_2 - a_1}, a_1 \leq x \leq a_2 \\ \dfrac{(x - a_2)l_2 + (a_3 - x)l_1}{a_3 - a_2}, a_2 \leq x \leq a_3 \\ l_2 \dfrac{x - a_4}{a_3 - a_4}, a_3 \leq x \leq a_4 \\ 0, \qquad \text{otherwise} \end{cases}$$

And its pictorial representation is given by,



When $l_1 = l_2 = 1$, the generalized quadrilateral fuzzy number becomes a trapezoidal fuzzy number. When $a_1 = a_2$ or $a_2 = a_3$ or $a_3 = a_4$, then the generalized quadrilateral fuzzy number becomes a generalized triangular fuzzy number.

Note: Classical Equivalent Fuzzy Mean (CEFM)

If $F(R)$ is a set of generalized quadrilateral fuzzy numbers. The classical equivalent fuzzy mean M assigns a real number for each fuzzy number in $F(R)$.

For $\widetilde{A} = [a_1, a_2, a_3, a_4; l_1, l_2] \in F(R)$, then the classical equivalent fuzzy mean M is defined as $M(\widetilde{A}) = \frac{(a_1 + a_2 + a_3 + a_4)(l_1 + l_2)}{8}$.

Also we define orders on $F(R)$ by, $M(\widetilde{A}) \geq M(\widetilde{B})$ if and only if $\widetilde{A} \underset{M}{\geq} \widetilde{B}$, $M(\widetilde{A}) \leq M(\widetilde{B})$ if and only if $\widetilde{A} \underset{M}{\leq} \widetilde{B}$, and $M(\widetilde{A}) = M(\widetilde{B})$ if and only if $\widetilde{A} \underset{M}{=} \widetilde{B}$.

This method is the way to defuzzify the generalized quadrilateral fuzzy numbers. We called this method as a classical equivalent fuzzy mean only because of the properties in the section-4. Also it is known that based on the above said notion, the unique arithmetic operations on GQFN's have been proposed as follows.

3. Arithmetic Operations on Generalized Quadrilateral Fuzzy Numbers Based on CEFM

Let $\widetilde{A} = [a_1, a_2, a_3, a_4; l_1, l_2]$, and $\widetilde{B} = [b_1, b_2, b_3, b_4; m_1, m_2]$, be two GQFN's.

Take $\sigma_l = l_1 + l_2$ and $\sigma_m = m_1 + m_2$.

(i) **Addition:**

$$\widetilde{A} + \widetilde{B} = \left[\frac{2(a_1\sigma_l + b_1\sigma_m)}{\sigma_l + \sigma_m}, \frac{2(a_2\sigma_l + b_2\sigma_m)}{\sigma_l + \sigma_m}, \frac{2(a_3\sigma_l + b_3\sigma_m)}{\sigma_l + \sigma_m}, \right.$$

$$\frac{2(a_4\sigma_l + b_4\sigma_m)}{\sigma_l + \sigma_m}, \frac{l_1 + m_1}{2}, \frac{l_2 + m_2}{2}\Bigg]$$

(ii) **Subtraction:**

$$\widetilde{A} - \widetilde{B} = \Bigg[\frac{2(a_1\sigma_l - b_4\sigma_m)}{\sigma_l + \sigma_m}, \frac{2(a_2\sigma_l - b_3\sigma_m)}{\sigma_l + \sigma_m}, \frac{2(a_3\sigma_l - b_2\sigma_m)}{\sigma_l + \sigma_m},$$

$$\frac{2(a_4\sigma_l - b_4\sigma_m)}{\sigma_l + \sigma_m}, \frac{l_1 + m_1}{2}, \frac{l_2 + m_2}{2}\Bigg].$$

(iii) Scalar multiplication:

If $k > 0$, $k\widetilde{A} = [ka_1, ka_2, ka_3, ka_4; l_1, l_2]$

If $k < 0$, $k\widetilde{A} = [ka_4, ka_3, ka_2, ka_1; l_1, l_2]$.

(iv) Multiplication:

If $M(\widetilde{B}) > 0$,

$$\widetilde{A} \cdot \widetilde{B} = \Bigg[\frac{2a_1\sigma_l}{\sigma_l + \sigma_m} M(\widetilde{B}), \frac{2a_2\sigma_l}{\sigma_l + \sigma_m} M(\widetilde{B}), \frac{2a_3\sigma_l}{\sigma_l + \sigma_m} M(\widetilde{B}),$$

$$\frac{2a_4\sigma_l}{\sigma_l + \sigma_m} M(\widetilde{B}); \frac{l_1 + m_1}{2}, \frac{l_2 + m_2}{2}\Bigg]$$

If $M(\widetilde{B}) < 0$,

$$\widetilde{A} \cdot \widetilde{B} = \Bigg[\frac{2a_4\sigma_l}{\sigma_l + \sigma_m} M(\widetilde{B}), \frac{2a_3\sigma_l}{\sigma_l + \sigma_m} M(\widetilde{B}), \frac{2a_2\sigma_l}{\sigma_l + \sigma_m} M(\widetilde{B}),$$

$$\frac{2a_1\sigma_l}{\sigma_l + \sigma_m} M(\widetilde{B}); \frac{l_1 + m_1}{2}, \frac{l_2 + m_2}{2}\Bigg].$$

(v) Division:

If $M(\widetilde{B}) > 0$,

$$\frac{\widetilde{A}}{\widetilde{B}} = \Bigg[\frac{2a_1\sigma_l}{(\sigma_l + \sigma_m)M(\widetilde{B})}, \frac{2a_2\sigma_l}{(\sigma_l + \sigma_m)M(\widetilde{B})}, \frac{2a_3\sigma_l}{(\sigma_l + \sigma_m)M(\widetilde{B})},$$

$$\frac{2a_4\sigma_l}{(\sigma_l+\sigma_m)M(\widetilde{B})};\ \frac{l_1+m_1}{2},\ \frac{l_2+m_2}{2}\Bigg].$$

If $M(\widetilde{B}) > 0$,

$$\frac{\widetilde{A}}{\widetilde{B}} = \Bigg[\frac{2a_4\sigma_l}{(\sigma_l+\sigma_m)M(\widetilde{B})},\ \frac{2a_3\sigma_l}{(\sigma_l+\sigma_m)M(\widetilde{B})},\ \frac{2a_2\sigma_l}{(\sigma_l+\sigma_m)M(\widetilde{B})},$$

$$\frac{2a_1\sigma_l}{(\sigma_l+\sigma_m)M(\widetilde{B})};\ \frac{l_1+m_1}{2},\ \frac{l_2+m_2}{2}\Bigg].$$

(vi) Square:

$$\widetilde{A}^2 = [a_1 M(\widetilde{A}),\ a_2 M(\widetilde{A}),\ a_3 M(\widetilde{A}),\ a_4 M(\widetilde{A});\ l_1,\ l_2].$$

(vi) Square root:

If $M(B) > 0$,

$$\sqrt{\widetilde{A}} = \Bigg[\frac{a_1}{\sqrt{M(\widetilde{A})}},\ \frac{a_2}{\sqrt{M(\widetilde{A})}},\ \frac{a_3}{\sqrt{M(\widetilde{A})}},\ \frac{a_4}{\sqrt{M(\widetilde{A})}}\ l_1,\ l_2\Bigg].$$

4. Classical EOQ Model with Backorders

Assumptions:

T-The whole period of the plan

D-The annual demand

Q-The order quantity per cycle

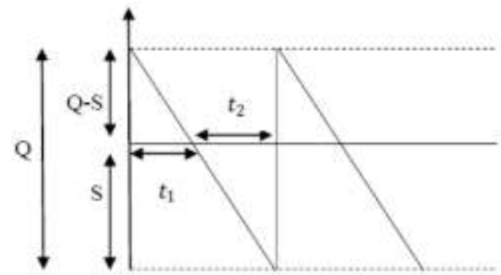
h-The holding cost per unit per cycle

a-The shortage (or backorder) cost

b-The order cost per cycle

S-Shortage (or backorder) quantity per cycle

t-The length of the inventory cycle.



The total cost $F(Q, S)$ during the planning period T is

$$F(Q, S) = \frac{(h+a)TS^2}{2Q} - hTS + \frac{hTQ}{2} + \frac{bD}{Q}, \; 0 < S < Q.$$

Therefore the optimal solution is,

$$Q^* = \left[\frac{2(h+a)bD}{haD}\right]^{1/2}, \; S^* = \left[\frac{2hbD}{a(h+a)}\right]^{1/2}, \; F(Q^*, S^*) = \left[\frac{2habDT}{h+a}\right]^{1/2}.$$

5. The Karush Kuhn-Tucker conditions [1]

Karush Kuhn-Tucker (KKT) conditions is a method to solve the nonlinear programming problems with differentiable functions. The theorem based on the KKT conditions referred from [1] is given as follows:

Theorem 1. *Assume that an objective function* $f(x)$ *and the constraints* $g_1(X), g_2(X), g_3(X), \ldots, g_n(X)$ *are differentiable satisfying certain regularity conditions. Then* $X^* = (x_1^*, x_2^*, x_3^*, \ldots, x_s^*)$ *is the optimal solution for the nonlinear programming problem only if there exists r numbers* $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r$ *such that all the following conditions are satisfied*:

$$\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \geq 0 \; at \; x = x^* \; for \; j = 1, 2, \ldots, s$$

1. $x_j^* \left(\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \right) = 0 \; at \; x = x^* \; for \; j = 1, 2, \ldots, s$

2. $g_i(x)^* - b_i \leq 0 \; for \; i = 1, 2, \ldots, r$

3. $\lambda_i(g_i(x)^* - b_i) = 0$ *for* $i = 1, 2, \ldots, r$

4. $x_j^* \geq 0$ *for* $j = 1, 2, \ldots, s$

$\lambda_i \geq 0$ *for* $i = 1, 2, \ldots, r.$

6. Fuzzy EOQ Model with Backorder

In this section, fuzzified economic inventory model with backorder through the KKT conditions presented in section-5 have been derived.

Now let as assume that the input parameters as a GQFN's. That is

The annual demand $\tilde{D}$ - $[d_1, d_2, d_3, d_4; \alpha_1, \alpha_2]$

The order quantity $\tilde{Q}$ - $[q_1, q_2, q_3, q_4; \omega_1, \omega_2]$

The holding cost $\tilde{h}$ - $[h_1, h_2, h_3, h_4; \beta_1, \beta_2]$

The shortage cost $\tilde{a}$ - $[a_1, a_2, a_3, a_4; \gamma_1, \gamma_2]$

The order cost $\tilde{b}$ - $[b_1, b_2, b_3, b_4; \theta_1, \theta_2]$

Shortage quantity $\tilde{S}$ - $[s_1, s_2, s_3, s_4; t_1, t_2]$.

The total cost $\tilde{F}(\tilde{Q}, \tilde{S})$ during the planning period T is

$$\tilde{F}(\tilde{Q}, \tilde{S}) = \frac{(\tilde{h} + \tilde{a})T\tilde{S}^2}{2\tilde{Q}} - \tilde{h}T\tilde{S} + \frac{\tilde{h}T\tilde{Q}}{2} + \frac{\tilde{b}\tilde{D}}{Q}. \tag{1}$$

By using this fuzzy total cost and the KKT conditions, we can find the optimal fuzzy order quantity, optimal fuzzy shortage quantity and the minimum fuzzy total cost.

By our assumptions $\tilde{Q}$ and $\tilde{S}$ are both the generalized quadrilateral fuzzy numbers. So we have $0 < q_1 \leq q_2 \leq q_3 \leq q_4$ and $0 < s_1 \leq s_2 \leq s_3 \leq s_4$.

To find the optimal solutions of this inventory model, we have to solve the non-linear programming problem given as follows.

Thus, the optimal solution of $\widetilde{F}(\widetilde{Q}, \widetilde{S})$ given in equation (1), subject to the following constraints:

$$q_1 - q_2 \le 0,\ q_2 - q_3 \le 0,\ q_3 - q_4 < 0,\ -q_1 \le 0 \text{ and}$$

$$s_1 - s_2 \le 0,\ s_2 - s_3 \le 0,\ s_3 \le 0,\ s_3 - s_4 \le 0,\ -s_1 < 0.$$

The above non-linear programming problem can be solved by using KKT conditions as given in section-5. The required KKT conditions of the above problem based on theorem-1 are,

$$S_1 \times \left\{ \frac{1}{\widetilde{Q}} \left[\frac{2(h_1\sigma_\beta + a_1\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_2\sigma_\beta + a_2\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_3\sigma_\beta + a_3\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_4\sigma_\beta + a_4\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{\beta_1 + \gamma_1}{2}, \frac{\beta_2 + \gamma_2}{2} \right] T\widetilde{S}(1, 0, 0, 0; t_1, t_2) - (h_1, h_1, h_1, h_1; \beta_1, \beta_2) T(1, 0, 0, 0; t_1, t_2) - \widetilde{u}_1 + \widetilde{u}_4 \right\} = \widetilde{0} \quad (2)$$

Where $\widetilde{0}$ represents (0, 0, 0, 0; 0, 0).

$$S_2 \times \left\{ \frac{1}{\widetilde{Q}} \left[\frac{2(h_1\sigma_\beta + a_1\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_2\sigma_\beta + a_2\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_3\sigma_\beta + a_3\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma} \frac{2(h_4\sigma_\beta + a_4\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{\beta_1 + \gamma_1}{2}, \frac{\beta_2 + \gamma_2}{2} \right] T\widetilde{S}(0, 1, 0, 0; t_1, t_2) - (h_1, h_1, h_1, h_1; \beta_1, \beta_2) T(0, 1, 0, 0; t_1, t_2) + \widetilde{u}_1 - \widetilde{u}_2 \right\} = \widetilde{0} \quad (3)$$

$$S_3 \times \left\{ \frac{1}{\widetilde{Q}} \left[\frac{2(h_1\sigma_\beta + a_1\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_2\sigma_\beta + a_2\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_3\sigma_\beta + a_3\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_4\sigma_\beta + a_4\sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{\beta_1 + \gamma_1}{2}, \frac{\beta_2 + \gamma_2}{2} \right] T\widetilde{S}(0, 0, 1, 0; t_1, t_2) - (h_1, h_1, h_1, h_1; \beta_1, \beta_2) T(0, 0, 1, 0; t_1, t_2) + \widetilde{u}_2 - \widetilde{u}_3 \right\} = \widetilde{0} \quad (4)$$

$$S_4 \times \left\{ \frac{1}{\widetilde{Q}} \left[\frac{2(h_1 \sigma_\beta + a_1 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_2 \sigma_\beta + a_2 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_3 \sigma_\beta + a_3 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \right.\right.$$

$$\left. \frac{2(h_4 \sigma_\beta + a_4 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{\beta_1 + \gamma_1}{2}, \frac{\beta_2 + \gamma_2}{2} \right] T\widetilde{S}(0, 0, 1, 0; t_1, t_2)$$

$$\left. - (h_1, h_1, h_1, h_1; \beta_1, \beta_2) T(0, 0, 0, 1; t_1, t_2) + \widetilde{u}_3 \right\} = \widetilde{0} \qquad (5)$$

$$q_1 \times \left(\frac{1}{\widetilde{Q}^2} \left[\frac{2(h_1 \sigma_\beta + a_1 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_2 \sigma_\beta + a_2 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_3 \sigma_\beta + a_3 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \right.\right.$$

$$\left.\left. \frac{2(h_4 \sigma_\beta + a_4 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{\beta_1 + \gamma_1}{2}, \frac{\beta_2 + \gamma_2}{2} \right]\right) T\widetilde{S}^2 + \frac{1}{2} (h_1, h_1, h_1, h_1; \beta_1, \beta_2) T$$

$$- \frac{1}{\widetilde{Q}^2} \left[\frac{2b_1 \sigma_\theta}{\sigma_\theta + \sigma_\alpha} M(\widetilde{D}), \frac{2b_2 \sigma_\theta}{\sigma_\theta + \sigma_\alpha} M(\widetilde{D}), \frac{2b_3 \sigma_\theta}{\sigma_\theta + \sigma_\alpha} M(\widetilde{D}), \right.$$

$$\left. \frac{2b_4 \sigma_\theta}{\sigma_\theta + \sigma_\alpha} M(\widetilde{D}), \frac{\theta_1 + \alpha_1}{2}, \frac{\theta_2 + \alpha_2}{2} \right] (1, 0, 0, 0, \omega_1, \omega_2) - \widetilde{u}_5 + \widetilde{u}_8 \Big\} = \widetilde{0} \qquad (6)$$

$$q_2 \times \left(\frac{1}{\widetilde{Q}^2} \left[\frac{2(h_1 \sigma_\beta + a_1 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_2 \sigma_\beta + a_2 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_3 \sigma_\beta + a_3 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \right.\right.$$

$$\left. \frac{2(h_4 \sigma_\beta + a_4 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{\beta_1 + \gamma_1}{2}, \frac{\beta_2 + \gamma_2}{2} \right] T\widetilde{S}^2 + \frac{1}{2} (h_1, h_1, h_1, h_1; \beta_1, \beta_2) T$$

$$- \frac{1}{\widetilde{Q}^2} \left[\frac{2b_1 \sigma_\theta}{\sigma_\theta + \sigma_\alpha} M(\widetilde{D}), \frac{2b_2 \sigma_\theta}{\sigma_\theta + \sigma_\alpha} M(\widetilde{D}), \frac{2b_3 \sigma_\theta}{\sigma_\theta + \sigma_\alpha} M(\widetilde{D}), \right.$$

$$\left. \frac{2b_4 \sigma_\theta}{\sigma_\theta + \sigma_\alpha} M(\widetilde{D}), \frac{\theta_1 + \alpha_1}{2}, \frac{\theta_2 + \alpha_2}{2} \right] (0, 1, 0, 0, \omega_1, \omega_2) + \widetilde{u}_5 - \widetilde{u}_6 \Big) \Big\} = \widetilde{0} \qquad (7)$$

$$q_3 \times \left(\frac{1}{\widetilde{Q}^2} \left[\frac{2(h_1 \sigma_\beta + a_1 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_2 \sigma_\beta + a_2 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_3 \sigma_\beta + a_3 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \frac{2(h_4 \sigma_\beta + a_4 \sigma_\gamma)}{\sigma_\beta + \sigma_\gamma}, \right.\right.$$

$$\left. \frac{\beta_1 + \gamma_1}{2}, \frac{\beta_2 + \gamma_2}{2} \right] T\widetilde{S}^2 + \frac{1}{2} (h_1, h_1, h_1, h_1; \beta_1, \beta_2) T$$

$$-\frac{1}{\widetilde{Q}^2}\left[\frac{2b_1\sigma_\theta}{\sigma_\theta+\sigma_\alpha}M(\widetilde{D}), \frac{2b_2\sigma_\theta}{\sigma_\theta+\sigma_\alpha}M(\widetilde{D}), \frac{2b_3\sigma_\theta}{\sigma_\theta+\sigma_\alpha}M(\widetilde{D}), \frac{2b_4\sigma_\theta}{\sigma_\theta+\sigma_\alpha}M(\widetilde{D}),\right.$$

$$\left.\frac{\theta_1+\alpha_1}{2}, \frac{\theta_2+\alpha_2}{2}\right](0, 0, 1, 0, \omega_1, \omega_2)+\widetilde{u}_6-\widetilde{u}_7\Bigg)\Bigg\} = \widetilde{0} \tag{8}$$

$$q_4\times\left(\frac{1}{\widetilde{Q}^2}\left[\frac{2(h_1\sigma_\beta+a_1\sigma_\gamma)}{\sigma_\beta+\sigma_\gamma}, \frac{2(h_2\sigma_\beta+a_2\sigma_\gamma)}{\sigma_\beta+\sigma_\gamma}, \frac{2(h_3\sigma_\beta+a_3\sigma_\gamma)}{\sigma_\beta+\sigma_\gamma}, \frac{2(h_4\sigma_\beta+a_4\sigma_\gamma)}{\sigma_\beta+\sigma_\gamma},\right.\right.$$

$$\left.\frac{\beta_1+\gamma_1}{2}, \frac{\beta_2+\gamma_2}{2}\right]T\widetilde{S}^2+\frac{1}{2}(h_1, h_1, h_1, h_1; \beta_1, \beta_2)T$$

$$-\frac{1}{\widetilde{Q}^2}\left[\frac{2b_1\sigma_\theta}{\sigma_\theta+\sigma_\alpha}M(\widetilde{D}), \frac{2b_2\sigma_\theta}{\sigma_\theta+\sigma_\alpha}M(\widetilde{D}), \frac{2b_3\sigma_\theta}{\sigma_\theta+\sigma_\alpha}M(\widetilde{D}), \frac{2b_4\sigma_\theta}{\sigma_\theta+\sigma_\alpha}M(\widetilde{D}),\right.$$

$$\left.\frac{\theta_1+\alpha_1}{2}, \frac{\theta_2+\alpha_2}{2}\right](0, 0, 0, 1, \omega_1, \omega_2)+\widetilde{u}_7\Bigg)\Bigg\} = \widetilde{0} \tag{9}$$

$$\widetilde{u}_i(s_i - s_{i+1}) = \widetilde{0}, \; i = 1, 2, 3 \tag{10}$$

$$\widetilde{u}_4(-s_1) = \widetilde{0} \tag{11}$$

$$\widetilde{u}_{4+i}(q_i - q_{i+1}) = \widetilde{0}, \; i = 1, 2, 3 \tag{12}$$

$$\widetilde{u}_8(-q_1) = \widetilde{0} \tag{13}$$

$$s_i \geq 0, \; q_i \geq 0, \; i = 1, 2, 3, 4 \text{ and } M(\widetilde{u}_j) \geq 0. \tag{14}$$

From the constraints (11) and (13) we have $\widetilde{u}_4 = \widetilde{u}_8 = \widetilde{0}$. Then solving the equations (2)-(14) we get the optimal solutions,

$$\widetilde{Q}^* = \left[\frac{4(h_1\sigma_\beta+a_1\sigma_\gamma)}{(3\sigma_\beta+\sigma_\gamma)M(\widetilde{h})}, \frac{4(h_2\sigma_\beta+a_2\sigma_\gamma)}{(3\sigma_\beta+\sigma_\gamma)M(\widetilde{h})}, \frac{4(h_3\sigma_\beta+a_3\sigma_\gamma)}{(3\sigma_\beta+\sigma_\gamma)M(\widetilde{h})},\right.$$

$$\left.\frac{4(h_4\sigma_\beta+a_4\sigma_\gamma)}{(3\sigma_\beta+\sigma_\gamma)M(\widetilde{h})}; \frac{3\beta_1+\gamma_1}{4}, \frac{3\beta_2+\gamma_2}{4}\right]\widetilde{S}^* \tag{15}$$

$$\widetilde{S}^* = \left[\frac{64b_1\sigma_\theta M(\widetilde{D})M(\widetilde{h})}{(M(\widetilde{h})+M(\widetilde{a}))M(\widetilde{a})T(8\sigma_\theta+8\sigma_\alpha+11\sigma_\beta+5\sigma_\gamma)},\right.$$

$$\frac{64b_2\sigma_\theta M(\widetilde{D})M(\widetilde{h})}{(M(\widetilde{h})+M(\widetilde{a}))M(\widetilde{a})T(8\sigma_\theta+8\sigma_\alpha+11\sigma_\beta+5\sigma_\gamma)}$$

$$\frac{64b_3\sigma_\theta M(\widetilde{D})M(\widetilde{h})}{(M(\widetilde{h})+M(\widetilde{a}))M(\widetilde{a})T(8\sigma_\theta+8\sigma_\alpha+11\sigma_\beta+5\sigma_\gamma)},$$

$$\frac{64b_4\sigma_\theta M(\widetilde{D})M(\widetilde{h})}{(M(\widetilde{h})+M(\widetilde{a}))M(\widetilde{a})T(8\sigma_\theta+8\sigma_\alpha+11\sigma_\beta+5\sigma_\gamma)}$$

$$\left.\frac{8\theta_1+8\alpha_1+11\beta_1+5\gamma_1}{32}, \frac{8\theta_2+8\alpha_2+11\beta_2+5\gamma_2}{32}\right]^{1/2}. \quad (16)$$

Therefore $\widetilde{F}(\widetilde{Q}^*, \widetilde{S}^*)$ can be obtained by substituting the values of $\widetilde{Q}^*$ and $\widetilde{S}^*$ in equation (1).

7. Numerical Examples

If the parameters of the classical inventory models having the values $h = 10, a = 20, b = 200, D = 2000$ and $T = 12$, then the optimal order quantity $\widetilde{Q}^* = 100$, the shortage quantity $\widetilde{S}^* = 33.333$ and the minimum total cost $\widetilde{F}(\widetilde{Q}^*, \widetilde{S}^*) = 8000$.

Table 1. Generalized quadrilateral fuzzy values of holding cost (h).

$\widetilde{h}$	$M(\widetilde{a})$	Change%
(5, 11, 20, 28; 0.4, 0.6)	8	– 20
(8, 12, 23, 29; 0.4, 0.6)	9	– 10
(0, 10, 30, 40; 0.4, 0.6)	10	0
(2, 9, 33, 44; 0.4, 0.6)	11	10
(5, 20, 26, 45; 0.4, 0.6)	12	20

Table 2. Generalized Quadrilateral fuzzy values of shortage cost (a).

$\widetilde{a}$	$M(\widetilde{a})$	Change%
(46, 63, 72, 75; 0.2, 0.3)	16	– 20
(57, 68, 79, 84; 0.2, 0.3)	18	– 10
(63, 76, 88, 93; 0.2, 0.3)	20	0
(69, 85, 96, 102; 0.2, 0.3)	22	10
(76, 94, 105, 109; 0.2, 0.3)	24	20

Table 3. Generalized Quadrilateral fuzzy values of order cost (b).

$\widetilde{b}$	$M(\widetilde{b})$	Change%
(210, 260, 320, 490; 0.3, 0.7)	160	– 20
(260, 300, 370, 510; 0.3, 0.7)	180	– 10
(300, 350, 400, 550; 0.3, 0.7)	200	0
(330, 390, 480, 560; 0.3, 0.7)	220	10
(370, 450, 510, 590; 0.3, 0.7)	240	20

Table 4. Generalized Quadrilateral fuzzy values of demand (D).

$\widetilde{D}$	$M(\widetilde{D})$	Change%
(200, 1000, 2900, 3900; 0.7, 0.9)	1600	– 20
(550, 1200, 3100, 4150; 0.7, 0.9)	1800	– 10
(800, 1500, 3300, 4400; 0.7, 0.9)	2000	0
(950, 1750, 3650, 4650; 0.7, 0.9)	2200	10
(1200, 1950, 3900, 4950; 0.7, 0.9)	2400	20

Table 5. The optimal fuzzy shortage quantity $(\widetilde{S}^*)$.

$\widetilde{S}^*$	$M(\widetilde{S}^*)$	Change%
(1088.435, 1347.587, 1658.568, 2539.683; 0.419, 0.653)	29.814	– 10.5570
(1347.587, 1554.908, 1917.719, 2643.343; 0.419, 0.653)	31.623	– 5.132
(1554.908, 1814.059, 2073.210, 2850.664; 0.419, 0.653)	33.333	0
(1710.398, 2021.380, 2487.852, 2902.494; 0.419, 0.653)	34.960	4.881
(1917.719, 2332.362, 2643.343, 3057.985; 0.419, 0.653)	36.515	9.545

Table 6. The optimal fuzzy order quantity $(\widetilde{Q}^*)$.

$\widetilde{Q}^*$	$M(\widetilde{Q}^*)$	Change%
(119.256, 181.014, 238.512, 278.974; 0.350, 0.525)	89.442	– 10.558
(146.57, 184.718, 250.976, 285.109; 0.350, 0.525)	94.869	– 5.131
(119.999, 182.855, 281.902, 329.521; 0.350, 0.525)	99.999	– 0.001
(132.576, 187.059, 294.209, 345.06; 0.350, 0.525)	104.880	4.880
(149.538, 233, 272.993, 346.023; 0.350, 0.525)	109.545	9.545

Table 7. The minimum fuzzy total cost $(\widetilde{F}(\widetilde{Q}^*, \widetilde{S}^*))$.

$\widetilde{F}(\widetilde{Q}^*, \widetilde{S}^*)$	$M(\widetilde{F}(\widetilde{Q}^*, \widetilde{S}^*))$	Change%
(– 1872, 5821.006, 15561.58, 25419.19; 0.402, 0.617)	5724.334	– 28.446
(775.575, 6577.381, 19142.79, 27116.38; 0.402, 0.617)	6830.520	– 14.619
(– 7750.68, 4120.732, 26424.6, 39996.62; 0.402, 0.617)	8000.000	0
(– 7580.13, 3533.008, 32104.71, 44384.05; 0.402, 0.617)	9229.518	15.369
(– 5087.27, 16192.98, 24755.23, 46680.33; 0.402, 0.617)	10516.273	31.453

As in the way we solve the problem in the crisp sense, here we found the optimal solutions of the fuzzy inventory model with the fuzzy parameters as given in the tables 1-4 and by using the formulas in equation (15) and (16), the required solutions of the problems, that is the optimal fuzzy shortage quantity, the optimal fuzzy order quantity and the minimum fuzzy total cost have been represented in the tables 4-6 respectively. In the tables we represent the generalized quadrilateral fuzzy values of each fuzzy parameter, their respective classical equivalent fuzzy mean values and its change percentage related to the values of the parameters in the crisp sense. In the previous literature of the fuzzy inventory models [3] with the trapezoidal fuzzy number has the minimum change percentage 5. But in this study we proposed the method to find optimal fuzzy solution with the change percentage nearest to 0. This is advantage of the proposed method, and for this benefit, this method be the suitable method for the fuzzy situations in the inventory theory.

8. Conclusion

In this paper, the generalized quadrilateral fuzzy numbers and its arithmetic operations are reviewed. We developed the concept of fully fuzzified economic inventory models by means of GQFN's. By using these techniques, we may solve the fuzzy economic inventory problems and produce the required solutions also in the fuzzy sense. With these concepts, the deterministic and also the probabilistic inventory models may be utilized in future.

References

[1] D. Duboisand and H. Prade, Operations on fuzzy numbers, International Journal of Systems Science 9(6) (1978), 613-626.

[2] F. S. Hilier and G. J. Jieberman, Introduction to Operations Research, McGraw-Hill, USA, 2001.

[3] N. Kazemi, E. Eshani and M. Y. Jaber, An inventory model with backorder for fuzzy parameters and decision variables, International Journal of Approximate Reasoning 5 (2010), 964-97.

[4] Kweimei Wu and Jing-Shing Yao, Fuzzy inventory with back order for fuzzy order quantity and fuzzy shortage quantity, European Journal of Operational Research 150 (2003), 320-352.

[5] T. Pathinathan and S. Santhoshkumar, Quadrilateral fuzzy number, International Journal of Engineering and Technology 7(4-10) (2018), 1018-1021.

[6] D. Stephen Dinagar and D. Abirami, On critical path project scheduling using TOPSIS ranking of more generalized interval valued fuzzy numbers, Malaya Journal of Matematik S(2) (2015), 485-495.

[7] D. Stephen Dinagar, and B. Christopar Raj, A method for solving fully fuzzy transportation problem with generalized quadrilateral fuzzy numbers, Malaya Journal of Matematik, S(1) (2019), 24-27.

[8] D. Stephen Dinagar and M. Manvizhi, A Study on Generalized Quadrilateral Fuzzy Numbers, Malaya Journal of Matematik, (Communicated).

[9] H. A. Taha, Operations Research, Prentice Hall, New Jersey, USA, 1997.

[10] J. S. Yao and H. M. Lee, Fuzzy inventory with backorder for fuzzy order quantity, Information Sciences (1996), 283-319.

[11] J. S. Yao and H. M. Lee, Fuzzy inventory with or without backorder for order quantity with trapezoidal fuzzy number, Fuzzy Sets and Systems 105 (1999), 311-337.

[12] J. S. Yao and J. S. Su, Fuzzy inventory with backorder for fuzzy total demand based on interval-valued fuzzy set, European Journal of Operational Research 124 (2000), 396-408.

[13] L. A. Zadeh, Fuzzy Sets, Information and Control 8 (1965), 338-353.