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NORMALIZATION, INACCURACY AND CRISPNESS

G. S. BUTTAR, VIKRAMJEET SINGH and HARPREET KAUR

Department of Mathematics
Chandigarh University
Mohali, India
E-mail: gurcharanbuttar@gmail.com

IKGPTU Campus Batala
Punjab Technical University, India
E-mail: vikram31782@gmail.com

Department of Mathematics
Khalsa College for Women
Guru Nanak Dev University
Amritsar, India
E-mail: harjas2702@gmail.com

Abstract

The conception of distance is pertinent to practically every field of Engineering and Science from a block of brick to a chromosome, which encourages en-route forth expansion of numerous mathematical models. It is usual that the conception can be protracted to contributions in further disciplines as well. In this communication, expressions for the normalized measures of divergence for fuzzy distributions are derived as normalized measures are more convenient for comparison, relation between the fuzzy entropy, divergence and inaccuracy between two fuzzy sets is obtained.

I. Introduction

Entropy is one of the tools to measure information. Zadeh [17] was the first to employ entropy to measure fuzziness. Fuzzy entropy is used to measure fuzziness of a set. Zadeh's [17] had been applied in various fields such as speech recognition, feature selection, aircraft control, pattern

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identification, etc. In the literature of distance measures, one of the most significant and worthwhile measure of relative entropy written below is owing to Kullback and Leibler [8]. Keeping the necessity of new distance measures, Kapur [6] introduced the various measure of directed divergence between two probability distributions. Ferreri [9] developed the probabilistic measure of relative entropy. The measure-theoretic explanation of Kullback-Leibler's [8] divergence measure plays an elementary role in outlining numerous classical information measures on general spaces. The work in the field of fuzzy set theory was initiated by Zadeh [17]. Thus, conforming to the probabilistic measure of relative entropy Bhandari and Pal [1] developed the measure of fuzzy relative entropy between fuzzy sets. Kapur [7] presented some axioms to describe the expressions of fuzzy relative entropy for fuzzy sets.

Tran and Duckstein [13] have established a new method based on relative entropy for ranking fuzzy numbers. With similar motive some new FD measures for continuous fuzzy distributions have been developed by Buttar, Sharma and Sharma [2-5], Bhatia [18] etc. The fuzzy measure of inaccuracy and its generalizations have been done by Verma and Sharma [16]. In second section to compare accurately various measures their normalized values are obtained. Third section deals with the relation between the fuzzy entropy, divergence and inaccuracy between two fuzzy sets. In section 4 relation of most fuzzy set and measure of crispness is established.

II. Normalized Weighted Measures of FD

The measures the probabilistic measures of directed divergence are to measure the degree of difference between two probability distributions. To compare two distributions, that which one is more uniform we required normalized measures in place of actual one. Thus to compare correctly the uniformity or equality or the uncertainty between them, we compare normalized measures. The normalized value of the measure lies between 0 and one, so two or more measures can be compared effortlessly. This induces us to normalize the measures the probabilistic measures of directed divergence. We consider probability distributions $P = (p_1, p_2, \dots, p_n)$ and $Q = (q_1, q_2, \dots, q_n)$, whereas $W = (w_1, w_2, \dots, w_n)$; $w_i \geq 0$ be any weighted distribution.

Theorem 1. $D(P : Q; W)$ *is a valid measure of fuzzy directed divergence.*

Proof:

(i) $D(P : Q; W) \geq 0$

(ii) $D(P : Q; W) = 0$ *iff* $P = Q$

(iii) $D(P : Q; W) = D(Q : P; W)$

(iv) $D(P : Q; W)$ *is a convex function of both distributions.*

Three normalizations are given below:

$$\overline{D}_1(P; Q; W) = \frac{D(P : Q; W)}{\max\limits_{P, Q} D(P : Q; W)}, \tag{1}$$

$$\overline{D}_2(P; Q; W) = \frac{D(P : Q; W)}{\max\limits_{P} D(P : Q; W)}, \tag{2}$$

and

$$\overline{D}_3(P; Q; W) = \frac{D(P : Q : W)}{\max\limits_{Q} D(P : Q; W)}. \tag{3}$$

In general for any values of weights, $\overline{D}_1(P; Q; W)$ and $\overline{D}_3(P; Q; W)$ normalizations are not worthy. We, consider the normalization for $\overline{D}_2(P; Q; W)$. For this, we take the assumption that no component of distribution Q is zero. Keeping Q fixed, we discover distribution P for the maximum value of $D(P : Q; W)$.

$$\max_{P} D(P : Q; W) = \max_{r} D(\Delta_{r} : Q; W). \tag{4}$$

Where $\Delta_1 = (1, 0, 0, \ldots, 0)$, $\Delta_2 = (0, 1, 0, \ldots, 0)$ and $\Delta_{r} = (0, 0, 0, \ldots, 1)$.

Thus

$$\overline{D}_2(P : Q; W) = \frac{D(P : Q; W)}{\max\limits_{r} D(\Delta_{r} : Q; W)}. \tag{5}$$

The normalized value of the measure takes between 0 and 1, zero when distributions are identical and unity for which P is degenerate distribution for the maximum value of $D_2(\Delta r : Q; W)$. Thus, for evaluation of $\overline{D}_2(P : Q; W)$, fixed the distribution Q

(i) Calculate relative entropy between P to Q.

(ii) Calculate maximum value of relative entropy.

(iii) Calculate the proportion of (i) and (ii).

Next, we normalized the fuzzy measures. The supreme value of the weighted fuzzy relative entropy $D(P : Q; W)$ has maximum for which fuzzy set A is crisp i.e. values of the set are either 0 or 1.

$$\bar{D}_2(A : B; W) = \frac{D(A; B; W)}{\max_r D(C_r : B; W)}, \tag{6}$$

where C_r; $r = 1, 2, \ldots n$ is any crisp set.

Thus, to normalize weighted fuzzy directed divergence, we take the ratio of FD to the maximum possible divergence. Let us consider fuzzy relative entropy equivalent to well known Kullback-Leibler's [8].

$$D(A : B; W) = \sum_{i=1}^{n} w_i \left[\mu_B(x_i) f\left\{\frac{\mu_A(x_i)}{\mu_B(x_i)}\right\} + (1 - \mu_B(x_i)) f\left\{\frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)}\right\}\right], \tag{7}$$

where $f(\cdot)$ is a convex function and twice differentiable, taking $f(1) = 0$.

Now

$$\begin{aligned} D(C_r : B; W) &= w_1 \mu_B(x_1) f(0) + w_2 \mu_B(x_2) f(0) + \ldots \\ &\quad + w_r \mu_B(x_r) f(0) + \ldots + w_n \mu_B(x_n) f(0) \\ &\quad + w_1 \mu_B(x_1) f\left(\frac{1}{\mu_B(x_1)}\right) + w_2 \mu_B(x_2) f\left(\frac{1}{\mu_B(x_2)}\right) + \ldots + w_r \mu_B(x_r) f\left(\frac{1}{\mu_B(x_r)}\right) \\ &\quad + \ldots + w_n \mu_B(x_n) f\left(\frac{1}{\mu_B(x_n)}\right) + w_1 (1 - \mu_B(x_1)) f\left(\frac{1}{1 - \mu_B(x_1)}\right) \\ &\quad + w_2 (1 - \mu_B(x_2)) f\left(\frac{1}{1 - \mu_B(x_2)}\right) \end{aligned}$$

$$+ \dots + w_r(1-\mu_B(x_r))f\left(\frac{1}{1-\mu_B(x_r)}\right) + \dots + w_n(1-\mu_B(x_n))f\left(\frac{1}{1-\mu_B(x_n)}\right)$$

$$+ w_1(1-\mu_B(x_1))f(0) + w_2(1-\mu_B(x_2))f(0) + \dots$$

$$+ w_r(1-\mu_B(x_r))f(0) + \dots + w_n(1-\mu_B(x_n))f(0)$$

$$= f(0)\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i\mu_B(x_i)f\left(\frac{1}{\mu_B(x_i)}\right) + \sum_{i=1}^{n} w_i(1-\mu_B(x_i))f\left(\frac{1}{1-\mu_B(x_i)}\right). \quad (8)$$

Let

$$\phi_w[\mu_B(x)] = f(0)\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i\mu_B(x_i)f\left(\frac{1}{\mu_B(x_i)}\right)$$

$$+ \sum_{i=1}^{n} w_i(1-\mu_B(x_i))f\left(\frac{1}{1-\mu_B(x_i)}\right) \quad (9)$$

$$\phi'_w[\mu_B(x)] = -w_i\frac{1}{\mu_B(x_i)}f'\left(\frac{1}{\mu_B(x_i)}\right) + w_i f\left(\frac{1}{\mu_B(x_i)}\right)$$

$$+ w_i\frac{1}{1-\mu_B(x_i)}f'\left(\frac{1}{1-\mu_B(x_i)}\right) - w_i f\left(\frac{1}{1-\mu_B(x_i)}\right) \quad (10)$$

and

$$\phi''_w[\mu_B(x)] = w_i\left[\frac{1}{\mu_B^3(x_i)}f''\left\{\frac{1}{\mu_B(x_i)}\right\} + \frac{1}{(1-\mu_B(x_i))^3}f''\left\{\frac{1}{1-\mu_B(x_i)}\right\}\right]. \quad (11)$$

Now since $f(\cdot)$ is convex, we must have

$$f''\left\{\frac{1}{\mu_B(x_i)}\right\} > 0 \text{ and } f''\left\{\frac{1}{1-\mu_B(x_i)}\right\} > 0.$$

Thus equation (11) implies that $\phi''_w[\mu_B(x)] > 0$.

Also $\phi_w[\mu_B(x)]$ is a convex function of $\mu_B(x)$.

From equations (8) and (9), we have

$$D(C_r : B; W) = \phi_w[\mu_B(x_r)]. \quad (12)$$

For maximum value at $r = k$, equation (6) provides

$$\max_r D(C_r : B; W) = \phi_w[\mu_B(x_k)]$$

$$= \sum_{i=1}^{n} w_i \left[f(0) + \mu_B(x_k) f\left\{\frac{1}{\mu_B(x_k)}\right\} + (1 - \mu_B(x_k)) f\left\{\frac{1}{1 - \mu_B(x_k)}\right\}\right]. \quad (13)$$

Equations (6), (7) and (13) give

$$\overline{D}_2(A : B; W) = \frac{\sum_{i=1}^{n} w_i \left[\mu_B(x_i) f\left\{\frac{\mu_A(x_i)}{\mu_B(x_i)}\right\} + (1 - \mu_B(x_i)) f\left\{\frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)}\right\}\right]}{\sum_{i=1}^{n} w_i \left[f(0) + \mu_B(x_k) f\left\{\frac{1}{\mu_B(x_k)}\right\} + (1 - \mu_B(x_k)) f\left\{\frac{1}{1 - \mu_B(x_k)}\right\}\right]}. \quad (14)$$

Next, we interpret special cases for divergence:

I. Kullback-Leibler's [8] weighted normalized FD

The measure of weighted fuzzy relative entropy can be obtained by substituting $f(x) = x \log x$ in equation (14).

$${}_1\overline{D}_2 (A : B; W) = \frac{\sum_{i=1}^{n} w_i \left[\mu_A(x_i) \log\left\{\frac{\mu_A(x_i)}{\mu_B(x_i)}\right\} + (1 - \mu_A(x_i)) \log\left\{\frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)}\right\}\right]}{-\log \mu_B(x_k) \sum_{i=1}^{n} w_i - \log (1 - \mu_B(x_k)) \sum_{i=1}^{n} w_i}$$

$$= \frac{\sum_{i=1}^{n} w_i \left[\mu_A(x_i) \log\left\{\frac{\mu_A(x_i)}{\mu_B(x_i)}\right\} + (1 - \mu_A(x_i)) \log\left\{\frac{1 - \mu_A(x_i)}{1 - \mu_B(x_i)}\right\}\right]}{-\log \mu_B(x_k)(1 - \mu_B(x_k)) \sum_{i=1}^{n} w_i}.$$

II. Havrada-Charvat's [15] weighted normalized FD

To find normalized weighted measure of *FD*, we take $f(x) = \frac{X^\alpha - X}{\alpha - 1}$ in (14).

$$ {}_2\overline{D}_2(A : B; W) = \frac{\sum_{i=1}^{n} w_i[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}(1 - \mu_B(x_i))^{1-\alpha} - 1]}{[\mu_B^{1-\alpha}(x_i) + (1 - \mu_B(x_i))^{1-\alpha} - 2]\sum_{i=1}^{n} w_i}. $$

III. Sharma-Taneja's [12] weighted normalized FD

To obtain an expression, we take

$$ f(x) = \frac{x^{\alpha} - x^{\beta}}{\alpha - \beta};\ 0 < \alpha < 1,\ \beta > 0 \text{ or } \alpha > 1,\ 0 < \beta < 1. $$

Thus, equation (14) gives

$$ {}_3\overline{D}_2(A:B;W) = \frac{\sum_{i=1}^{n} w_i \begin{bmatrix} \mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) - \mu_A^{\beta}(x_i)\mu_B^{1-\beta}(x_i) + \\ (1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha} - (1-\mu_A(x_i))^{\beta}(1-\mu_B(x_i))^{1-\beta} \end{bmatrix}}{[\mu_B^{1-\alpha}(x_i) - \mu_B^{1-\beta}(x_i) + (1-\mu_B(x_i))^{1-\alpha} - (1-\mu_B(x_i))^{1-\beta}]\sum_{i=1}^{n} w_i}, $$

which gives for Sharma and Taneja's [12] weighted normalized FD.

III. Relation Between Fuzzy Entropy, Fuzzy Inaccuracy and Fuzzy Measure of Divergence

It was Kerridge [10], who proposed and explained the concept of inaccuracy in probability spaces and derived the following formula for its calculations:

$$ I(P : Q) = -\sum_{i=1}^{n} p_i \log q_i, \tag{15} $$

where $P = (p_1, p_2, \ldots, p_n)$ is the true probability distribution and $Q = (q_1, q_2, \ldots, q_n)$ is the predicted probability distribution. If A and B are different then the fuzzy set A is ambiguous because of the difference between A and B. In this case, the difference or the directed distance $D(A, B)$ of A

from B is considered as reason for this ambiguity. Thus, we conclude that the total inaccuracy or total ambiguity is given by the following expression:

$$I(A, B) = \text{Fuzzy entropy of the set } A + D(A, B).$$

By using above expression, we can measure the total inaccuracy present in any fuzzy set and consequently develop measures of total ambiguity. We have developed the following measures of total inaccuracy for the following measure of fuzzy entropy:

$$H_\alpha(A) = -\sum_{i=1}^{n}[\mu_A^\alpha(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))^\alpha \log(1-\mu_A(x_i))], \alpha \neq 1, \alpha > 1. \quad (16)$$

And corresponding measure of fuzzy relative entropy:

$$D_\alpha(A, B) = \sum_{i=1}^{n}\left[\mu_A^\alpha(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i))^\alpha \log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right]. \quad (17)$$

From equations (16) and (17), we have the following expression for the total fuzzy inaccuracy:

$$_1I_a(A, B) = H_a(A) + D_a(A, B).$$

Thus, we have the following measure of fuzzy inaccuracy:

$$_1I_a(A, B) = -\sum_{i=1}^{n}[\mu_A^\alpha(x_i)\log\mu_B(x_i) + (1-\mu_A(x_i))^\alpha \log(1-\mu_B(x_i))]. \quad (18)$$

II. Next we consider the measure of fuzzy entropy of order α by Buttar, Sharma and Sharma [2-4]

$$H^\alpha(A) = -\sum_{i=1}^{n}[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))]$$

$$+ \frac{2^{\alpha-1}}{1-\alpha}\sum_{i=1}^{n}[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha - 1]; \ \alpha \neq 1, \ \alpha > 1. \quad (19)$$

For this measure of entropy, Buttar, Sharma and Sharma's [2, 4] measure of FD:

$$D^{\alpha}(A, B) = \sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i))\log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right]$$

$$+\frac{2^{\alpha-1}}{\alpha-1}\sum_{i=1}^{n}[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}-1]. \quad (20)$$

From (19) and (20), we have the following expression for the total fuzzy inaccuracy:

$${}_2I_{a}(A, B) = H^{\alpha}(A) + D^{\alpha}(A, B).$$

Thus, we have the following measure of fuzzy inaccuracy:

$${}_2I_{a}(A, B) = -\sum_{i=1}^{n}[\mu_A(x_i)\log\mu_B(x_i) + (1-\mu_A(x_i))\log(1-\mu_B(x_i))]$$

$$+\frac{2^{\alpha-1}}{1-\alpha}\sum_{i=1}^{n}[\mu_A^{\alpha}(x_i) + (1-\mu_B(x_i))^{1-\alpha} + (1-\mu_A(x_i))^{\alpha}\{1-(1-\mu_B(x_i)^{1-\alpha})\}]. \quad (21)$$

III. Buttar, Sharma and Sharma's [2] measure of fuzzy entropy of order α:

$${}_{\alpha}H(A) = -\sum_{i=1}^{n}[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))]$$

$$+\frac{1}{1-\alpha}\sum_{i=1}^{n}\log[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}],\ \alpha\neq 1,\ \alpha > 0. \quad (22)$$

For this measure of entropy, Buttar, Sharma and Sharma's [2, 4] introduced the following measure of FD:

$${}_{\alpha}D(A, B) = \sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i))\log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right]$$

$$+\frac{1}{\alpha-1}\sum_{i=1}^{n}\log[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}]. \quad (23)$$

From equations (22) and (23), we have the following measure of fuzzy inaccuracy:

$$ {}_3I_d(A, B) = -\sum_{i=1}^{n} [\mu_A(x_i)\log\mu_B(x_i) + (1-\mu_A(x_i))\log(1-\mu_B(x_i))] $$

$$ + \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\frac{\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}}{\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}}\right]. \quad (24) $$

This relation between the fuzzy entropy, divergence and inaccuracy between two fuzzy sets is obtained, can be applied in corporate problems. In these problems corporation has to deal with clients and significant number of issues to be negotiated. This relation is helpful and can be applied to similar business situations for selecting the best client, where degree of negotiation (inaccuracy) should be minimum on common issues.

IV. Relationship Between Most Fuzzy Distribution and Measure of Crispness

Conceptually, fuzzy entropy gives a measure of fuzzy uncertainty of a fuzzy distribution while FD gives a measure of distance of one fuzzy set from another fuzzy set. Fuzzy uncertainty measures how close the fuzzy distribution is from the most fuzzy vector distribution and how far it is from the distribution of crisp sets. Thus, the most fuzzy distribution vector $F = (1/2, 1/2, \ldots, 1/2)$ represents maximal uncertainty fuzzy distribution, since the uncertainty is maximum on account of all outcomes being equally likely. Using the same concept, crispness can be measured for a fuzzy set. We take any monotonic non-decreasing function, which measure distance of A from F. For larger distance the set A is less fuzzy and crispier, below, we discuss and obtain measures of crispness by converting probabilistic measures into fuzzy distribution. These measures can be applied in many fields such as fuzzy reasoning, image processing and pattern recognition.

I. Weighted measure of crispness corresponding to Kullback and Leibler's [8] FD measure:

We have

$$ D_1(A : B; W) = \sum_{i=1}^{n} w_i\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} + (1-\mu_A(x_i))\log\frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))}\right]. \quad (25) $$

Taking $\mu_B(x_i) = \frac{1}{2}$ for each i, equation (31) gives

$$D_1(A : F; W) = \sum_{i=1}^{n} w_i [\mu_A(x_i) \log 2\mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i))]$$

$$= \log 2 \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log 2(1 - \mu_A(x_i))].$$

Thus, the first weighted measure of crispness is given by

$$C_1(A; W) = \log 2 \sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i))]. \quad (26)$$

II. Weighted measure of crispness corresponding to Havrada-Charvat's [15] FD measure:

$$D_2^{\alpha}(A : B; W) = \frac{1}{\alpha(\alpha - 1)} \sum_{i=1}^{n} w_i [\mu_A^{\alpha}(x_i) \mu_B^{1-\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha}$$

$$(1 - \mu_B(x_i))^{1-\alpha} - 1]; \ \alpha \neq 0, 1, \alpha > 0. \quad (27)$$

Taking $\mu_B(x_i) = \frac{1}{2}$ for each i, equation (33) gives

$$D_2^{\alpha}(A : F; W) = \frac{1}{\alpha(\alpha - 1)} \sum_{i=1}^{n} w_i [\mu_A^{\alpha}(x_i) 2^{\alpha-1} + (1 - \mu_A(x_i))^{\alpha} 2^{\alpha-1} - 1]$$

$$= \frac{2^{\alpha-1}}{\alpha(\alpha - 1)} \sum_{i=1}^{n} w_i [\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha} - 1]$$

$$+ \frac{2^{\alpha-1}}{\alpha(\alpha - 1)} \sum_{i=1}^{n} w_i - \frac{1}{\alpha(\alpha - 1)} \sum_{i-1}^{n} w_i.$$

Thus, the weighted measure of crispness corresponding to (33) is

$$C_2(A; W) = \frac{2^{\alpha-1} - 1}{\alpha(\alpha - 1)} \sum_{i=1}^{n} w_i + \frac{2^{\alpha-1} - 1}{\alpha(\alpha - 1)} \sum_{i=1}^{n} w_i [\mu_A^{\alpha}(x_i) + (1 - \mu_A(x_i))^{\alpha} - 1]. \quad (28)$$

III. Weighted crispness measure due to Burg's [14] measure:

$$D_3(A:B;W)=\sum_{i=1}^{n} w_i\left[\frac{\mu_A(x_i)}{\mu_B(x_i)}-\log\frac{\mu_A(x_i)}{\mu_B(x_i)}+\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}-\log\frac{1-\mu_A(x_i)}{1-\mu_B(x_i)}-2\right]. \quad (29)$$

Taking the set B as most fuzzy, we have

$$D_3(A:F;W)=\sum_{i=1}^{n} w_i[-2\log 2-\log\mu_A(x_i)+\log(1-\mu_A(x_i))].$$

Thus, the weighted measure of crispness is

$$C_3(A;W)=2\log\frac{1}{2}\sum_{i=1}^{n} w_i-\sum_{i=1}^{n} w_i[\log\mu_A(x_i)+\log(1-\mu_A(x_i))]. \quad (30)$$

IV. Weighted measure of crispness corresponding to directed divergence measure due to Renyi's [11]:

Renyi's [11] measure of weighted directed divergence is given by

$$D_5(A:B;W)=\frac{1}{\alpha-1}\sum_{i=1}^{n} w_i\log[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}]. \quad (31)$$

Corresponding to this measure (31), the measure of crispness is

$$C_5(A;W)=\log 2\sum_{i=1}^{n} w_i+\frac{1}{\alpha-1}\sum_{i=1}^{n} w_i\log[\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}]. \quad (32)$$

V. Weighted measure of crispness corresponding to Kapur's [7] *FD* measure:

Kapur's [7] measure is given by

$$D_6(A:B;W)=\frac{1}{\alpha-\beta}\sum_{i=1}^{n} w_i\left[\frac{\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}}{\mu_A^{\beta}(x_i)\mu_B^{1-\beta}(x_i)+(1-\mu_A(x_i))^{\beta}(1-\mu_B(x_i))^{1-\beta}}\right]. \quad (33)$$

The corresponding measure of crispness is

$$C_6(A;W)=\frac{2^{\alpha-\beta}}{\alpha-\beta}\sum_{i=1}^{n} w_i\left[\frac{\mu_A^{\alpha}(x_i)+(1-\mu_A(x_i))^{\alpha}}{\mu_A^{\beta}(x_i)+(1-\mu_A(x_i))^{\beta}}\right]. \quad (34)$$

V. Conclusion

In Section 2 we have established the conception of normalized measures of fuzzy directed divergence. It has been scrutinized that the efficiency of the process is raised if redundancies and overlapping in comparable situations are removed. "Inaccuracy" measure has not to be based on randomness but on fuzziness. Relation of entropy, divergence and inaccuracy is applicable to client negotiation situations, etc. The relation between most fuzzy set and crispness can be applied to fuzzy reasoning, image processing and pattern recognition.

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