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2-CONNECTEDNESS OF DEZA GRAPHS

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Abstract

A graph G is said to be a Deza graph with parameters (n, λ, μ) if and only if, any two adjacent vertices of G have λ common neighbours and any two non-adjacent vertices of G have μ common neighbours. The present paper explores the 2-connectedness property of a connected Deza graph which is not isomorphic to a fan graph or a star graph.

1. Introduction

The graph G is said to be strongly regular [2] with parameters (n, k, λ, μ) if the following conditions hold.

(1) each vertex has k neighbours;

(2) any two adjacent vertices of G have λ common neighbours;

(3) any two non-adjacent vertices of G have μ common neighbours.

In literature there are two weaker notions of strongly regular graphs. A graph satisfying the conditions (1) and (2) is called an edge-regular graph [3] with parameters (n, k, λ). A graph satisfying the conditions (2) and (3) is called a Deza graph [3] with parameters (n, λ, μ). It is evident that strongly regular graphs are Deza graphs, but the converse does not hold. In fact, the class of strongly regular graphs is far more restricted than that of Deza graphs, with a rich structure that has attracted numerous researchers. For excellent overviews on the topic, see [4, 5, 7].

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The concept of connectivity in graph theory plays an important role in network analysis. The vertex connectivity of a connected graph G is the minimum k for which there exists a k-vertex cut is called the vertex connectivity or simply the connectivity of G, it is denoted by $\kappa(G)$ or simply κ. The edge connectivity of a connected graph G is the smallest k for which there exists a k-edge cut. And is denoted by $\lambda(G)$ or simply λ. A graph G is r-connected if $\kappa(G) \geq r$. Also, G is r-edge connected if $\lambda(G) \geq r$. Good overviews on the topic can be found in [1], [9], [8], [11], [16] and [15]. The present paper explores the 2-connectedness property of a connected Deza graph which is not isomorphic to a fan graph and a star graph.

In this paper the following definitions are used.

Two triangles sharing a common edge i.e.; $K_4 - e$, is called a diamond. An n-fan is n triangles sharing a common vertex [10]. In this paper for convenience, complete graphs K_n (n fixed) sharing a common vertex is also termed as a fan graph. Let S and S' be two non empty subsets of $V(G)$ then $[S, S']$ denote the set of all edges of G that have one end vertex in S and the other in S' [1]. All graph theoretic notations and terminology not mentioned here are from [1].

2. Deza Graphs

This section studies about the 2-connectedness property of Deza graphs. To study this the following theorems are used.

Theorem 2.1. [1] *A graph G with at least three vertices is 2-connected if and only if any two vertices of G lie on a common cycle.*

Theorem 2.2. [1] *A graph G with at least three vertices is 2-edge connected if and only if any two distinct vertices of G are connected by at least two edge-disjoint paths in G.*

Theorem 2.3. *Let $G = (n, \lambda, \mu)$ be a connected Deza graph with $\mu \geq 2$, then G is 2-connected.*

Proof. Consider any two vertices u and v in G. Either $uv \in E(G)$ or $uv \notin E(G)$. If $uv \in E(G)$, then there are two possibilities.

1. $\lambda \geq 1$;

2. $\lambda = 0$.

If $\lambda \geq 1$ and $uv \in E(G)$, then clearly u and v belong to at least one C_3. If not, since the graph is connected, $\mu \geq 2$ and $\lambda = 0$ there exist at least two distinct vertices u_1 and v_1 such that u_1 is adjacent to u and v_1 is adjacent to v. If $u_1v_1 \in E(G)$, then uvv_1u_1u is a cycle of length 4 containing u and v. On the other hand if $u_1v_1 \notin E(G)$, since $\mu \geq 2$ and $\lambda = 0$ there exist at least two vertices other than u and v common to both u_1 and v_1. Let p be a vertex common to both u_1 and v_1. Then uvv_1pu_1u is a cycle of length 5 containing u and v.

Now for the second case $uv \notin E(G)$, since $\mu \geq 2$, clearly there exists a cycle C_4 containing the vertices u and v.

Thus any two vertices of G belong to a cycle. Hence G is 2-connected. □

Note. By Theorem 2.3, if $G = (n, \lambda, \mu)$ be a connected Deza graph with $\mu \geq 2$, then G is 2-connected. Also, when we consider a Deza graph with $\mu = 0$, the graph is the disjoint union of complete graphs, since there exists no induced $K_{1, 2}$. Hence if G is a Deza graph with parameters $(n, \lambda, 0)$ then, G is 2-connected and $\lambda = n - 2$. Hence it is enough to consider the case, i.e., $\mu = 1$.

Lemma 2.4. *Let G be a connected Deza graph with parameters $(n, 0, 1)$ where $n > 3$. Then either $G \cong K_{1, n-1}$ or any two non-adjacent vertices belong to a C_5.*

Proof. *Clearly $G \cong K_{1, n-1}$ is a Deza graph with parameters $(n, 0, 1)$. If* not, since $n > 3$ the graph G has at least 4 vertices. Consider any two vertices u and v in G. Since $\lambda = 0$ and $\mu = 1$, C_4's and K_3's are forbidden in G. Also $G \ncong K_{1, n-1}$ implies that, there exists an induced P_4 containing u and v in G. Consider the end vertices of this induced P_4. Since $\mu = 1$ and $\lambda = 0$, these end vertices must have a common vertex other than the vertices belong to the above mentioned P_4. Hence u and v belong to a C_5. □

Theorem 2.5. *If G be a connected Deza graph with parameters $(n, 0, 1)$ where $n > 3$ then, either $G \cong K_{1, n-1}$ or G is a 2-connected graph.*

Proof. By Lemma 2.4, if $G \ncong K_{1, n-1}$, then any two vertices of G belong to a C_5. Hence the result follows from Theorem 2.1. □

Theorem 2.6. *Let $G = (n, 1, \mu)$ be a connected Deza graph. If $u \in V(G)$, then $d(u)$ is an even number and u together with its neighbours form a $\frac{d(u)}{2}$-fan, where $d(u)$ denotes the degree of the vertex u.*

Proof. Consider an arbitrary vertex u in G. Let e be an edge incident on u. Since $\lambda = 1$, the edge e belongs to a K_3. If the degree of u is two, there are no more edges incident on u. Hence the neighbours of u form a 1-fan. If not there exists a third edge incident on u. Since $\lambda = 1$, diamonds are forbidden in G therefore there exist a fourth edge incident on u and the third and the fourth edges together form a K_3. If the degree of u is four, there are no more edges incident on u. Hence the neighbours of u form a 2-fan. Proceeding like this we can find $d(u)$ is an even number and u together with its neighbours form a $\frac{d(u)}{2}$-fan.

Theorem 2.7. *Let $G = (n, 1, 1)$ be a connected Deza graph. If G is not isomorphic to a fan graph then G is 2-connected.*

Proof. Consider two vertices u and v belong to G. If u is adjacent to v then u and v belong to a K_3. If not, since G is a connected graph with $\mu = 1$ there exists a vertex x adjacent to both u and v. Now by Lemma 2.6, x together with its neighbours form a $\frac{d(u)}{2}$-fan. Since G is not isomorphic to a fan graph, there exists a vertex z adjacent to any one of the neighbouring vertices of x. Let y be that vertex. Then there are three possibilities:

1. $y = u$ or $y = v$.

2. y is adjacent either to u or v;

3. y is not adjacent to both u and v.

For the first case suppose $y = u$, i.e., z is adjacent to u. Since $\mu = 1$, z and v are non-adjacent. So by hypothesis there exists a vertex w adjacent to both z and v other than the vertex x. Then $wzyuxvw$ form a C_5 in G.

For the second case, without loss of generality we assume y is adjacent to u. Since c_4's and diamond's are forbidden in G, there exists an induced P_4 connecting z and v. Since $\mu = 1$, there must exists a vertex w adjacent to both z and v. Then $wzyuxvw$ is a cycle of length 6 containing the vertices u and v in G. Now for the third case, since $\mu = 1$, the vertex z is also not adjacent to both u and v. So by hypothesis there exist two distinct vertices w_1 and w_2 such that the common vertex of u and z is w_1 and the common vertex of v and z is w_2. Hence uw_1zw_2vxu is a cycle of length 6 containing the vertices u and v in G.

Thus any two vertices of G belong to a cycle. Hence by Theorem 2.1, G is 2-connected. □

Lemma 2.8. *Let $G = (n, \lambda, 1)$ be a connected Deza graph with $\lambda \geq 2$. Then every vertex is a central vertex of at least one wheel.*

Note. In the above Lemma, wheel is considered only as a subgraph not as an induced subgraph.

Proof. Consider an arbitrary vertex u in G and an edge e incident to u. Since $\lambda \geq 2$ there exists two other edges e_1 and e_2 incident to u such that e_1, e, e_2 must induce a diamond with central edge e. Now by hypothesis there exists two diamonds with central edges e_1 and e_2. Proceeding like this the process will terminate only when u together with some set of edges form a wheel. Also it may happen that there exists another edges incident to u which is not in the above wheel. Then by the same argument we can identify another wheel with u as the central vertex. □

Theorem 2.9. *If $G = (n, \lambda, 1)$ be a connected Deza graph with $\lambda \geq 2$.* If *then G is not isomorphic to a fan graph then G is 2-connected.*

Proof. As in the previous proof, consider two vertices u and v belong to G.

If u is adjacent to v then u and v belong to a K_3. If not, since G is a connected graph with $\mu = 1$ there exists a vertex x adjacent to both u and v. Now by Lemma 2.8, x together with some set of edges form a wheel W_k. If $u, v \in V(W_k)$ then clearly u and v belong to a C_k. If not by Lemma 2.8 there exist two wheels W_k and W_l such that $u \in V(W_k)$ and $v \in V(W_l)$. Since G is not isomorphic to a fan graph there exists at least one vertex not adjacent to x. Let p denote that vertex. Since $\mu = 1$, the vertex p is not adjacent to both u and v. Without loss of generality let p is adjacent to u. since $\mu = 1$ there exist a vertex r such that r is adjacent to both v and p. Then uprvxu is a cycle of length 5 connecting u and v. On the other hand if p is not adjacent to both u and v, since $\mu = 1$ there exist two distinct vertices q and r such that q is adjacent to both u and p and q is adjacent to both v and p. Then uqprvxu is a cycle of length 6 connecting u and v. Hence by Theorem 2.1, G is 2-connected.

□

Theorem 2.10. *Let G be a connected Deza graph not isomophic to a fan graph or a star graph, then G is* 2*-connected*.

Proof. The result follows from the above theorems and Theorem 2.1. □

Theorem 2.11. *Let G be a connected Deza graph not isomophic to a fan graph or a star graph, then G is* 2*-edge connected*.

Proof. From the above theorems, any two vertices $u, v \in V(G)$ belong to at least one cycle C. Since C is the union of two edge disjoint $u - v$ paths, the result follows from Theorem 2.2. □

References

[1] R. Balakrishnan and K. Ranganathan, A text book of graph theory, Springer, (1999).

[2] R. B., Bapat, Graphs and Matrices, Springer, (2010).

[3] L. Beineke, R. Wilson and P. Cameron, Topics in Algebraic Graph Theory: 102 (Encyclopedia of Mathematics and its Applications, Series Number 102), Cambridge University Press, (2004), 1-54.

[4] A. Brouwer and H. Van Maldeghem, Strongly regular graphs, https://homepages.cwi.nl/~aeb/math/srg/rk3/srgw.pdf.

[5] P. Cameron, Strongly regular graphs, Topics in Algebraic Graph Theory 102 (2004), 203-221.

[6] D. Cvetcovic, Peter Rowlinson and Slobodan Simic, An Introduction to the Theory of Graph Spectra, London Mathematical Society Students Texts 75 (2010).

[7] C. Godsil and G. Royle, Algebraic graph theory, Springer Science and Business Media, 207 (2001).

[8] F. Harary, Graph Theory, Narora publishing House, (1988).

[9] F. Harary, The maximum connectivity of graph, Proceedings of the National Academy of Sciences of the United States of America 48(7) (1962), 1142-1146.

[10] Jeepamol J. Palathingal, Aparna Lakshmanan S., Gallai and anti-Gallai graph operators, Electron. Notes Discrete Math. 6(3) (2017), 447-453.

[11] J. Harant, An upper bound for the radius of a 3-connected graph, Discrete Mathematics 122(1-3) (1993), 335-341.

[12] E. Prisner, Graph Dynamics, Longman, (1995).

[13] Le, V. B., Gallai graphs and anti-Gallai graphs, Discrete Math. 159 (1996), 179-189.

[14] E. Sampathkumar and S. B. Chikkodimath, The Semi total graphs of a graph-II, J. Karnatak Univ. Sci. 18 (1973), 281-284.

[15] B. West, Introduction to Graph Theory, Prentice-Hall, (1999).

[16] S. Mukwembi, The radius triangle-free graphs with prescribed edge connectivity, Utilitas Mathematica 77 (2008), 135-144.